AMERICAN UNIVERSITY OF BEIRUT

# ON SENDOV'S CONJECTURE

by
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# Abstract
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Sendov's Conjecture states that if a complex polynomial of degree $n \geq 2$ has all its zeros inside the closed unit disk, then each zero is at distance no more than one from at least one critical point. This conjecture is known for $n < 9$, but only partial results are available for higher $n$. In 2020, Prof. Terence Tao proved this conjecture for sufficiently high degree polynomials in a singular contribution that departs from conventional approaches. This thesis studies his work after examining the existing ground of theorems that relate the zeros and critical points of complex polynomials.

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# Chapter 1

# Introduction

**Sendov's Conjecture states that if a complex polynomial of degree $n \geq 2$ has all its zeros inside the closed unit disk $\overline{D(0,1)}$, then each zero is at distance no more than one from at least one critical point.**

This conjecture is known for $n < 9$, but only partial results are available for higher $n$. In his paper "Sendov's conjecture for sufficiently high degree polynomial"[1], that was released in 2020, Terence Tao shows the existence of a constant $n_0$ such that Sendov's conjecture holds for $n \geq n_0$. The proof of Tao relies on compactness methods and so does not easily give an effective value of $n_0$.

Sendov's conjecture was included in the collection of Research Problems in Function Theory, published in 1967 by Professor Hayman. Since it had been brought to Hayman's attention by Professor Ilyeff, it became known as "Ilyeff's Conjecture". Actually, this conjecture was due to the Bulgarian mathematician B. Sendov who had acquainted the mathematician Morris Marden and possibly others with it in 1962 at the International Congress of Mathematicians held in Stockholm.

There is a theorem, which was implied in a note of Gauss (1837) and stated and proved explicitly by Lucas (1874), that states that all the critical points of a non-constant polynomial lie in the convex hull of the set of its zeros. In particular, if a polynomial's roots lie all in the closed unit disk $\overline{D(0,1)}$, then so do all of its critical points. Now, and Instead of considering the relative positions of *all* the zeros and *all* the critical points, Sendov's conjecture provides a possible answer to the following question: At most, how far from a zero of a complex polynomial that has all its roots in $\overline{D(0,1)}$ does the nearest critical point lie?

In chapter 2, we provide a physical interpretation of the critical points of a complex polynomial. This interpretation, we show, can contribute to proving Gauss-Lucas theorem, which is a pivotal theorem in the study of relations between zeros and critical points of complex polynomials, as well as to proving other theorems that come in later chapters of this research. Also, in the same vein, Sendov's conjecture acquires physical implications that we discuss briefly along other kind of implications.

In chapter 3, we carry our previous discussion of Gauss-Lucas theorem into a wider discussion of major theorems that exist on the relation between zeros and critical points of complex polynomials. Of these theorems are a generalization of Gauss-Lucas theorem, by Laguerre, and the famous Grace's Theorem. The last two theorems are proved and concepts like Polar derivatives and Apolarity are introduced and discussed in support of their proofs. The chapter concludes with a proof of Grace-Heawood theorem, which relies on the mentioned theorems of Grace and Laguerre, and stands as quite a good analogue of Rolle's theorem in the complex plane.

In chapter 4, an introduction into the work of Tao is laid, the main theorem (Theorem 4.0.1) of his paper is introduced, and some partial results towards his theorem, and thus towards Sendov's Conjecture, are studied to provide a sense of the existing contributions in the literature.

In Chapter 5, the approach of Tao is studied and Theorem 4.0.1 is proved in the case $a^{(n)} = o(\frac{1}{\log n})$. The outstanding position that Tao's approach takes in the existing literature on Sendov's conjecture is stressed through attempted elaborations of different concepts that play major roles in compiling his proof as the Logarithmic potential, Brownian Motion, Fourier Analysis and others.

# Chapter 2

# Gauss-Lucas Theorem and What If Sendov's Conjecture was True

For this chapter, we mostly relied on Morris Marden's Book "Geometry of Polynomials" [2], as well as his paper titled "Conjectures on the Critical Points of a Polynomial"[3].

## 2.1 The critical points of a polynomial as equilibrium positions of force fields

Consider the following generic expression of a polynomial $P(z)$ of degree $n$

$$P(z) = \alpha \prod_{i=1}^{p} (z - z_i)^{m_i}$$

where $n = \Sigma_{i=1}^{p} m_i$.

The derivative of $P(z)$ could be written as

$$P'(z) = \Sigma_{i=1}^{p} \frac{m_i P(z)}{z - z_i}.$$

For $z \neq z_i \;\; i = 1, .., p$, we have

$$\frac{P'(z)}{P(z)} = \Sigma_{i=1}^{p} \frac{m_i}{z - z_i}$$

There are possibly zeros of $P'(z)$ that are zeros of $P(z)$ too. Else, all the other zeros of $P'(z)$ are exactly the zeros of $\Sigma_{i=1}^{p} \dfrac{m_i}{z - z_i} := F(z)$.

If we were to consider the complex conjugate of the expression of $F(z)$, which certainly possesses the same zeros as that of $F(z)$, a physical interpretation of the zeros of $F(z)$, which are but the unrepeated zeros of $P(z)$, would better reveal itself.

If we write $z - z_i = \rho_i e^{i\phi_i}$, then

$$\overline{F(z)} = \Sigma_{i=1}^{p} \frac{m_i}{\overline{z - z_i}} = \Sigma_{i=1}^{p} \frac{m_i}{\rho_i} e^{i\phi_i}$$

The $i$-th term of the later sum can therefore be represented by a vector whose direction is from $z_i$ to $z$ and the magnitude of $m_i$ times the reciprocal of the distance between $z_i$ and $z$. In physical words, the $i$-th term is the force which a fixed mass (or electric charge) $m_i$ at $z_i$ repels a movable unit mass (or charge) at $z$, the law of repulsion being the inverse distance law.

Therefore, The zeros of the function $F(z) = \Sigma_{i=1}^{p} \frac{m_i}{z - z_i}$ are the points of equilibrium in the field of force due to the system of $p$ masses (point charges) $m_i$ at the fixed points $z_i$ repelling a movable unit mass at $z$ according to the inverse distance law.

## 2.2 Proving Gauss-Lucas Theorem - A Mathematical Approach

We prove that all critical points of a non-constant polynomial $P$, of degree $n$, lie in the convex hull[1] of the zeros of $P$.

The points $z$ such that

$$z = \Sigma_{i=1}^{n} \lambda_i z_i$$

where $\Sigma_{i=1}^{n} \lambda_i = 1$, and $\lambda_i \leq 1$ for $i = 1, .., n$, constitute the convex hull of $z_1, .., z_n$.

Speaking of the critical points of $P$, if they were already zeros of $P$ then they satisfy lying in the convex hull of zeros of $P$ as a set lies in its convex hull. For critical points $z$ that are not zeros of $P$, the following is available

$$\frac{P'(z)}{P(z)} = \Sigma_{i=1}^{n} \frac{1}{z - z_i}$$

And eventually, $\Sigma_{i=1}^{n} \frac{1}{z - z_i} = 0$ is satisfied.

We try to prove that such $z$ could be written as linear combination of $z_i$ where the coefficients are non negative and sum up to one. A good step towards this goal is rationalizing the fractions present in the above sum to get

$$\Sigma_{i=1}^{n} \frac{\bar{z} - \bar{z}_i}{|z - z_i|^2} = 0$$

[1]The convex hull of a given set is the (unique) minimal convex set containing it

This implies that

$$\Sigma_{i=1}^{n} \frac{\bar{z}}{|z - z_i|^2} = \Sigma_{i=1}^{n} \frac{\bar{z}_i}{|z - z_i|^2}$$

Dividing by the sum $\Sigma_{i=1}^{n} \frac{1}{|z - z_i|^2}$ and barring on both sides, we obtain

$$z = \Sigma_{i=1}^{n} \left( \frac{\frac{1}{|z - z_i|^2}}{\Sigma_{j=1}^{n} \frac{1}{|z - z_j|^2}} \right) z_i$$

as desired.

## 2.3 Proving Gauss Lucas Theorem - A Physical Approach

We have already made the equivalence between the critical points of $P$ (that are not repeated zeros) and the points where an electric field created by point charges placed at the zeros of $P$ vanishes. So, proving that the equilibrium positions of such an electric field lie in the convex hull of the point charges is proving the Gauss-Lucas theorem.

We prove hereby that the electric field produced by positively charged particles cannot cancel at a point that is not in the convex hull of the charges. Let $z$ be a point that is outside the convex hull of the charged points. In fact, positively charged particles create an electric field that points radially outward.

By the hyperplane separation theorem, we can draw a line with $z$ on one side and the charged points $z_1, .., z_n$ on the other. And then a normal vector $v$ to that line, pointing to the side of $z$, would satisfy $v.(z - z_i) > 0$ for all $i = 1, .., n$.

As the electric field created by the $ith$ point charge on $z$ is a positive multiple of $z - z_i$, we can say that the total electric field created on $z$ has a positive dot product with $v$ and thus cannot be zero.

## 2.4 Physical and Geometrical Interpretations of Sendov's Conjecture

The available connection between the critical points and the equilibrium positions provided a possible approach of proving Gauss-Lucas theorem. This is a physical knowledge contributing to the development of a mathematical theory. If we proved true a mathematical conjecture like the one of sendov that relates the critical points to the zeros more intimately, than merely saying they are within the convex hull, we would be equivalently proving a result on the equilibrium points and their position with respect to the positions of charge points, and thus using mathematical knowledge to develop a physical theory this time.

Sendov's conjecture would imply that if all the particles were of positive unit charge and situated on the closed unit disk, then at least one equilibrium point will lie within unit distance of each particle.

The conjecture might also be given geometric interpretations. A polynomial $P(z)$ of the form

$$P(z) = (z - z_1)^{m1}(z - z_2)^{m2}(z - z_3)^{m3}, m1, m2, m3 > 0$$

has two critical points excluding $z_1, z_2, z_3$. These points, we know, are the foci of the ellipse inscribed in the triangle $z_1z_2z_3$ which touches its sides $z_1z_2, z_2z_3, z_3z_1$ at the points that divide these segments in the ratios $m1/m2, m2/m3, m3/m1$ respectively. Sendov's conjecture implies that if the vertices $z_1, z_2, z_3$ of the triangle were to lie in the closed unit disk, then each vertex is within one unit distance far from one of the foci of the inscribed ellipse.

# Chapter 3

# The Quest of an Analogue of Rolle's Theorem in the Complex Plane

As Bojanov puts it in his paper "Extremal Problems for Polynomials in the Complex Plane"[4], Sendov made his conjecture trying to find an analogue to Rolle's Theorem in the Complex Plane. Rolle's theorem says that between any two distinct zeros of a polynomial with real coefficients there is at least one zero of the derivative of that polynomial. In the case of polynomials with real coefficients, Gauss-Lucas theorem does not say as much as Rolle's theorem does. Even Gauss-Lucas extensions, which we could look at in this study, do not say as much as Rolle's theorem does.

Grace-Heawood Theorem, however, is a more similar statement to Rolle's theorem for polynomials with complex coefficients. It is more similar in the sense that it proves the existence of at least one critical point in a specific domain that is determined by two zeros of the polynomial.

The proof of Grace-Heawood theorem makes use of Grace's Theorem. Of the studies on the comparative location of zeros of related polynomials is the study on the comparative location of zeros of Apolar polynomials, and this is where Grace's theorem takes place. We present below a proof of Grace's theorem as it not only paves the way for proving Grace-Heawood theorem, but contributes as well to the proofs of other results that matter to this research.

## 3.1 On a Proof Grace's Theorem

When two polynomials are apolar, Grace's theorem asserts that a circular region that contains all the zeros of one polynomial has to contain at least one zero of the other. Here by circular region we mean the closure of not merely the interior of a circle but also the exterior of a circle or half plane.
Grace's theorem deploys a theorem by Laguerre.

As of the above, an introduction of the notions of polar derivatives and apolar polynomials with some exposure of their grounds is beneficial.

### 3.1.1 *Polar Derivatives*

**Definition 3.1.1.** The polar derivative $f_1(z)$ of the nth degree complex polynomial f(z) with respect to $z_0$ is defined as follows

$$f_1(z) = nf(z) - (z - z_0)f'(z)$$

Note that the polynomial $f_1(z)$ is of degree $n-1$. The function $f_1(z)$ has been called by Laguerre the "emanant of $f(z)$" and by Polya-Szego "the derivative of $f(z)$ with respect to the point $z_0$", but we shall call $f_1(z)$ the polar derivative of $f(z)$ with respect to $z_0$ following the terminology of Morris Marden.

By virtue of Gauss-Lucas theorem, any circle which encloses all the zeros of a polynomial also encloses all the zeros of its derivative. Polar derivatives come in the context of generalizing the Gauss-Lucas result by use of conformal mappings. For example, since the closed interior of a circle could be conformally mapped to the closed exterior of any other, we ask whether the case of the zeros of a polynomial function being exterior to some circle would imply that the zeros of its derivative are as well. In general, the last question has a negative answer.

What polar derivatives have is their good behavior under linear transformations. The zeros of the polar derivative of a polynomial $f$ map to the zeros of the polar derivative of the transform of $f$ under non-singular linear transformations [1]of the form

$$z = \frac{\alpha Z + \beta}{\gamma Z + \delta}, \alpha\delta - \gamma\beta \neq 0$$

also called Mobius Transformations, where the transform of $f(z)$ is

$$F(Z) = (\gamma Z + \delta)^n f(\frac{\alpha Z + \beta}{\gamma Z + \delta})$$

This is while the zeros of the logarithmic derivative of a polynomial are not necessarily carried to the zeros of the logarithmic derivative of $F(z)$. Note that the points into which the zeros of $f$ are transformed are the zeros of $F(Z)$. With Polar derivatives, a generalization of Gauss-Lucas in the sense already mentioned is available, and is referred to as the invariant form of Gauss-Lucas due to Laguerre or Laguerre's theorem.

As we said, the usual derivative or the logarithmic derivative of a polynomial $f$ does not necessarily satisfy that its zeros map to the zeros of the derivative or logarithmic derivative of the transform of $f$. This is going to be highlighted below

[1]Which are conformal maps

as we compute the logarithmic derivative of the transform of a polynomial, and the conditions for attaining the desired behavior will then be investigated. In due course, more sense is made on the way Polar derivatives are defined. What is important to get of the foregoing argument, especially for apprehending the approach for Laguerre's Theorem proof, is the essence behind Polar derivatives' advantage.

The logarithmic derivative of the transform $F(z)$ is

$$\frac{F'(Z)}{F(Z)} = \frac{\gamma n}{\gamma Z + \delta} + \frac{\alpha\delta - \gamma\beta}{(\gamma Z + \delta)^2} f'\left(\frac{\alpha Z + \beta}{\gamma Z + \delta}\right) \left[f(\frac{\alpha Z + \beta}{\gamma Z + \delta})\right]^{-1}$$

So, for $z'$ a zero of the logarithmic derivative of $f$, and $Z'$ such that $z' = \dfrac{\alpha Z' + \beta}{\gamma Z' + \delta}$, we have

$$\frac{F'(Z')}{F(Z')} = \frac{\gamma n}{\gamma Z' + \delta}$$

The necessary and sufficient condition for $F'(Z') = 0$, if $Z' \neq \infty$, is that $\gamma n = 0$. To satisfy the condition when $Z' \neq \infty$ and $\gamma \neq 0$[2], $n$ must be zero. We deduce that functions of the form $f(z) = \prod_{i=1}^{P}(z - z_i)^{m_i}$, with $n = \sum_{i=1}^{p} m_i = 0$, even where $m_i$ may be negative, satisfy that the zeros of their logarithmic derivatives are invariant under the (general) linear transformations.

In an approach to observe the logarithmic derivative of a polynomial within the expression of its polar derivative, and to find the reasoning for why the zeros of polar derivatives are invariant under general linear transformations, we note the following. For $z \neq z_0$ and $f(z) \neq 0$,

$$\frac{f_1(z)}{(z - z_0)f(z)} = \frac{n}{z - z_0} - \frac{f'(z)}{f(z)}$$

This communicates $\dfrac{f_1(z)}{(z - z_0)f(z)}$ as the logarithmic derivative of the function

$$-(z - z_0)^{-n} f(z)$$

which meets the condition for the invariance of zeros of its logarithmic derivative under Mobius transformations. The zeros of $f_1(z)$ are the pole $z_0$ if $f(z_0) = 0$, the multiple zeros of $f$, and the zeros of $\dfrac{f_1(z)}{(z - z_0)f(z)}$. Hence, the zeros of the polar derivative of a polynomial are invariant under general linear transformations.

More exposure of the mathematical ground of polar derivatives could be relevant; not only to acquaint it with a more entrenched interpretation but also to reengage its invariant property, and even to support the next discussion on apolar polynomials. We discuss hereby the polar derivative in the complex projective line and its

[2] $\gamma = 0$ means the transformation is a translation, rotation, or that of similitude.

expression using homogeneous coordinates.

We are concerned with the complex projective space of dimension 1, denoted $\mathbb{CP}^1$, but the presented ideas could lead a certain understanding of general complex projective spaces. The one dimensional complex projective space is defined through the two dimensional complex vector space $\mathbb{C}^2$ by an equivalence relation on $\mathbb{C}^2\backslash(0,0)$ as follows

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$$

for all $\lambda \in \mathbb{C}^*$. So, each point in the one dimensional complex projective space correspond to a one dimensional complex subspace of $\mathbb{C}^2$. Each point is written in the so called homogeneous coordinates as $[z_1 : z_2]$, which means the "line" spanned by $(z_1, z_2)$. $\mathbb{CP}^1$ is identified with the extended complex plane, or the Riemann sphere. The point $[z_1, z_2]$ in $\mathbb{CP}^1$ where $z_1$ is not zero is identified with the complex number $\frac{z_2}{z_1}$. When $z_1$ is zero, the point $[z_1, z_2]$ is identified with $\infty$. Mobius transformations correspond to linear maps in homogeneous coordinates.

We introduce the homogeneous coordinates into polar derivatives by substituting $z = \frac{\xi}{\eta}$ into $f(z)$ and $f_1(z)$ as follows

$$F(\xi, \eta) = \eta^n f\left(\frac{\xi}{\eta}\right)$$

$$F_1(\xi, \eta) = \eta_0 \eta^{n-1} f_1\left(\frac{\xi}{\eta}\right)$$

assuming the homogeneous representation of the pole $z_0$ is $\frac{\xi_0}{\eta_0}$. By Euler's identity, and as a homogeneous polynomial[3] of degree $n$, $F(\xi, \eta)$ satisfies

$$nF(\xi, \eta) = \xi \frac{\partial F}{\partial \xi} + \eta \frac{\partial F}{\partial \eta}$$

Subsequently,

$$F_1(\xi, \eta) = \xi_0 \frac{\partial F}{\partial \xi} + \eta_0 \frac{\partial F}{\partial \eta_0}$$

This implies that upon the introduction of homogeneous coordinates, the polynomial $f(z)$ transforms into a homogeneous function $F(\xi, \eta)$ and its polar derivative $f_1(z)$ into $F_1(\xi, \eta)$, the (classical) first polar of $F(\xi, \eta)$. The first polar of a homogeneous polynomial $F(\xi, \eta)$ with respect to $(\xi_0, \eta_0)$ could be realized as the directional derivative of $F(\xi, \eta)$ with respect to $(\xi_0, \eta_0)$. The first polar is covariant under linear transformations in homogeneous coordinates.

[3] A homogeneous polynomial is a polynomial whose nonzero terms all have the same degree

### 3.1.2 *Apolar Polynomials*

**Definition 3.1.2.** Two polynomials

$$f(z) = \Sigma_{k=0}^{n} a_k z^k, and \;\; g(z) = \Sigma_{k=0}^{n} b_k z^k$$

are said to be apolar if their coefficients satisfy the equation:

$$\Sigma_{k=0}^{n} \frac{(-1)^k}{C(n, n-k)} a_k b_{n-k} = 0$$

In an attempt of understanding the subtlety of this equation and the context where it presumably originates, we include the following ideas that run mainly through chapters one and eleven of the book “The Algebra of Invariants” by John Hilton Grace and Alfred Young [5].

If in the expressions

$$a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$$
$$a_0^{'} x_1^2 + 2a_1^{'} x_1 x_2 + a_2^{'} x_2^2$$

we write

$$x_1 = \xi_1 X_1 + \eta_1 X_2$$
$$x_2 = \xi_2 X_1 + \eta_2 X_2$$

They become

$$A_0 X_1^2 + 2A_1 X_1 X_2 + A_2 X_2^2$$
$$A_0^{'} X_1^2 + 2A_1^{'} X_1 X_2 + A_2^{'} X_2^2$$

With the following:

$$A_0 A_2^{'} - 2A_1 A_1^{'} + A_2 A_0^{'} = (a_0 a_2^{'} - 2a_1 a_1^{'} + a_2 a_0^{'})(\xi_1 \eta_2 - \xi_2 \eta_1)^2$$

The above identity claims that some function of the coefficients of the two expressions has its new value, i.e its value after their transformation, differing from its original value by a factor depending only on the transformation involved. This is giving account to the theory of functions of coefficients possessing properties analogous to the described above; those are functions denoted by invariants, and more specifically, joint invariants of binary forms.

*Remark.* In the classical literature, homogeneous polynomials are called forms. The adjectives ”binary”, ”ternary”, etc. refer to the number of variables that the form depends on.

If two binary forms $f_1$ and $f_2$ be changed by a linear transformation into new forms $F_1$ and $F_2$, and a function $I$ of the coefficients of $F_1$ and $F_2$ be equal to the same function of the coefficients of $f_1$ and $f_2$ multiplied by a factor depending solely on the transformation, then $I$ is called a joint invariant of the two binary forms $f_1$

and $f_2$.

Two binary forms of the same order are said to be apolar when the joint invariant which is linear in the coefficients of both is zero. Suppose that the two forms are:

$$f_1 = a_0 x_1^n + a_1 x_1^n x_2 + ... + a_n x_2^n$$

$$f_2 = a_0 x_1^n + a_1 x_1^n x_2 + ... + a_n x_2^n$$

Then the so called lineo-linear joint invariant is

$$\Sigma_{k=0}^{n} \frac{(-1)^k}{C(n, n-k)} a_k a_{n-k}^{'}$$

and this vanishes when the forms are apolar. That the above is the only linear joint invariant visit "The Invarient Theory of Binary Forms".

The discussion of apolar forms may be regarded as development of the theory of the simplest type of invariant; the fact that each coefficient only occur to the first degree in the invariant renders such a discussion simple and accounts for the relative importance of the allied geometrical properties. If two quadratic forms are apolar they are harmonic so we may regard apolarity as being, in a certain way, the generalization of harmonic properties. You can refer to "The minima of a pair of indefinite, harmonic, binary quadratic forms"[6], by Kathleen Ollerenshaw, for a look on when two quadratic binary forms are harmonic.

### 3.1.3 *Laguerre's Theorem*

**Theorem 3.1.1.** *If all the zeros $z_i$ of the nth degree polynomial $f(z)$ lie in a circular region $C$ and if $Z$ is any zero of the polar derivative of $f(z)$*

$$f_1(z) = nf(z) + (\xi - z)f'(z)$$

*then not both points $Z$ and $\xi$ may lie outside of $C$.Furthermore, if $f(Z) \neq 0$, any circle $K$ that passes through $Z$ and $\xi$ either passes through all the zeros of $f(z)$ or separates them.*

*Proof.* Suppose that $Z$ and $\xi$ lie outside the circular region $C$. This means that $Z$ and $\xi$ are not possibly zeros of $f(z)$, as all zeros of $f(z)$ happen to be in $C$. Also, we could make the point that $Z$ and $\xi$ are not equal because that would make of $Z$ a zero of $f(z)$ which is not.

Through $Z$ a circle $\gamma$ could be drawn which separates $\xi$ from the region $C$.

That $Z \neq \xi$ and $f(Z) \neq 0$ allows the following observation

$$0 = \frac{f_1(Z)}{(Z-\xi)(f(Z))} = \frac{-n}{Z-\xi} + \frac{f'(Z)}{f(Z)} = \frac{-n}{Z-\xi} + \Sigma_{i=1}^{p} \frac{m_i}{Z-z_i}$$ [4]

And this says that $Z$ is an equilibrium point in a plane force field due to electric charges of total charge zero; the electric charges being situated at the zeros of $f(z)$, each of charge equal to the multiplicity of the respective zero, and at $\xi$ is one with charge equal to negative $n$.

The pair consisting of $m_i$ at $z_i$ and $(-m_i)$ at $\xi$ acts in the direction of the circular arc $z_i Z \xi$ and hence towards the side of the circle $\gamma$ not containing $C$. The vectors contributing to the resultant electric force created at $Z$ are all drawn from $Z$ to points on the same side of the tangent to $\gamma$ at $Z$, and hence they cannot sum to zero. This means $Z$ could not be an equilibrium point. This contradiction to our assumption on $Z$ proves the first part of Laguerre's Theorem, the statement of which is equivalent to saying that a circular region containing all the zeros of $f(z)$, but not the point $\xi$, contains all the zeros of the polar derivative $f_1(z)$.

Concerning the other part of the theorem, assuming the existence of a circle $K$ that passes through $Z$ and $\xi$ such that the zeros of $f(Z)$ either lie on its circumference or in its interior (equivalently exterior) would forbid $Z$ from being an equilibrium point of the aforementioned electric field which it should be, actually, if $f(Z)$ were to be not zero. In other words, and for more elaboration, if $f(z) \neq 0$ and there is a circle $K$ through $Z$ and $\xi$ that has at least one zero $z_i$ in its interior, no zero $z_i$ in its exterior, and the remaining $z_i$ on its circumference, the forces at $Z$ are directed along the tangent line to $K$ at $Z$ or to one side of this line and hence cannot sum to zero. A contradiction would similarly follow if $K$ were assumed to have at least one $z_i$ in its exterior and no $z_i$ in its interior, so we conclude that $K$ has to separate the $z_i$ unless it passes through all of them.

□

### 3.1.4 *Grace's Theorem*

**Theorem 3.1.2.** *If $f(z)$ and $g(z)$ are apolar polynomials and if one of them has all its zeros in a region $C$, then the other would have at least one zero in $C$.*

*Proof.* For the convenience of computations, we consider the following forms for $f(z)$ and $g(z)$

$$f(z) = \Sigma_{i=1}^{n} C(n,i) A_i z^i$$

$$g(z) = \Sigma_{i=1}^{n} C(n,i) B_i z^i$$

Denote by $z_1, z_2, ..., z_n$ the zeros of $f(z)$ and by $\xi_1, \xi_2, ..., \xi_n$ the zeros of $g(z)$. Let us prove the theorem on the assumption that all the zeros of $f(z)$ lie in a circular region

[4] This is the logarithmic derivative that appeared in the discussion on polar derivatives and it was corresponding to the function that has a zero degree.

$C$. As we know, and in simple terms, apolarity is a condition on the coefficients of two polynomials. Our first step towards proving a zero of $g(z)$ has to lie in $C$ is constructing the following sequence of polar derivatives

$$f_k(z) = (n-k+1)f_{k-1} + (\xi_k - z)f'_{k-1}(z), \qquad k = 1, 2, ..., n$$

with $f_0(z) = f(z)$. The poles $\xi_k$ are but the zeros of $g(z)$. For reassurance, $f_k(z)$ is a polynomial of degree $n-k$. Just as the position of the zeros of $f^{(k)}(z)$ may be determined by repeated application of the Lucas Theorem, the position of the zeros of $f_k(z)$ may be determined by repeated application of Laguerre's Theorem. This is to say that, if the zeros of $g(z)$ were all to lie exterior to $C$, the zeros of each polar derivative $f_k(z), k = 1, 2, ..., n$ would lie in $C$.

Following the forms we considered for $f(z)$ and $g(z)$, we write $f_k(z)$ in the form

$$f_k(z) = \Sigma_{i=0}^{n-k} C(n-k, i) A_i^k z^i$$

Where we define $A_i^{(k)} = 0$ for $i < 0$ and $i > n-k$. Proceeding, we relate the coefficients of $f^k(z)$ to those of $f(z)$. As of the above form of $f_k(z)$,

$$f_{k-1}(z) = \Sigma_{i=0}^{n-k+1} C(n-k+1, i) A_i^{(k-1)} z^i$$

$$f'_{k-1}(z) = \Sigma_{i=0}^{n-k} (i+1) C(n-k+1, i+1) A_{i+1}^{(k-1)} z^i$$

Substituting into the equation of construction of $f_k(z)$ for $f_{k-1}(z)$ and $f_k(z)$, and equating the coefficients of $z^i$ on the right and left sides of the resulting equation, we find

$$\begin{aligned}
&C(n-k,i)A_i^{(k)} \\
&= (n-k+1)C(n-k+1,i)A_i^{(k-1)} + \xi_k(i+1)C(n-k+1,i+1)A_{i+1}^{(k-1)} - iC(n-k+1,i)A_i^{(k-1)} \\
&= (n-k+1)(C(n-k+1,i)A_i^{(k-1)} + \xi_k C(n-k,i)A_{i+1}^{(k-1)} - C(n-k,i-1)A_i^{(k-1)}) \\
&= (n-k+1)(C(n-k,i)A_i^{(k-1)} + \xi_k C(n-k,i)A_{i+1}^{(k-1)}) \\
&= (n-k+1)(C(n-k,i))(A_i^{(k-1)} + \xi_k A_{i+1}^{(k-1)})
\end{aligned}$$

And thus

$$A_i^{(k)} = (n-k+1)(A_i^{(k-1)} + \xi_k A_{i+1}^{(k-1)})$$

By repeated application of the last equation, we may derive the formula

$$A_i^{(k)} = n(n-1)...(n-k+1)\Sigma_{j=0}^{k} \sigma(k,j) A_{j+i}$$

where $\sigma(k,j)$ is the symmetric function consisting of the sum of all possible products of $\xi_1, \xi_2, ..., \xi_k$. For $k = 1$, the formula is the same as the settled equation. We show that if the formula is valid for $k$, it is as well valid for $k+1$. In fact,

$$\begin{aligned}
A_i^{(k+1)} &= (n-k)(A_i^{(k)} + \xi_{k+1} A_{i+1}^{(k)}) \\
&= n(n-1)...(n-k)\Sigma_{j=0}^{k}(\sigma(k,j)A_{j+i} + \xi_{k+1}\sigma(k,j)A_{j+i+1}) \\
&= n(n-1)...(n-k)\Sigma_{j=0}^{k}(\sigma(k,j) + \xi_{k+1}\sigma(k,j-1))A_{j+i} \\
&= n(n-1)...(n-k)\Sigma_{j=0}^{k}\sigma(k+1,j)A_{j+i}
\end{aligned}$$

Hence, the desired formula is established by mathematical induction. Note that the elementary symmetric functions $\sigma(k,j)$ relate to the coefficients $B_j$ of $g(z)$ in the manner below

$$C(n,j)B_{n-j} = (-1)^j \sigma(n,j)B_n$$

This, and the representation of the coefficients of $f_k(z)$ in terms of those of $f(z)$, will allow for the usage of apolarity in proving that $\xi_n$ is in fact a zero of $f_{n-1}(z)$, and therefore has to lie in $C$.
We have that

$$f_{n-1} = A_0^{(n-1)} + A_1^{(n-1)} z$$

where

$$A_0^{(n-1)} = n!\Sigma_{j=0}^{(n-1)} \sigma(n-1,j)A_j$$

$$A_1^{(n-1)} = n!\Sigma_{j=0}^{n-1} \sigma(n-1,j)A_{j+1}$$

Bearing that $\sigma(n-1,k) + \sigma(n-1,k-1)\xi_n = \sigma(n,k)$, we observe the following

$$f_{n-1}(\xi_n) = n!\Sigma_{j=0}^{n} \sigma(n,j)A_j = \frac{n!}{B_n}\Sigma_{j=0}^{n}(-1)^j C(n,j)B_{n-j} = 0$$

The last equal is implied since $f(z)$ and $g(z)$ are apolar. Whence, it can be said that at least one of the zeros $\xi_1, \xi_2, ..., \xi_n$ must lie in any circular region $C$ containing all the zeros of $f(z)$. Similarly, at least one of the zeros $z_1, z_2, ..., z_n$ of $f(z)$ lies in any circular region containing all the zeros of $g(z)$.

□

## 3.2 On a proof of Grace-Heawood Theorem

Having laid the adequate ground for introducing Grace-Heawood Theorem, the following is Grace-Heawood Theorem and its proof:

**Theorem 3.2.1.** *If $z_1$ and $z_2$ are any two zeros of an nth degree polynomial $f(z)$, at least one zero of its derivative $f'(z)$ will lie in the circle $C$ with center at point $\frac{z_1+z_2}{2}$ and with a radius of $\frac{1}{2}|z_1 - z_2| \, cot(\frac{\pi}{n})$.*

*Proof.* We may without loss of generality assume that $z_1 = 1$ and $z_2 = -1$; if that was not the case, it could be made to be through rotations, translations, and/or dilations [5]. Then, proving a zero of $g(z)$ to exist in the circle of center 0 and radius $cot(\frac{\pi}{n})$ would equivalently mean the existence of one in the circle $C$.

[5] Namely, through the consideration of $g(z) = f(\frac{|z_1 - z_2|}{2} e^{i(arg(z_2 - z_1) - \pi)} z + \frac{z_1 + z_2}{2})$

Let $f'(z) = a_0 + a_1 z + ... + a_{n-2}z^{n-2} + z^{n-1}$; similarly, assuming the derivative is monic does not affect the generality of the argument. As of the assumptions made, we have that

$$0 = f(1) - f(-1) = \int_{-1}^{1} f'(t)dt = 2a_0 + 2\frac{a_2}{3} + 2\frac{a_4}{5} + ...$$

With this linear relation of the coefficients of $f'(z)$ in hands, we could distinguish the following polynomial which $f'(z)$ is apolar to

$$g(z) = \int_{-1}^{1} (t-z)^n dt = \frac{1}{n}((1-z)^n - (-1-z)^n)$$

To understand more the reasoning behind this choice of $g(z)$, check theorem 15,2 in Morris Marden's Geometry of Polynomials or follow the discussion that will be included ahead of this theorem's in a generalization of the previous lines. Well, the zeros of $g(z)$ are $z_k = icot(\frac{\pi}{n})$, $k = 1, 2, ..., n-1$. By Grace's Theorem, at least one zero of $f'(z)$ lies in any circle containing all the zeros of $g(z)$. Particularly, the circle of center the origin and radius $cot(\frac{\pi}{n})$ could be considered.

That the radius in Grace-Heawood Theorem may not be replaced by a smaller number may be seen from the polynomial

$$\begin{aligned} f(z) &= \int_{-1}^{z} (t - icot(\frac{\pi}{n}))^{n-1} dt \\ &= \frac{1}{n}((z - icot(\frac{\pi}{n})^n - (-1 - icot(\frac{\pi}{n}))^n) \end{aligned}$$

which has the zeros $-1$ and 1 and the derivative of which $f'(z) = (z - icot(\frac{\pi}{n}))^n$ has its only zero at $z = icot(\frac{\pi}{n})$.

In v.11(1900-1902) of the Proceedings of the Cambridge Philosophical Society, J.H Grace provides, in his paper titled "The Zeros of a Polynomial", a discussion on how $f'$ would be apolar to the form $(\alpha - z)^n - (\beta - z)^n$ if $f$ had $\alpha$ and $\beta$ as zeros, and $n$ its degree. Namely, let $f'$ be given as

$$f'(z) = p_0 z^{n-1} + p_1 z^{n-2} + ... + p_{n-1}$$

So that

$$f(z) = \frac{p_0}{n} z^n + \frac{p_1}{n-1} z^{n-1} + .... + p_{n-1}z + p_n$$

where $p_n$ is an arbitrary constant.

If $f$ was to have  and $\beta$ of its roots, we have that

$$\frac{p_0}{n}(\alpha^n - \beta^n) + \frac{p_1}{n-1}(\alpha^{n-1} - \beta^{n-1}) + .... + p_{n-1}(\alpha - \beta) + p_n = 0$$

which constitutes a linear relation between the coefficients of $f'$. It could be seen from the above relation that $f'$ is actually apolar to the following form

$$\frac{\alpha^n - \beta^n}{n} - z\frac{n-1}{n-1}(\alpha^{n-1} - \beta^{n-1}) + z^2\frac{(n-1)(n-2)}{2}\frac{1}{n-2}(\alpha^{n-2} - \beta^{n-2}) + ... = 0$$

Or to

$$(\alpha_n - \beta_n) - nz(\alpha_{n-1} - \beta_{n-1}) + \frac{n(n-1)}{2}z^2(\alpha^{n-2} - \beta^{n-2}) + ...$$

That is the form of degree $n-1$

$$(\alpha - z)^n - (\beta - z)^n$$

Therefore, if $f(z)$ has two given zeros $\alpha$ and $\beta$, then $f'$ is apolar to the form $(\alpha - z)^n - (\beta - z)^n$.Concerning the zeros of this latter form, they are easily constructed, for they are given by

$$(\alpha - z) = w(\beta - z)$$

where $w$ is any nth root of unity that is not unity itself.

$$\frac{z - \alpha}{z - \beta} = e^{\frac{2k\pi}{n}i}, k = 1, 2, ..., n-1$$

implies that

$$\frac{z - \frac{\alpha + \beta}{2}}{\frac{\alpha - \beta}{2}} = i$$

which is

$$z = \frac{\alpha + \beta}{2} + icot(\frac{k\pi}{n})\frac{\alpha - \beta}{2}$$

Denoting by $A$, $B$ the points in the complex plane referring to the complex numbers $\alpha$ and $\beta$ respectively, and letting $O$ be the midpoint of the segment $AB$, the latest equation says that to arrive at $z$ we have to travel a distance of $OAcot(\frac{k\pi}{n})$ from $O$ along a line at right angles to $AB$.

□

# Chapter 4

# On the Works of Rubinstein and Degot: A Step into Tao's Approach

As mentioned before, the main theorem Tao proves in his paper [2] is that Sendov's conjecture is true for all sufficiently large n. He starts by providing an equivalent asymptotic contradiction form of this theorem

**Theorem 4.0.1.** *Let $n$ range over a sequence of natural numbers going to infinity. For each $n$ in this sequence, let $f = f^{(n)}$ be a monic polynomial of degree $n$ with all zeros in $\overline{D(0,1)}$, and let $a = a^{(n)} \in [0,1]$ be such that $f(a) = 0$. Suppose also that for every $n$ in this sequence, $f'$ has no zeros in $\overline{D(a,1)}$. Then one can derive a contradiction.*

As is tradition in the literature on this conjecture, Tao have normalized f to be monic by multiplying with a constant and $a$ to be a real number between 0 and 1 by applying a rotation around the origin. These steps do not intrigue in changing the relative distances between the zeros and the critical points of f and thus the premise of our investigation is not harmed.

We always reserve the right to pass to a subsequence of n as necessary to improve the convergence as n goes to infinity. For instance, by Blzano-Weierstrass theorem, and passing to a subsequence if necessary, we may assume that $a = a^{(n)}$ converges to a fixed limit $a^{(\infty)}$. Upon this comprehension, Tao separated the following three cases

I. $a^{(\infty)} = 0$

   In this case, we are saying that the sequence of zeros $a = a^{(n)}$ converges to zero.

   It might be that a=0, i.e all the considered zeros of the polynomials indexed by $n$ are equal to zero. Guass-Lucas theorem takes full care of this situation.

If not all the terms are zero, the theorem is confirmed when $a \leq \frac{1}{n-1}$ by Bojanov.

II. $0 < a^{(\infty)} < 1$
Tao mentions Degot to have established the theorem for this case.

III. $a^{(\infty)} = 1$
Rubinstein proves the theorem if a=1.
Chijiwa establishes the theorem when $a \geq 1 - \frac{1}{2n^9 4^n}$.

In his paper, Tao supplements the above partial results by proving Theorem 4.0.1 in the cases $a = o(\frac{1}{\log n})$ and $1 - o(1) \leq a \leq 1 - \epsilon_0^n$ for some fixed $\epsilon_0 > 0$. These two results, combined with the existing results, furnish a complete proof of the theorem. In the next chapter, we provide a thorough treatment of Tao's argument in the case $a = o(\frac{1}{\log n})$. In this chapter, the works of Rubinstein and Degot are studied in an attempt to gain a sense of the existing approaches in the literature towards Theorem 4.0.1.

## 4.1 Rubinstein's Work

Given a polynomial $P(z)$ whose zeros $z_1, z_2, ..., z_n (n \geq 2)$ lie in $|z| \leq 1$, Rubinstein proves that $P'(z)$ always has a zero in $|z - z_1| \leq 1$ if one of the following cases occur

Case 1: $|z_1| = 1$, $n$ is arbitrary
Case 2: $|z_1| < 1$, $n = 2, 3, 4$

We start the discussion with the first case

Rubinstein assumes, without loss of generality, that $z_1 = 1, z_k \neq 1 fork = 2, 3, ..., n, and P'(1) = 1$.

First, if $z_1 \neq 1$ we compose $P$ with $e^{\theta i}$ where $\theta$ is the negative of the argument of $z_1$. The resulting function, $g(z) := f(e^{\theta i})$, is the one then considered for treatment. It has one as a zero. Proving the result for g is equivalent to proving it for f as the relative positions between the zeros and the critical points of f does not change upon rotation; this was noted earlier in the approach of Tao.

Second, if the multiplicity of $z_1 = 1$ was strictly bigger than one, then we expect to meet 1 again as a zero of $P'$; i.e, as a critical point. In fact, this means that the case is closed as $P'$ is found to have a zero in $|z - z_1| \leq 1$ which is 1.

Third, let $P'(1) = m$. If $m$ is zero then this closes the case. If not, we consider multiplying $P$ by $\frac{1}{m}$. The last step does not mess with the zeros and critical points of $P$, and so the work could be equivalently carried on $g := \frac{1}{m}P$ instead of $P$ where $g'(1) = 1$.

Having checked that the three assumptions considered by Rubinstein does not affect the generality of the argument, we continue with the proof.

Rather than proving $P'(z)$ has at least one zero in $|z_1 - 1| \leq 1$, Rubinstein goes for proving $P'(z+1)$ has a zero in the closed unit disk. Suppose to the contrary that $P'(z+1)$ has no zeros in the closed unit disk.

We will use the following corollary from Silverman's Book (Corollary 7.52 page 240):

If $f$ is non zero and analytic in a simply connected domain D, then an analytic branch of $(f(z))^{1/n}$ ( $n$ is a positive integer) can be defined in D.

In our situation, $P'(z+1)$ is non zero and analytic in $|z| < 1$, then an analytic branch $g(z)$ of $P'(z+1)$ in $|z| < 1$ exists such that

$$P'(z+1) = g(z)^{n-1}$$
$$g(0) = 1$$

From its Taylor series, $g(z)$ could be represented as $g(z) = 1 - zf(z)$ where $f(z)$ is analytic and less than one in modulus as $P'(z+1)$ has no zeros in the closed unit disk. Bearing in mind the degree of $P'(z)$ is $n-1$, we could moreover say that $f(z)$ is a constant function .

Having $P(1)$ and $P'(1)$ known, Rubinsten finds $P''(1)$ on side of them (in terms of $f$)

$$P''(1) = (1-n)f(0)$$

If we examine the steps that have been done so far, we notice that the assumption that 1 is a zero of $P$ is still not exploited. In fact, for the representation that we put up of $P'(z+1)$, we did use the facts that $P'(z+1)$ has no zeros in the closed unit disk and that $P'(1) = 1$ (which is behind the 1 residing in the representation of $g$ as $1 - zf(z)$ ). So, as 1 is a zero of $P$, the following representation is available for $P(z)$: $P(z) = (z-1)Q(z)$. Note that 1 is not a possible zero of $Q$ as part of our assumptions is that $P$ has 1 as a zero of multiplicity 1.

$Q(1)$ and $Q'(1)$ will be found from the occurring data we have on $P'$ and $P''$ which are due the assumption that dispute the result we want. Then, $\frac{Q(1)}{Q'(1)}$ will be one time computed in a way that do not rely on any of these data and another time computed out of the results on $Q(1)$ and $Q'(1)$. Equating will create a contradiction.

First way: $\dfrac{Q'(1)}{Q(1)} = \dfrac{1}{1-z_2} + \dfrac{1}{1-z_3} + ... + \dfrac{1}{1-z_n}$

Second way: $\dfrac{Q'(1)}{Q(1)} = \dfrac{P''(1)}{2P'(1)} = \dfrac{(1-n)f(0)}{2}$

From the latter way, we obtain the following

$$\left|\frac{Q'(1)}{Q(1)}\right| < \frac{n-1}{2}$$

From the former way, however, we have that

$$Re\frac{Q'(1)}{Q(1)} = Re\frac{1}{1-z_1} + ... + Re\frac{1}{1-z_n}$$

And as (for $n \geq 2$)

$$\frac{1}{1-z_n} = \frac{1}{1-Rez_n - Imz_n i} = \frac{1-Rez_n + Imz_n i}{(1-Rez_n)^2 + (Imz_n)^2}$$

We get

$$Re\frac{1}{1-z_n} = \frac{1-Rez_n}{(1-Rez_n)^2 + (Imz_n)^2}$$
$$\geq \frac{1-Rez_n}{(1-Rez_n)^2 + 1 - (Rez_n)^2} = \frac{1}{1-Rez_n+1+Rez_n} = \frac{1}{2}$$

So, $Re\dfrac{Q'(1)}{Q(1)} \geq \dfrac{n-1}{2}$. This creates a contradiction.

Furthermore, Rubinstein argues that the disk $|z - z_1|$ would always contain a zero of $P'(z)$ except when $P(z) = c(z^n - e^{i\theta})$, theta being any real number. We observe that $|f(z)| \leq 1$ even if $P'(z+1) \neq 0$ only for $|z| < 1$, so that we also obtain a contradiction except possibly when all the $z_k$ lie on the unit circumference and $f(z)$ is a constant of absolute value one; only if all $z_k$ were to lie on the unit circumference does the real part of $\dfrac{1}{1-z_k}$ equate to $\dfrac{1}{2}$ for every $k$, or equivalently, the real part of $\dfrac{Q'(1)}{Q(1)}$ to $\dfrac{n-1}{2}$. Respectively, only if $f(z)$ was to be (constant) of absolute value one does the modulus of$\dfrac{Q'(1)}{Q(1)}$, in the case that $P'(z+1)$ does not have any zero in $|z| < 1$, be equal to $\dfrac{n-1}{2}$.

$f(z)$ being a constant of absolute value one, is $P'(z)$ having a zero of multiplicity $n-1$ that lies on the circle $|z-1| = 1$. For $n \geq 3$, and if all the zeros of $P(z)$ were to lie on the unit circumference, the $n--1$ fold zero of $P'(z)$, call it $w$, is necessarily equal to zero as the zeros of $P(z)$ verifies by simple computation that

$$z_k = w + re^{i\phi_k}$$

for every $k = 1, ..., n$, where $r$ is a real number that is independent of $k$ and dependent on the constant term of the polynomial $P(z)$, and $\phi_k$ is a real number dependent on $k$ such that $\phi_k$ and $\phi_{k+1}$ differ by $\frac{2\pi}{n}$. Hence, w is the center of the circle passing by all the zeros $z_k$ of $P(z)$ which is, necessarily, the unit circle . In light of this, and for $n \geq 3$, there exist no zero of $P'(z)$ in the disk $|z-1| < 1$ if $P(z)$ is exactly of the following form $P(z) = c(z^n - e^{i\theta})$.
For n=2, however, the two fold zero of $P'(z)$, w, could be other than zero while all the zeros of $P(z)$ still lying on the unit circumference as the two zeros of $P(z)$ do not distinguish a unique circle. In case $w$ is not equal to zero, and as $arg(z_1 - w, z_2) = \pi$, $w$ lies necessarily on the arc of the circle $|z-1| = 1$ enclosed by the unit circle, and $P'(z)$ will satisfy having a zero $(w)$ in the circle $|z-1| = 1$ as $|w-1| = \frac{L}{2}$ where $L$ is the length of some secant of the unit circle that is not a diameter. So, and for $n = 2$ as well, only $P(z)$ of the form $P(z) = c(z^n - e^{i\theta})$ would verify not having a zero of its derivative in the disk $|z-1| < 1$. This establishes the stated argument.

On the second case,

Rubinstein made use of the results he proved for the first case to prove that for a polynomial $P(z)$ of degree three or four whose zeros lie in the closed unit disk, and for any zero $z_1$ of $P(z)$, there exists a zero of $P'(z)$ that lies in $|z - z_1| \leq 1$.

For $|z_1| = 1$, the above follows by case the first case. For $|z_1| < 1$, Rubinstein proves that the polynomial $P'(z + z_1)$ has a zero in $|z| < 1$. Supposing, to the contrary, that $P'(z + z_1) \neq 0$ in $|z| < 1$, and considering that $P(z) = (z - z_1)Q(z)$ where the zeros $z_2, z_3, ..., z_n$ of $Q(z)$ lie in $|z| \leq 1$, Rubinstein finds a lower bound to the distances $|z_k - z_1|$, $k = 2, 3, ..., n$, through a representation of the zeros of $Q(z + z_1)$ that is linked to the zeros of the polynomial $P'(z + x)$ by a result due to *Szego*. More precisely, the assumption on the zeros of $P'(z + z_1)$ having a modulus greater than or equal to one will yield a bound on the the moduli of the zeros of $Q(z)$. Subsequently, a bound on the zeros of the polynomial $Q_1(z)$, where $P(z - 1 + x_1) = (z - 1)Q_1(z)$, is deduced, which produces a contradiction to the results of first case. We have that

$$P'(z + z_1) = \Sigma_{k=0}^{n} (k + 1) \frac{Q^{(k)}(z_1)}{k!} z^k$$

$$Q(z + z_1) = \Sigma_{k=0}^{n} \frac{Q^{(k)}(z_1)}{k!} z^k$$

By a result due to Szego, every zero $\gamma$ of $Q(z + z_1)$ can be written as $\gamma = -\alpha\beta$, where alpha belongs to a circular region containing all the zeros of $P'(z + z_1)$ and $\beta$ is a zero of the polynomial $g(z) = \Sigma_{k=0}^{n} \frac{1}{k+1} C(n, k) z^k$. The zeros of $g(z)$ have the form $\beta = -1 + w$, where $w$ is such that $w^{n+1} = 1$ and $\beta \neq 0$. Here is where the specialty of degrees three and four prevails, as for these degrees we have that $|\beta| \geq \sqrt{2}$. If $P'(z + z_1) \neq 0$ in $|z| < 1$, we deduce that $|\alpha| \geq 1$. And therefore, a zero $\gamma$ of $Q(z + z_1)$ satisfies that $|\gamma| \geq \sqrt{2}$, which implies that $|z_k - z_1| \geq \sqrt{2}$ for every $k = 2, 3, .., n$.

The polynomial $R(z) = P(z - 1 + z_1) = (z - 1)Q_1(z)$, $Q_1(z) = Q(z - 1 + z_1)$, is such that no zero of $R'(z)$ lies in $|z - 1| < 1$ (following the assumption that no zeros of $P'(z + z_1)$ lie in $|z| < 1$). Furthermore, the zeros of $Q_1(z)$ satisfy the following inequalities $|z - 1 + z_1| \leq 1$ and $|z - 1| \geq \sqrt{2}$. With 1 a zero of $R(z)$, a contradiction is obtained if it was proved that all the zeros of $Q_1(z)$ lie in $|z| < 1$, as then by case 1 a zero of $R'(z)$ has to exist in $|z - 1| < 1$ which is not the case. This follows by straight forward calculations.

## 4.2 Degot's work

In his paper titled "Sendov Conjecture for High Degree Polynomials"[7], Jerome Degot proves that for $0 < a < 1$, there exists an integer $N$, such that the disk $|z - a| \leq 1$ contains a critical point of $P$, $P$ being any polynomial of degree greater than $N$ having "$a$" as its zero and all its other zeros lie in the closed unit disk. His proof relies on estimating below and above the positive real number $|P(c)|$, for some real number $c$ satisfying $0 < c < a$, whenever $P$ does not possess a zero in the disk $|z - a| \leq 1$ . The estimation yields an upper bound for the degree $n$ of the polynomial $P$.

The upper estimation provided for $|P(c)|$, $c$ being any real number that belongs to $(0, a)$, is $|P(c)| \leq 1 + a$. The proof of the latter inequality utilized two theorems that we mention below. These theorems could be implied from theorems that we have already concerned ourselves with in this research.

**Theorem 4.2.1.** *Let $f$ be a polynomial of degree $n \geq 2$. For a complex number $Z$ that is neither a zero nor a critical point of $f$, every circle $C$ that passes through $Z$ and $Z - n\dfrac{f(Z)}{f'(Z)}$ separates least two zeros of $f$ unless all the zeros lie on $C$.*

*Proof.* As for the proof of the above theorem, we recall Laguerre's theorem, and specifically the second part of Laguerre's theorem, where we proved a circle passing by $\xi$ and a zero $Z$ of the polar derivative of $f$ with respect to $\xi$ that is not a zero of $f$, is a circle that separates the zeros of $f$ or passes through all of them. In our case, and if we were to prescribe $Z$ and determine $\xi$ such that $nf(Z) + (\xi - Z)f'(Z) = 0$, we get $\xi = Z - n\dfrac{f(Z)}{f'(Z)}$. Hence, by direct application of Laguerre, we secure the desired result.

□

**Theorem 4.2.2.** *For $f$ a polynomial of degree $n \geq 2$, the perpendicular bisector of a line segment joining two distinct zeros of $f$ separates the critical points of $f$, i.e each of the closed half planes whose boundary is the perpendicular bisector of the segment contains a zero of the derivative $f'$ of the polynomial.*

*Proof.* The proof of this theorem employs Grace's theorem. In fact, not only does this proof employ Grace's thoerem, but it also does it in a manner analogous to how Grace-Heawood theorem do. First, and without loss of generality, it may be assumed that the two zeros of $f$ are $z_1 = 0$ and $z_2 = 1$ respectively. By the same argument adopted in the proof of Grace-Heawood, which was even inspected more generally in the tail of that proof, we get that $f'(z)$ and $q(z) = (z-1)^n - z^n$ are apolar with the zeros of $q(z)$ being situated on the line $x = \frac{1}{2}$. Then, and by Grace's theorem, each of the half planes $x \geq \frac{1}{2}$ and $x \leq \frac{1}{2}$ contains a zero of $f'$.

□

# Chapter 5

# Tao's Approach

Terence Tao treats the problem from a relatively new perspective, where random variables are used to denote a zero of the polynomial $f$ in Theorem 4.0.1, or a zero of its derivative, that are chosen uniformly at random. The asymptotic distributions of the zeros of $f$ and $f'$, or the limiting behavior of the mentioned random variables, is put under study in the cases of $a^{(\infty)}$ being zero, with potential theory and Balayage theory getting involved. Obtaining some results on the limiting distributions of the random variables, further conditions are put on the sequence $a$, of the kind $a = o(\frac{1}{logn})$, that will yield absurd results.

## 5.1 The introduction of Random Variables and Some of Their (General) Limiting Properties

For every $n$ in the ambient sequence, let $\lambda = \lambda^{(n)}$ denote a zero of $f$ chosen uniformly at random of the $n$ zeros of f, and $\zeta = \zeta^{(n)}$ a zero of $f'$ chosen uniformly at random from the $n-1$ zeros of $f'$ and independently of $\lambda$. Denote by $\Omega^{(n)}$ the probability space used to define $\lambda^{(n)}$ and $\zeta^{(n)}$, and by $\mathbb{E} = \mathbb{E}^{(n)}$ and $\mathbb{P} = \mathbb{P}^{(n)}$ the expectation and probability with respect to this space respectively.

### 5.1.1 *That $\lambda$ and $\zeta$ have the same expectation*

Denote by $\lambda_1, \lambda_2, ..., \lambda_n$ the zeros of $f$ and by $\zeta_1, \zeta_2, ..., \zeta_{n-1}$ the zeros of $f'$ counting multiplicity. The expectations of $\lambda$ and $\zeta$ are

$$\mathbb{E}(\lambda^{(n)}) = \sum_{i=1}^{n} \frac{1}{n}\lambda_i$$

$$\mathbb{E}(\zeta^{(n)}) = \sum_{i=1}^{n} \frac{1}{n-1}\zeta_i$$

Writing $f$ and $f'$ in the following forms

$$f(z) = \sum_{i=1}^{n} a_i z^i = \prod_{i=1}^{i=n} (z - \lambda_i)$$

$$f'(z) = \sum_{i=1}^{n-1} a'_i z^i = n \prod_{i=1}^{i=n-1} (z - \zeta_i)$$

When we carry out the multiplications and equate coefficients, we obtain the following expressions for the coefficients $a_{n-1}$ and $a'_{n-2}$ of $f$ and $f'$ respectively

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i$$

$$a'_{n-2} = -n \sum_{i=1}^{n-1} \zeta_i$$

By differentiating $f$, we know that $(n-1)a_{n-1} = a'_{n-2}$, this yields

$$\sum_{i=1}^{n} \frac{1}{n} \lambda_i = \sum_{i=1}^{n-1} \frac{1}{n-1} \zeta_i$$

which amounts to $\mathbb{E}(\lambda^n) = \mathbb{E}(\zeta^n)$.

### 5.1.2 *That subsequences of $\lambda$ and $\zeta$ converging in distribution exist and their limits have equal moments and Balayages*

We recall some convergence concepts regarding the convergence of a sequence of random variables. This **recallment** is supported by Patrick Billingsley's Book, "Convergence of Probability Measures"[8], as well as his other book "Probability and Measure"[9].For this, Consider a metric space $S$, and let $\mathcal{S}$ be the Borel $\sigma - field$, i.e the one generated by the open sets[1].

#### 5.1.2.1 Recalling Some Convergence Concepts in Probability Theory

**Definition 5.1.1.** *Weak convergence of probability measures on metric spaces.* Recall that a probability measure $\mathcal{P}$ on $\mathcal{S}$ is a nonnegative, countably additive set function satisfying $\mathcal{P}(\mathcal{S}) = 1$. We say that a sequence of probability measures $\mathcal{P}^{(n)}$ converges weakly to $\mathcal{P}$ if

$$\int f d\mathcal{P}^{(n)} \longrightarrow \int f d\mathcal{P}$$

for every bounded continuous complex-valued function on $S$. As to what $\int f d\mathcal{P}$ means, we recall the way the integral with respect to a measure is defined for a non-negative measurable function g on $(S, \mathcal{S}, \mathcal{P})$, and then the definition of the integral

[1] I would like to say something about the borel choice

of a complex function follows. The integral of $g$ is defined as the supremum of the sums

$$\int g d\mathcal{P} = sup \sum_i (\inf_{w \in A_i} g(w))\mathcal{P}(A_i)$$

The supremum extends over all finite decompositions $A_i$ of $S$ into $\mathcal{S} - sets$.

**Definition 5.1.2.** *The distribution of a random variable.* A mapping $X$ from some probability space $(\Omega, \mathcal{F}, P)$, to $(S, \mathcal{S})$, that is $\mathcal{F}/\mathcal{S}$ measurable is called a random variable. The distribution or law of $X$ is the probability measure $\mathcal{P} = PX^{-1}$ on $(S, \mathcal{S})$. The expected value of $X$ on $(\Omega, \mathcal{F}, P)$, in case it was complex valued, is the integral of $X$ with respect to the measure $P$. Namely, $E(X) = \int XdP$[2].

**Definition 5.1.3.** *Convergence in distribution of a sequence of random variables.* A sequence $X^{(n)}$ of random variables is said to converge in distribution to the random variable $X^{(\infty)}$ if the sequence of distributions $\mathcal{P}^{(n)}$, associated to $X^{(n)}$, converges weakly to $\mathcal{P}$, the distribution of $X$. Equivalently,

$$E^{(n)} f(X^{(n)}) \longrightarrow E^{(\infty)} f(X^{(\infty)})$$

for every bounded continuous function $f : \mathbb{C} \longrightarrow \mathbb{C}$. This is considering $X^{(n)}$ to be defined on a probability space $\Omega^{(n)}$ and $X^{(\infty)}$ on a probability space $\Omega^{(\infty)}$ with ranges $\mathbb{C}$ and expectations $E^{(n)}$ and $E^{(\infty)}$ respectively.

*Remark.* Although the above definition makes no sense unless the range (and the topology on it) is the same for all $X^{(\infty)}, X^{(1)}, X^{(2)}, ...$, the underlying probability spaces (the domains) may all be distinct.

**Definition 5.1.4.** *Tightness of a family of probability measures.* A family $\Pi$ of probability measures on $(S, \mathcal{S})$ is called tight if for $\epsilon > 0$ there exists a compact set[3] $K \subset S$ such that $\mathcal{P}(k) > 1 - \epsilon$ for all $\mathcal{P} \in \Pi$. If $S$ is separable and complete, then each probability measure on $(S, \mathcal{S})$ is tight.

**Definition 5.1.5.** *Relatively compact family of probability measures.* A family of probability measures $\Pi$ on $(S, \mathcal{S})$ is said to be relatively compact if any sequence of its elements contains a weakly convergent subsequence. The limiting probability measures might be different for different subsequences and lie outside $\Pi$.

**Theorem 5.1.1.** *Prokhorov's Theorem. If a family of probability measures* $\Pi$ *on* $(S, \mathcal{S})$ *is tight, then it is relatively compact.*

[2] The integral could also be writtes as $E(X) = \int_\Omega X(w)dP(w)$ which, through a change of variable, leads $E(X) = \int_S xd\mathcal{P}(x)$

[3] It is compact so it is measurable with respect to the Borel $\sigma - field$.

**This marks the end of the recallment.**

In our case, $\lambda$ and $\zeta$ are mappings with domain $\Omega^{(n)}$ and range $\mathbb{C}$, or more precisely, with range $\overline{D(0,1)}$. The complex plane, along with the metric $d$ induced by the norm given by the complex modulus, forms a complete and separable metric space, and so does $\overline{D(0,1)}$ with the induced metric. This is to say that, by Prokhorov's Theorem, any family of probability measures on $(\overline{D(0,1)}, d)$ has a weakly convergent subsequence. In particular, after passing to a subsequence, the random variables $\lambda$, $\zeta$ converge in distribution to (fixed) random variables $\lambda^{(\infty)}$, $\zeta^{(\infty)}$ respectively on some probability space $\Omega^{(\infty)}$ taking values in $\overline{D(0,1)}$. Thus,

$\mathbb{E}f(\lambda) = \mathbb{E}f(\lambda^{(\infty)}) + o(1)$ and $\mathbb{E}f(\zeta) = \mathbb{E}f(\zeta^{(\infty)}) + o(1)$

For any (fixed) continuous function $f : \mathbb{C} \longrightarrow \mathbb{C}$.

#### 5.1.2.2 Bringing in the Logarithmic Potential and Stieltjes Transform

In order to establish the equality between the moments of the limiting random variables, as well as between their Balayages, we consider **bringing in the logarithmic potential and the steiltjes transform** of the random variables which, as we will convey, have a set of helpful properties that will contribute to the understanding of the behavior of the random variables in the asymptotic limit.

**Definition 5.1.6.** Given a complex random variable $\eta$ taking values in a fixed compact set, we define the *logarithmic potential* $U_\eta$ by

$$U_\eta(z) = \mathbb{E}\log\frac{1}{|z-\eta|}$$

and the $stieltjes\ transform$ or $Cauchy\ transform$ $s_\eta$ by

$$s_\eta(z) = \mathbb{E}\frac{1}{z-\eta}$$

These functions are defined for almost every $z \in \mathbb{C}$. In the literature, the logarithmic potential and the Stieltjes transform are more likely to be approached from a measure-theoretic angle and written as functions of the law $\mu_\eta$ of $\eta$. We render, hereby, a brief exposition of this approach pointing up the general motivation to the introduction of these functions in the literature.

Books on potential theory or logarithmic potentials, such as "Potential Theory in the Complex Plane"[10] by Ranford or "Logarithmic Potentials with External Field"[11] by Totik and Saff, adopt an electrostatic motivation to the introduction of logarithmic potentials. The fundamental electrostatics problem concerns the distribution of charge on a conductor. If the conductor is regarded as a compact set in the complex plane and the charges repel each other according to an inverse distance law then

$$\int \log \frac{1}{|z-t|} d\mu(t)$$

represents the potential energy at $z$ due to $\mu$, with $\mu$ being the charge distribution on the conductor. In mathematical terms, $\mu$ is a finite borel measure on $C$ with compact support. The terms "potential function" and "potential" applied to integrals of the form $\int \frac{1}{|z-t|}\mu(t)$, which is analogous to the form we introduced for the Cauchy transform, were first used by G. Green in 1828 and Gauss in 1840. $\frac{1}{|z-t|}$ is sometimes referred to as the Newton Kernel and it depicts the inverse of the distance between two masses or charges. It could be said, at least from a physics perspective, that the logarithmic potential arises from the potential with Newton kernel [4]. In contrast to the Newton kernel, the logarithmic kernel has a singularity not only as $|z-t| \longrightarrow 0$, but also as $|z-t| \longrightarrow \infty$, which causes some differences in the behavior of the logarithmic potential compared to the Newton potential.

**Some properties of $U_\eta$ and $s_\eta$.** The integral, which $U_\eta$ stands for, is well defined on $\mathbb{C}$ (may be $\infty$, for $z$ in the support of the distribution of $\eta$)[5]. By Young's inequality, we can prove the local integrability of $U_\eta$ with respect to Lebesgue measure, and thus that it is finite a.e. in $\mathbb{C}$ with respect to this measure. The potential $U_\eta$ is superharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\backslash(supp\mathcal{P})$[6]. $\mathcal{P}$ refers to the distribution or law of $\eta$. For the superharmonicity of $U_\eta$ on $\mathbb{C}$, we direct the reader to the following book where he could find a verification [11].

To verify the harmonicity of $U_\eta$ in $\mathbb{C}\backslash(supp\mathcal{P})$, we observe that for a fixed $t$, the function $\log(\frac{1}{|z-t|})$ is harmonic in $\mathbb{C}\backslash\{t\}$. Hence for $z \notin supp(\mathcal{P})$ the Laplacian of $U_\eta$ satisfies

$$\Delta U_\eta(z) = \int \Delta \log \frac{1}{|z-t|} d\mathcal{P}(t) = 0$$

The integral representing $s_\eta$ is well defined outside the support of $\mathcal{P}$. $s_\eta$ is locally integrable with repect to Lebesgue measure, and consequently is finite a.e on $\mathbb{C}$. $s_\eta(z)$ is essentially the (distributional) gradient of $U_\eta(z)$,

$$s_\eta(x+iy) = (-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})U_\eta(x+iy)$$

[4]consult foundations of potential theory for the significance of introducing log to the newton kernel

[5]with $t \longrightarrow \log \frac{1}{|z-t|}$ being a measurable (extended) function with respect to the law of $\eta$.

[6]Notice that the support of $\mathcal{P}$ is a set of measure zero with respect to Lebesgue measure, this contributes to the a.e. finiteness of $_\eta$ on $C$, along with the local integrability of $U_\eta$ on $\mathbb{C}\backslash(supp\mathcal{P})$.

Away from the support of $\eta$, $s_\eta$ is holomorphic.

The above properties apply to the logarithmic potentials and Stieltjes transforms of general random variables. In particular, they apply for the logarithmic potentials and Stieltjes transforms of $\lambda$ and $\zeta$. Moreover, we will be able to relate the logarithmic potentials and Stieltjes transforms of the just mentioned random variables to the polynomial $f$ and its first two derivatives $f', f''$, paving the way for setting up relations between the logarithmic potentials and Stieltjes transforms themselves.

**Relational Properties.** For almost all $z$, one has

$U_\lambda(z) = \mathbb{E}\log\frac{1}{|z-\eta|} = \frac{1}{n}\sum_{i=1}^{n}\log\frac{1}{|z-\lambda_i|} = \frac{1}{n}\log\frac{1}{|f(z)|} = -\frac{1}{n}\log|f(z)|$

$U_\zeta(z) = \mathbb{E}\frac{1}{|z-\zeta|} = \frac{1}{n-1}\sum_{i=1}^{n-1}\log\frac{1}{|z-\zeta_i|} = \frac{1}{n-1}\log\frac{n}{|f'(z)|} = \frac{\log n}{n-1} - \frac{1}{n-1}\log|f'(z)|$

$s_\lambda(z) = \mathbb{E}\frac{1}{z-\lambda} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{z-\lambda_i} = \frac{1}{n}\frac{f'(z)}{f(z)}$

Similarly, $s_\zeta(z) = \frac{1}{n-1}\frac{f''(z)}{f'(z)}$.

From the first three relations, we could set up the following one

$$U_\lambda(z) - \frac{n-1}{n}U_\zeta(z) = \frac{1}{n}\log|s_\lambda(z)|.$$

**The above relation could serve in proving that $U_{\lambda^\infty}(z) = U_{\zeta^\infty}(z)$, for $z$ in a specific region of the complex plane, as follows**

For $z$ in the region $\overline{D(0,1)}^c$, $s_\lambda(z)$ is bounded. This is as for such $z$, $z-\lambda \in D(z,1)$, and $|z-\lambda|$ is bounded by $|z|-1$ from below and $|z|+1$ from above. Consequently, $\frac{1}{z-\lambda}$ lies in a disk where $\frac{1}{|z|+1}$ and $\frac{1}{|z|-1}$ bound it from below and above respectively. By convexity, $\mathbb{E}\frac{1}{z-\lambda} = s_\lambda(z)$ lies within the same disk and has the same bounds. hence,

$$-\frac{\log(|z|+1)}{n} \leq U_\lambda(z) - \frac{n-1}{n}U_\delta(z) \leq -\frac{\log(|z|-1)}{n}$$

Since $\lambda, \zeta$ converge in distribution to $\lambda^{(\infty)}, \zeta^{(\infty)}$, and as $t \longrightarrow \log\frac{1}{|z-t|}$ is continuous and bounded in $\overline{D(0,1)}$ for fixed $z \in \overline{D(0,1)}^c$, we have that $U_\lambda(z)$ and $U_\zeta(z)$ converge to $U_{\lambda^{(\infty)}}$ and $U_{\zeta^{(\infty)}}$ respectively as $n \longrightarrow \infty$ for $z \in \overline{D(0,1)}^c$. Taking limits as $n$ tends to $\infty$ for the above inequalities, we conclude that

$$U_{\lambda^{(\infty)}}(z) = U_{\zeta^{(\infty)}}(z)$$

for $z \in \overline{D(0,1)}^c$.

#### 5.1.2.3 **Proving the Equality Between the Moments**

The equality between the moments of $\lambda^{(\infty)}, \zeta^{(\infty)}$ could be deduced from the above equality by considering the taylor expansion of $\log\dfrac{1}{|z-t|}$ which will allow for the emergence of powers of $t$ in the expression of $\log\dfrac{1}{|z-t|}$, and subsequently the emergence of the moments in the expressions of the logarithmic potentials after expectation is applied to those expressions. In fact, if we fix $z \in \overline{D(0,1)}^c$, a Taylor expansion of $\log\dfrac{1}{|z-t|}$, as a function of $t$, could be obtained in the disk $D(0,|z|)$ by an attempt of realizing the real logarithm of the complex modulus as the real part of the complex logarithm and then implementing the Taylor expansion of the complex logarithm evaluated at one minus a complex number that's less than one in modulus as follows

$$\log\frac{1}{|z-t|} = -\log|z-t| = -\Re\log(z-t) = -\Re\log(z(1-\frac{t}{z}))$$

$$= -\Re\log z - \Re\log(1-\frac{t}{z}) = -\log|z| + \Re\sum_{k=1}^{\infty}\frac{1}{k}(\frac{t}{z})^k = -\log|z| + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left((\frac{t}{z})^k + \overline{(\frac{t}{z})^k}\right)$$

By further manipulation, we arrive at the following

$$\log\frac{1}{|z-t|} = -\log|z| + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\overline{z}^k t^k + z^k\overline{t}^k}{|z|^{2k}}\right)$$

Conducive to visualizing this Taylor Series as a Fourier Series, bearing the periodicity of the function with respect to the arguemnt of $z$, and the interest in preserving the powers of $t$, we write $z$ in the form $z = Re^{\theta i}$and obtain

$$\log\frac{1}{|z-t|} = -\log R + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{e^{-ik\theta}t^k}{R^k} + \frac{e^{ik\theta}\overline{t}^k}{R^k}\right)$$

$$= -\log R + \frac{1}{2}\sum_{k<0}\frac{t^{|k|}e^{ik\theta}}{|k|R^{|K|}} + \frac{1}{2}\sum_{k>0}\frac{\overline{t}^k e^{ik\theta}}{kR^K}$$

In his paper, Tao suggests a one sum over $k \in \mathbb{Z}, k \neq 0$ which disregards the conjugation associated with $t$ in the case of positive $k$ and we believe is thus not true. So, and with the last expression being available for all $t \in D(o,R)$, we get by Fubini-Tonelli Theorem that

$$U_{\lambda^\infty}(z) = -\log R + \frac{1}{2}\sum_{k<0}\frac{e^{ik\theta}}{|k|R^{|K|}}\mathbb{E}(\lambda^{(\infty)})^{|k|} + \frac{1}{2}\sum_{k>0}\frac{e^{ik\theta}}{kR^K}\mathbb{E}(\overline{\lambda^{(\infty)}})^k$$

Similarly for $U_\zeta^{(\infty)}$. Now, as the logarithmic potentials of $\lambda^{(\infty)}, \zeta^{(\infty)}$ are equal on the exterior of the closed unit disk, the property of Fourier uniqueness yields that the random variables $\lambda^{(\infty)}, \zeta^{(\infty)}$ have equal moments. Namely,

$$\mathbb{E}(\lambda^{(\infty)})^k = \mathbb{E}(\zeta^{(\infty)})^k$$

for all natural numbers $k$.

An equality which Tao tries further to approve of, and which is going to be exploited extensively in the attempts of understanding better the asymptotic profiles of the random variables of our interest, is the equality between the Balayages of these random variables, and is just another way of communicating the equality between their moments.

**Balayages**, as real functions that we will define on the boundary of a disk $D(0, R)$ -$R$ in this case is just a positive number and is not necessarily greater than one- **will deploy the poisson kernel's formula on this disk and will have a Brownian motion related interpretation that is important to our research work.** So, we take the step to introduce and discuss some of the properties of the Poisson kernel and Brownian motion.

#### 5.1.2.4 On the Poisson Kernel

The Poisson kernel appears in the study of Harmonic functions, and more specifically, in the study of the Dirichlet problem that is concerned with finding a harmonic function on a domain with prescribed boundary values. Consider the disk $D(0, R)$ and let $f : \partial D(0, R) \longrightarrow \mathbb{R}$ be a continuous function. The Dirichlet problem is about finding a harmonic function, call it $H_f$, on $D(0, R)$ such that $\lim_{z\to\zeta} H_f(z) = f(\zeta)$. By expressing the function on the boundary as a Fourier series, in the sense $f(Re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$, a harmonic extension in $D(0, R)$ of $f$ could be guessed to be $H_f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k \left(\frac{r}{R}\right)^{|k|} e^{ik\theta}$, for $0 \leq r < R$. The latest series could be rewritten as a boundary integral through considering a substitution of the coefficeints $c_k$ by their integral form and then swapping the sum and integral. $H$ is then represented as follows

$$\begin{aligned} H_f(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} \left( \sum_{k=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|k|} e^{i(\theta-\phi)} \right) f(Re^{i\phi})d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\left(\frac{r}{R}\right)^2}{1-2\left(\frac{r}{R}\right) cos(\theta-\phi)+\left(\frac{r}{R}\right)^2} f(Re^{i\phi})d\phi \end{aligned}$$

Setting $P^R_{re^{i\theta}}(Re^{i\phi}) = \sum_{k=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|k|} e^{i(\theta-\phi)}$, the function $P^R_{re^{i\theta}} : \partial D(0,R) \longrightarrow \mathbb{R}$ denotes the Poisson kernel for the disk $D(0,R)$, and $H_f$, as defined, solves the Dirichlet problem. $H_f$ is referred to as the Poisson integral of $f$. By the Maximum principle, it could be proven that this solution is unique.

When it comes to the history of approaching the Dirichlet problem and trying to solve it, and after the solution was given as the Poisson integral, the solution did acquire some advanced interpretations. Fix $x \in D := D(0,R)$. As a continuous function $f : \partial D \longrightarrow \mathbb{R}$ determines a unique harmonic function $H_f$ that solves the Dirichlet problem, and by Riesz–Markov–Kakutani representation theorem and the maximum principle, $H_f(x)$ determines a probability measure $w(x,D)$ on $\partial D$ by

$$H_f(x) = \int_{\partial D} f(y) dw(x,D)(y)$$

The measure $w(x,D)$ is called the harmonic measure (of the domain $D$ with pole at $x$). With this, the harmonic measure has density equal to the Poisson kernel with respect to arc length on the boundary. Said differently, the harmonic measure is absolutely continuous with respect to arc length measure, and the Radon-Nikodym derivative is the Poisson kernel.

**In fact, the Poisson kernel, or lets say the Harmonic Measure, acquired after time of its introduction to the literature an interesting interpretation related to the theory of Brownian motion** with the works of Katakana (1944) and later Doob (1950s). As we said before, we consider providing an overview on Brownian motion and the related probabilistic derivation of the solution to the Dirichlet problem. Our overview on brownian motion follows some insights from the book "Brownian Motion and classical Potential Theory"[12].

#### 5.1.2.5 **On Brownian Motion**

A Brownian motion is a collection of random variables that satisfy a set of properties.[7] Bearing our interests in this research, we consider the random variables to be as follows

$$\{X_t : \Omega \longrightarrow \mathbb{C}\}_{t\in[0,\infty)}$$

where $\Omega$ denotes the set of all continuous functions from $[0,\infty)$ to $\mathbb{C}$ [8] and for $w \in \Omega$, $X_t(w) = w(t)$. Denote by $\mathcal{F}_\infty$ the smallest sigma field on $\Omega$ such that $X_t$ is measur-

[7] Instead of saying a collection of random variables we could have said a "Stochastic Process" in more formal terms.

[8] This choice of the probability space leads the so called canonical definition of Brownian motion, while there are other definitions where the Probability sapce is arbitrary. The arbitrary sample space makes defining some concepts that we are interested in like hitting time harder. Wiener who put the first rigorous definition of Brownian motion used this space of continuous functions, and the measure $P_z$ is then called the Wiener measure

able for all $t \in [0, \infty)$.

Let $s \in \mathbb{C}$. The above collection of random variables is said to be a Brownian motion starting at $s$ if there exists a probability distribution $P_s$ on $\Omega$ such that the joint distribution of $X_{t_1}, X_{t_2}, ..., X_{t_n}$ is

$$f(t_1, z_1 - s)f(t_2 - t_1, z_2 - z_1)...f(t_n - t_{n-1}, z_n - z_{n-1})$$

evaluated at $z_1, z_2, ..., z_n \in \mathbb{C}$, for $n \in \mathbb{N}$ and $0 < t_1 < ... < t_n$. This is where $f(t, z), t > 0$ denotes the normal distribution on $\mathbb{C}$ defined by

$$f(t, z) = \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}}$$

An element of $\Omega$ is referred to as a realization of the Brownian motion. A realization is one outcome that the simulation of the process $\{X_t\}_t$ could generate. $t$ is referred to as an index of time; that each simulation is run through time and for each instant of time the simulation of the random variable indexed by that instant occurs.

Chapter 1 of the book " Brownian Motion and Classical Potential Theory" proves the existence of such a probability distribution $P_s$. The proof starts with considering an index set of time $T$ that is dense and countable in $[0, \infty)$, and then defining the measures $\mu_{t_1,...,t_n}$ on $\mathbb{C}^n$ that encode the joint behavior of the hypothetical process at times $t_1 < t_2 < ... < t_n$ as follows

$$\mu(dz_1, dz_2, .., dz_n) = f(t_1, z_1 - s)f(t_2 - t_1, z_2 - z_1)...f(t_n - t_{n-1}, z_n - z_{n-1})dz_1 dz_2...dz_n$$

These are consistent probability distributions, and thus by Kolmogorov's extension theorem [9], there exists a probability space $(\Omega', \mathcal{F}, P)$ and random varibales $Y(t), t \in T$ such that for all $t_1 < ... < t_n$ the joint law of $(Y(t_1), ..., Y(t_n))$ is $\mu_{t_1,...,t_n}$ with $X(0) = s$. Then, and with the necessary lemmas, it was proven that $Y(t), t \in T$ extend uniquely to $Y(t), t \geq 0$ which is continuous in $t$ for $w' \in \Omega'$. By the continuity of $f$ in t the joint distributions of $Y(t), t > 0$, are as desired[10]. Consider the map $M$ from $\Omega'$ to $\Omega$ defined by $(Mw')(t) = Y(t, w'), t > 0$. Let $P_s$ be the distribution on $(\Omega, \mathcal{F}_\infty)$ induced by $M$ and $(\Omega', \mathcal{F}, P)$. Then $P_s$ corresponds to Brownian motion starting at $s$.

In the sequel, we define the **exit distribution** $H_D(s, .)$ of an open disk $D$, associated to Brownian motion starting at $s$, by

$$H_D(s, A) = P_s(\tau_{D^c} < \infty, X(\tau_{D^c}) \in A)$$

[9] One reason why the definition of Brownian motion relying on joint densities was considered and not others is that aligns with Kolmogorov's extension theorem more directly.

[10] If we were not interested in the canonical version of definition of Brownian motion, we would have stopped the proof at here

where $A$ belongs to the Borel sigma algebra of $\mathbb{C}$ and $\tau_{D^c}$ is a random variable on $\Omega$, called the hitting time of $D^c$, and defined by $\tau_{D^c} = \inf\{t > 0 : X(t) \in D^c\}$ [11]. Notice that $H_D(s, A)$ is the (Wiener) measure of the set in $\Omega$ where the random variable $\tau_{D^c}$ takes finite values, intersected with the set in $\Omega$ where the random variable $X(\tau_{D^c})$ takes values in $A$. For even more clarification and precision, the random variable $X(\tau_{D^c})$ maps each Brownian path $w$ to the complex number $X_{\tau_{D^c}(w)}(w)$ which is the value the simulation or the path $w$ takes when it first exits the disk $D$.

As defined, $H_D(s, .)$ has total measure $P_z(\tau_{D^c} < \infty)$. Thus, $H_D(s, .)$ is a probability measure as $P_z(\tau_{D^c} < \infty) = 1$ [12]. Actually, the definition of $H_D(s, .)$ could be simplified to $H_D(s, A) = P_s(X(\tau_{D^c}) \in A)$.

By the continuity of paths $X(\tau_{D^c}) \in \partial D$ on $\{0 < \tau_{D^c} < \infty\}$, so $H_{D^c}(s, .)$ is concentrated on $\partial D$. For a measurable function $f$ that is bounded on the boundary of the unit disk, define $Hf(s) = \int f(z) H_D(s, dz)$.

For a continuous function $\varphi$ on $\partial D$, $H_D\varphi$ is a solution to the Dirichlet problem for $\varphi$.

Upon these illustrations, one can interpret $P_s^R(Re^{i\theta})$ as the probability density of the Harmonic measure of $s$ on the circle $\partial D(0, R)$ with respect to the normalized arc length measure on $\partial D(0, R)$, or the probability density of the distribution of the first location at which a Brownian Motion originating at $s$ exits the disk $\overline{D(0, R)}$ with respect to the normalized arc length measure on $\partial D(0, R)$.

#### 5.1.2.6 **Proving the Equality Between the Balayages**

For a Random variable $\eta$ taking values in $D(0, R)$, Define the Balayage $Bal_R(\eta)$ : $\partial D(0, R) \longrightarrow \mathbb{R}$ by

$$Bal_R(\eta)(Re^{i\theta}) = \mathbb{E}P_\eta^R(Re^{i\theta})$$

In turn, one can interpret $Bal_R(\eta)$ as the probability density of the first location where a Brownian motion originating at $\eta$ exits the disk $\overline{D(0, R)}$.

By substituting the expression of the Poisson kernel that involves a summation, we are able to express the Balayage as follows

$$Bal_R(\eta)(Re^{i\theta}) = 1 + 2\Re \sum_{k=1}^{\infty} R^{-k}\mathbb{E}\eta^k$$

Therefore, and as a cosequence of the equality betweem the moments of the limiting random variables, we have that for $R > 1$

[11] $\tau_{D^c} = \infty$ if $x(t) \notin D^c$. In words, the random variable $\tau_{D^c}$, as a measurable function on $\Omega$, takes the value $\infty$ when applied to $w \in \Omega$ where $X_t(w)$, as a measurable function from $[0, \infty)$ to $\mathbb{C}$, does not take a value in $D^c$ for all $t > 0$

[12] propositions 2.7 and 2.8 of ch2 of the book we're using proves this about $D^c$

$$Bal_R(\lambda^{(\infty)}) = Bal_R(\zeta^{(\infty)})$$

This is saying that the Brownian motions originating from $\lambda^{(\infty)}$ and $\zeta^{(\infty)}$ behaves Probabilistically similar in exiting the disk $\overline{D(0,1)}$ for the first time.

## 5.2 (Special) Properties of the Limiting Random Variables

The latest equality which was established between the Balayages of the limiting random variables is a fairly rich equality in the geometric or interpretative sense considering the points we made regarding the Brownian Motion related interpretations of the Poisson kernel. This allows for predicting forth, through intuition some assumptions on the distributions of $\lambda^{(\infty)}$ and $\zeta^{(\infty)}$ as to maintain this equality, like having to take values as near as to not lose the property of a Brownian motion starting at any of them having to exit the unit disk with the same distribution.

### 5.2.1 *The case* $a^{(\infty)} = 0$

**We prove that in this case $\lambda^{(\infty)}$ and $\zeta^{(\infty)}$ almost surely lie in the semicircle $C := \{e^{i\theta} : \pi/2 \leq \theta \leq 3\pi/2\}$ and have the same distribution.**

#### 5.2.1.1 That $\zeta^{(\infty)}$ is Supported on $C$

When it comes to $\zeta$, it is relatively obvious or intuitive that the limiting random variable $\zeta^{(\infty)}$ will almost surely lie in the semicircle $C$ having that $\zeta$ takes values in the lune $\overline{D(0,1)} \backslash \overline{D(a,1)}$. In fact, we use the results of Portmanteau Theorem, which is concerned with the convergence of Probability measures, and specifically in providing equivalent conditions to weak convergence, to prove that the support of $\zeta^{(\infty)}$ lies within $C$ by observing that the limit of the sequence of supports $S_n := Supp\zeta^{(n)}$ is $C$.

Below is a cut-down version of the Portmanteau Theorem which suffices to insure that our desired conclusion follows. The full version of the Portmanteau Theorem could be revised in the book of Billingsley which was referenced earlier titled "Convergence of Probability Measures".

**Theorem 5.2.1.** ***Portmanteau Theorem.*** *Consider $(S, \mathcal{S}, d)$ a metric space with the sigma field generated by open sets. For $P_n$ and $P$ probability meaures on $(S, \mathcal{S})$, we have*

*If $P_n$ converges weakly to $P$, then $limsup_n P_n F \leq PF$ for every $F$ closed, and $PG \leq liminf_n P_n G$ for every $G$ open.*

*Proof.* The weak convergence contributes to validating the above inequalities by first approximating the indicator function $I_F$ by the following continuous bounded functions

$$f_\epsilon(x) = (1 - \frac{d(x,F)}{\epsilon})^+$$

We have that $I_F \leq f_\epsilon$ and $f_\epsilon \leq I_{F^\epsilon}$ where $F^\epsilon = \{x : d(x,F) < \epsilon\}$.

From $I_F \leq f_\epsilon$, the following inequality holds

$$limsup_n P_n F \leq limsup_n f_\epsilon$$

Now, by the given weak convergence of $P_n$ to $P$, and as $f_\epsilon$ is but a continuous bounded function, we have that $limsup_n P_n f_\epsilon = P f_\epsilon$. This yields

$$limsup_n P_n F \leq P f_\epsilon$$

From $f_\epsilon \leq I_{F^\epsilon}$, it is guaranteed that $P f_\epsilon \leq P F_\epsilon$. Taking $\epsilon$ to zero, it follows that

$$limsup_n P_n F \leq PF$$

The second implication, which is for open sets, follows through complementation.

□

Back to our study of the support of $\zeta^{(\infty)}$, notice that $S_n$ is a decreasing sequence of closed sets, and $lim_n S_n = \bigcap_n S_n = C$. To prove that $\zeta^{(\infty)}$ lies in $C$ almost surely, we prove that $supp\zeta^{(\infty)} \subseteq C$.

For $z \notin C$, we have that $z \notin lim_n S_n$. This is saying that there exists an open neighborhood $U$ of $z$, such that for $n$ greater than or equal to some $N$, $S_n \cap U = \phi$. Hence, for $n \geq N$, $\mathcal{P}^{(n)}(U) = 0$, where $\mathcal{P}$ refers to the law of $\zeta^{(n)}$. According to Portmanteau Theorem, $\mathcal{P}^{(\infty)} U \leq liminf_n \mathcal{P}^{(n)} U = 0$.

Therefore, $U \cap supp\zeta^{(\infty)} = \phi$, and $z \notin supp\zeta^{(\infty)}$. Having started with $z \notin C$ and ended with $z \notin supp\zeta^{(\infty)}$, we have that $supp\zeta^{(\infty)} \subseteq C$ as desired.

#### 5.2.1.2 That $\lambda^{(\infty)}$ is also Supported on $C$

The other claim that we verify in the subsequent discussion is not as obvious as the last, and it is that $\lambda^{(\infty)}$ is also supported on $C$. As we tried to say in the introduction of this section, the equality between the Balayages of the limiting random variables $\zeta^{(\infty)}$ and $\lambda^{(\infty)}$ is crucial in establishing this claim.

We made the point in the previous section that the Poisson kernel $P_s^R(Re^{i\theta})$ is the probability density of the exist distribution $H_D(s,.)$ of $D = D(0,R)$ associated to Brownian Motion starting at $s$, with respect to the normalized arc length measure on $\partial D(0,R)$, call it $\sigma(R,.)$[13]. Meaning, for $A \subseteq \partial D(0,R)$

[13] It could be denoted as the uniform probability distribution on $\partial D(0,R)$ or as the surface area on $\partial D(0,R)$ normalized to have total measure one

$$H_D(s,A) = \int_A P_s^R(z)\sigma(R,dz)$$

We don't know much about $\lambda$ as to "polish" its range $\overline{D(0,R)}$ like we did with $\zeta$ when we said it takes values in $\overline{D(0,R)}/\overline{D(0,1)}$ - this we knew from the assumption that $f$ does not satisfy Sendov's conjecture at $a$ so no critical points are to be in $\overline{D(0,1)}$.

The approach is inspired by the observation that the probability a Brownian motion originating at $\zeta^{(\infty)}$ exists $\overline{D(0,R)}$ in a closed arc $A$ subset of the semicircle $C' := \{e^{i\theta} : -\pi/2 < \theta < \pi/2\}$ tends to zero as $R$ tends to one. The probability that a Brownian motion starting at $\lambda^{(\infty)}$ exists $\overline{D(0,R)}$ in the same arc, we deduce, should tend to zero as well (by the equality of Balayages). An upper bound for the probability of $\lambda^{(\infty)}$ taking values in compact subsets of $\overline{D(0,1)}/C$, by the probability that a Brownian motion originating from $\lambda^{(\infty)}$ exits $\overline{D(0,1)}$ in a closed arc of $C'$ (where the compact set intersects $\partial D(0,1)$), is made. This bound concludes that the probability of $\lambda^{(\infty)}$ lying in any compact subset of $\overline{D(0,1)}/C$ is zero, and thus that $\lambda^{(\infty)}$ is supported on $C$ (by inner regularity).

First, we prove that probability a Brownian motion originating at $\zeta^{(\infty)}$ exists $\overline{D(0,R)}$ in a closed arc $A := \{Re^{i\theta} : \theta \in I\}$, where $I = [a,b]$ is a compact interval subset of $(-\pi/2, \pi/2)$, tends to zero as $R$ tends to one. Denoting by $H(\zeta^{(\infty)}, A)$ the just mentioned probability, we know that

$$H(\zeta^{(\infty)},A) = \int_I Bal_R(\zeta^{(\infty)})(Re^{i\theta})\frac{d\theta}{2\pi} = \int_I \mathbb{E}P^R_{\zeta^{(\infty)}}(Re^{i\theta})\frac{d\theta}{2\pi}$$

For $w = e^{i\alpha} \in C$,

$$\int_I P_w^R(Re^{i\theta})\frac{d\theta}{2\pi} = \int_I \frac{R^2-1}{|Re^{i\theta}-e^{i\alpha}|^2}\frac{d\theta}{2\pi} \leq \frac{R^2-1}{c^2}\frac{l(I)}{2\pi}$$

where $c = min\{|e^{-\pi/2i} - e^{ai}|, |e^{\pi/2i} - e^{bi}|\}$.[14]

And so, taking $R$ to one, we obtain[15]

$$\lim_{R\longrightarrow 1^+}\int_I P_w^R(Re^{i\theta})\frac{d\theta}{2\pi} = 0$$

By setting $w = \zeta^{(\infty)}$ and taking expectations, we conclude

$$\lim_{R\longrightarrow 1^+} H(\zeta^{\infty)}, A) = 0$$

[14]Note that as $I$ is a compact interval in $(-\pi/2, \pi/2)$, the minimum as defined is strictly positive. Figures here would be helpful.

[15]we would like to stress here on the positive nature of the Poisson kernel which allowed for the viability of sandwich theorem.

Second, and under the additional assumption that $R$ satisfies $1 < R \leq m$ for some $m$, we prove that for $K$ a compact subset of $\overline{D(0,1)} \backslash C$ and $I$ being chosen as a compact interval in $(-\pi/2, \pi/2)$ such that it contains, within its interior, the set $E := \{\theta \in (-\pi/2, \pi/2) : e^{i\theta} \in K\}$[16], there exists $c = c(K, I) > 0$, a non zero number that can depend on $K$ and the choice of $I$, such that

$$\int_I P_w^R(Re^{i\theta}) \frac{d\theta}{2\pi} \geq c(K, I)$$

uniformly for all $w \in K$ and $1 < R \leq m$. In fact, it is not hard to notice that as long as $w = re^{i\alpha} \in K$ is such that $|w| \leq 1 - \epsilon$ for some constant $0 < \epsilon < 1$, we have

$$\int_I P_w^R(Re^{i\theta}) \frac{d\theta}{2\pi} = \int_I \frac{R^2 - r^2}{|Re^{i\theta} - w|^2} \frac{d\theta}{2\pi} \geq \frac{1 - (1-\epsilon)^2}{(2m)^2} \left( \frac{l(I)}{2\pi} \right) := c_1$$

For $(1 - \epsilon) \leq |w| < 1$, however, It is does not look easy how we would maintain this non zero (uniform) lower bound. In this case, we first make it clear that when $m \geq R \geq 1 + \delta$ for some $\delta > 0$, then a (uniform) lower bound could be attained for all $w \in K$

$$\int_I P_w^R(Re^{i\theta}) \frac{d\theta}{2\pi} \geq \frac{(1+\delta)^2 - 1}{(2(\delta+1))^2} \left( \frac{l(I)}{2\pi} \right) := c_2$$

The case $1 < R < 1 + \delta$ and $|w| \geq 1 - \epsilon$ is what remains untackled. We seek proving a (uniform) upper bound for the integral

$$\int_{[-\pi/2, 3\pi/2] \backslash I} P_w^R(Re^{i\theta}) \frac{d\theta}{2\pi}$$

to treat this case.

We observe that for $R > 1$, as the set $\{Re^{i\theta} : \theta \in [-\pi/2, 3\pi/2] \backslash I\}$ is compact and does not intersect $K$, the two sets are positively separated. In fact, we have that for all $R > 1$,

$$\min_{\theta \in [-\pi/2, 3\pi/2] \backslash I} d(Re^{i\theta}, K) \geq \min_{\theta \in [-\pi/2, 3\pi/2] \backslash I} d(e^{i\theta}, K) := d > 0$$

Accordingly, for all $1 < R < 1 + \delta$ and $w \in K$ such that $|w| \geq 1 - \epsilon$, we have

$$\int_{[-\pi/2, 3\pi/2] \backslash I} P_w^R(Re^{i\theta}) \frac{d\theta}{2\pi} \leq \frac{(1+\delta)^2 - (1-\epsilon)^2}{d^2} \left( \frac{l([-\pi/2, 3\pi/2] \backslash I)}{2\pi} \right)$$

[16]This is valid as the set $E$ is a compact subset of the open interval $(-\pi/2, \pi/2)$ and thus can be contained in the interior of a compact interval $I$ that still lies in $(-\pi/2, \pi/2)$.

For $\delta$ sufficiently small and $\epsilon$ sufficiently large, we could insure that the above integral is as small as we desire. Let $\delta_0$ and $\epsilon_0$ be such that

$$\int_{[-\pi/2,3\pi/2]\backslash I} P_w^R(Re^{i\theta})\frac{d\theta}{2\pi} \leq \frac{1}{2}$$

Consequently, since the Poisson kernel has mean one, and for the prescribes values of $\epsilon$ and $\delta$, we obtain

$$\int_I P_w^R(Re^{i\theta})\frac{d\theta}{2\pi} \geq \frac{1}{2} := c_3$$

Therefore, one secures the lower bound

$$\int_I P_w^R(Re^{i\theta})\frac{d\theta}{2\pi} \geq \min_{1\leq i\leq 3}\{c_i\} := c$$

uniformly for all $w \in K$ and $1 < R \leq m$ [17].

Setting $w = \lambda^{(\infty)}$ and taking expectations, we conclude

$$\frac{1}{c}\int_I Bal_R(\lambda^{(\infty)})\frac{d\theta}{2\pi} \geq \mathbb{P}(\lambda^{(\infty)} \in K)$$

Since $Bal_R(\lambda^{(\infty)}) = Bal_R(\zeta^{(\infty)})$ for all $R > 1$, we infer that $\mathbb{P}(\lambda^{(\infty)} \in K) = 0$ for all compact sets $K$ of $\overline{D(0,1)}\backslash C$

This means that $\lambda^{(\infty)}$, as $\zeta^{(\infty)}$, almost surely takes values in $C \subset \partial D$. And as a result, the moments can be viewed as Fourier coefficients of the laws of $\lambda^{(\infty)}$ and $\zeta^{(\infty)}$. **By Fourier Uniqueness, we conclude that $\zeta^{(\infty)}$ and $\lambda^{(\infty)}$ have the same distributions as claimed. Below is a relevant detour on Fourier analysis of functions and measures that will support the previous claim.**

#### 5.2.1.3 Fourier Analysis Detour

In classical Fourier analysis, complex valued Lebesgue integrable functions defined on the circle are written in terms of functions on the circle, indexed by $n, n \in \mathbb{N}$, and defined as follows

$$\gamma_n : e^{i\theta} \longrightarrow e^{in\theta}$$

These functions compose an orthogonal basis of $L^2(\mathbb{T})$, $\mathbb{T}$ denoting the circle group (it will become more clear why we referred to it as a group soon). As $n$ increases, the functions wind around the circle faster. In other words, as $n$ increases, $\gamma_n$ completes more full revolutions as $\theta$ goes from 0 to $2\pi$. $\gamma_n$ oscillates at a rate

[17] If you review the way the $c_i$s were defined , while considering $\delta = \delta_0$ and $\epsilon = \epsilon_0$, you notice that there definition depends on the choice of $I$ which as well has dependence on $K$, but they do not depend on $R$ or $w$. The value of $\delta_0$ and $\epsilon_0$ are chosen depending on $I$ as well.

$n$, and different rates of oscillation (different $n$) are orthogonal; they capture independent modes of behavior. The n-th Fourier coefficient tells you how much of the n-frequency oscillation is present in your "signal" and is defined as

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} g(e^{i\theta})d\theta$$

assuming the signal function is referred to as $g$. So in the classical setting, we usually work on $L^1(\mathbb{T})$ or even $L^2(\mathbb{T})$, but many signals of interest do not lie within these spaces. This we are saying to pave the way for talking about Fourier analysis on $M(\mathbb{T})$, the space of bounded measures on $\mathbb{T}$. Signals of interest in science do not always spread out smoothly over space, if we could say, but sometimes are concentrated at discrete points like in electrostatics when charge is modeled by Dirac delta or in the study of the periodic atomic arrangements in crystals where the atomic density is a sum of Dirac deltas as well.

Concerning the later example, and to illustrate how Fourier analysis on measures has applications in this area of science, crystallographers study how the atoms are arranged inside a crystal by shining waves at it. The produced (observed) diffraction pattern is analyzed to deduce the internal atomic arrangement of the crystal. The factor $(\gamma(k))^x$ encodes the phase shift introduced by the position $x$ in the crystal at frequency $k$. The total scattered wave is a superposition of these contributions. Intuitively speaking, the superposition weighted by the density is the Fourier transform of the density and knowing it allows for knowing the density itself.

Now we lay forth the rigorous ideas regarding Fourier coefficients of measures.

**Definition 5.2.1.** For $n$ in $\mathbb{Z}$, the Fourier Coefficient $\hat{\mu}(n)$ of a measure $\mu$ on $\mathbb{T}$ is given by

$$\hat{\mu}(n) = \int_{\mathbb{T}} z^n d\mu(z)$$

In his book "Fourier Analysis on Groups"[13], Rudin discusses the uniqueness theorem which implies our results. In fact, Rudin treats measures on general LCA, or locally compact abelian, groups. Classical Fourier analysis implicitly relied on Lebesgue measure. Haar's measure generalized the Lebesgue measure and provided the right foundation to extend Fourier analysis. The functions $\gamma_n$ on $\mathbb{T}$, were generalized to what are called *characters* on the *LCA* group.

## 5.3 Getting a contradiction in the case $a = o(\frac{1}{logn})$

The last section has just alluded to a (non-trivial) piece of information on the limiting random variable $\lambda^{(\infty)}$ [18]. It is that $\lambda^{(\infty)}$ is supported on $C$. The equality of

[18]We included the description non trivial as a contrast with the triviality of knowing $\zeta^{(\infty)}$ is supported on $C$ and to highlight the potential of extracting further (non trivial) knowledge of this piece of information.

Balayages, which was inscribed by (preceding) equalities of the logarithmic potentials and moments, is a profit, if one could say, of the facts that $f$ is a polynomial and that its zeros are in the closed unit disk - which contributed to defining the potentials, relating them, and bounding them. These two facts are the assumptions of Sendov's Conjecture. The knowledge on the distributions of the limiting random variables, however, was a profit of assuming additionally the negation of Sendov's conjecture while the limit of the sequence $a^{(n)}$ is zero, and so from here we believe the contradiction will spring [19].

Notice that while each $\lambda^{(n)}$ represents a zero of $f^{(n)}$ chosen uniformly at random, and $a^{(n)}$ represents a zero of $f^{(n)}$, we believe that the (limiting) random variable $\lambda^{(\infty)}$ have lost this kind of attachment to $a^{(\infty)}$ and is only supported on $C$; $\lambda^{(\infty)}$ does not stand to the same definition attributed to the random variables $\{\lambda^n, n \in \mathbb{N}\}$ anymore as no polynomial stand to the infinite degree anymore.

When it will be assumed that $a = o(\frac{1}{logn})$, this will affect the probability $\mathbb{P}(\lambda \in K)$, eventually dictating the convergence behavior of $s_\lambda$ to $s_{\lambda^{(\infty)}}$. Why talking $s_\lambda$ ? A reason is that the Stieltjes transform encapsulates information on both $f$ and $f'$. The argument used will be employing the hypothesis that $f^{(n)}$ does not satisfy Sendov's Conjecture at $a^{(n)}$, and the proved results, at $\infty$, which extend from this hypothesis. This will then reproduce a contradiction under the supposition that $a = o(\frac{1}{logn})$.

The point that $f$ has a zero at $a$ while $f'$ has no zeros in the Disk $\overline{D(a,1)}$, will be employed as a condition on the winding number of $s_\lambda(\gamma)$ around the origin for some contour $\gamma$. The result on the distribution of $\lambda^{(\infty)}$ will be employed as the holomorphicity of the limiting Stieltjes transform on $\mathbb{C} \backslash C$ and subsequently as a condition on the winding number, around 0, of $s_{\lambda^{(\infty)}}$ this time. The last two conditions will not come along under the convergence behavior that will be impelled by $a = o(\frac{1}{\log n})$, causing a contradiction. The Steiltjes transform represents the logarithmic derivative which has long been a crucial quantity in the studies on Sendov's conjecture.

### 5.3.1 *The hypothesis that $f$ does not satisfy Sendov's Conjecture at $a$ (an $s_\lambda$ perspective)*

For $n$ sufficiently large, $a^{(n)}$ is as near to the origin as desired. So, for $n$ sufficiently large, $0 < r < 1$, we have that $a \in D(0, r)$ and

[19] In fact, we will see that a contradiction will spring with the additional assumption that $a^{(n)}$ converges to zero more rapidly than $\frac{1}{logn}$, which is to say $a = o(\frac{1}{logn})$

$$D(0,r) \subset \overline{D(a,1)}$$[20]

This is to say, for $n$ sufficiently large, the disk $D(0,r)$ contains no zeros of $f'$ and at least one zero of $f$ which is $a$.

In consequence, and if the radius satisfied that no zeros of $f$ lie in $\partial D(0,r)$, the winding number of the contour

$$\theta \longmapsto \frac{1}{n}\frac{f'}{f}(re^{i\theta}), 0 \leq \theta \leq 2\pi$$

around the origin, is negative and at most -1, by the Argument principle. Since $s_{(\lambda)}(z)$ is equal to $\frac{1}{n}\frac{f'}{f}(z)$, for $z$ not in the support of $\lambda$, we get that the winding number of the contour

$$\theta \longmapsto s_\lambda(re^{i\theta}), 0 \leq \theta \leq 2\pi$$

(anti-clockwise) around the origin is negative and at most -1.

#### 5.3.1.1 ***The results at infinity extending from this hypothesis (an $s_{\lambda^{(\infty)}}$ perspective)***

We proved, of what we proved at $\infty$, that $\lambda^{(\infty)}$ is supported on $C$. And so, we infer that the steiltjes transform $s_{\lambda^{(\infty)}}$ of $\lambda^{(\infty)}$ is holomorphic on the (connected) domain $\mathbb{C}\backslash C$. We are interested in producing a statement on winding numbers related to this trasform that will fit in a later application of Rouche's Theorem.

In fact, since $s_{\lambda^{(\infty)}}$ is holomorphic on the connected domain $\mathbb{C}\backslash C$, and is not identically zero, the zeros of this transform on $\mathbb{C}\backslash C$ are isolated. From this, we can draw out the existence of an annulus $\overline{D(0,r_2)}\backslash D(0,r_2)$, for some $0 < r_1 < r_2 < \frac{1}{2}$ [21], on which $s_{\lambda^{(\infty)}}$ is bounded away from zero, meaning

$$|s_{\lambda^{(\infty)}}(z)| > c > 0$$

uniformly for all $r_1 \leq |z| \leq r_2$.

By the argument principle, and denoting by $m$ the number of zeros of $s_{\lambda^{(\infty)}}$ in $D(0,r_1)$, we get that for any radius $r_1 \leq r \leq r_2$ the contour

$$\theta \mapsto s_{\lambda^{(\infty)}}(re^{i\theta}), 0 \leq \theta \leq 2\pi$$

stays at a distance from the origin and winds exactly $m$ times anti-clockwise around the origin. Note that as $m$ represents a number of zeros, it is a non-negative number.

[20] We remind that when we say $a$ we mean $a = a^{(n)}$.

[21] The choice of $\frac{1}{2}$ is not special, we could have considered another number that is strictly less than one.

### 5.3.2 Assuming $a = o(\frac{1}{logn})$ and getting a contradiction (a Rouche's Theorem Approach)

There appear a potential contradictory matter if one was to prove that, for $n$ large enough, $s_{\lambda^{(n)}}(z)$ gets "near enough" to $s_{\lambda^{(\infty)}}(z)$ on a certain contour that lies within the mentioned annulus. By "near enough" on the contour, we mean as (uniformly) near at this contour as to drop below $c$, the positive lower bound of $|s_{\lambda^{(\infty)}}|$ on the annulus, and allow for the application of Rouche's theorem. This "near enough" will be attained by the help of the assumption $a = o(\frac{1}{logn})$, and a "considerable" contour will be shown to exist. By "considerable" we mean that it avoids the poles of $s_{\lambda^{(n)}}$ at the $n$ where we are.

Since $\lambda$ converges in distribution to $\lambda^{(\infty)}$, we benefit from this convergence to capture the convergence, not of $s_{\lambda^n}(z) = \mathbb{E}\frac{1}{z-\lambda}$ exactly, because $\frac{1}{z-\lambda}$ is not necessarily defined on the annulus, but of $\mathbb{E}\frac{1_{|\lambda|>1/2}}{z-\lambda}$. We write

$$s_{\lambda^{(n)}}(z) = \mathbb{E}\frac{1}{z-\lambda} = \mathbb{E}\frac{1_{|\lambda|>1/2}}{z-\lambda} + \mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{z-\lambda}$$

Then we say, as $\lambda^{(\infty)}$ is supported on $C$, and as $1_{|\lambda|>1/2}$ could be approximated by continuous functions in the usual fashion, the convergence in distribution leads that

$$\mathbb{E}\frac{1_{|\lambda|>1/2}}{z-\lambda} = \mathbb{E}\frac{1}{z-\lambda^{(\infty)}} + o(1) = s_{\lambda^{(\infty)}}(z) + o(1)$$

uniformly for $z \in D(0, r_2)$. Concerning the term $\mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{z-\lambda}$, however, we consider the following treatment:

if $|z| = r$, we have that

$$\left|\mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{z-\lambda}\right| \leq \mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{|r-|\lambda|}$$

The search for the desired contour, is the search for an $r \in [r_1, r_2]$ that satisfies the desired aspects, and it will be done through a "probabilistic method" (check the standard proof of the Hadamard factorization theorem). In fact, for $w \in \mathbb{C}$ such that $|w| \leq 1/2$, we have

$$\int_{r_1}^{r_2} \frac{dr}{max(|r-|w|, n^{-10})} \lesssim logn$$

This could be easily shown by examining the case when $[|w| - n^{-10}, |w| + n^{-10}]$ intersects $[r_1, r_2]$, and observing that an integral like $\int_{r_1}^{|w|-n^{-10}} \frac{dr}{|r-|w||}$ for example, when $r_1 < |w| - n^{-10}$, is such that

$$\int_{r_1}^{|w|-n^{-10}} \frac{dr}{|r-|w||} \leq \int_{r_1}^{|w|-n^{-10}} \frac{dr}{r-|w|} = \log(|w|-r_1) + 10\log(n) \lesssim \log(n)^{22}$$

This implies the following

$$\mathbb{E}\left(1_{|\lambda|\leq 1/2}\int_{r_1}^{r_2} \frac{dr}{max(|r-|\lambda|, n^{-10})}\right) \lesssim \log n \mathbb{E}1_{|\lambda|\leq 1/2} \lesssim \log n \mathbb{P}(|\lambda| \leq 1/2)$$

By Fubini-Tonelli,

$$\int_{r_1}^{r_2} \mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{max(|r-|\lambda|, n^{-10})} dr \lesssim \log n \mathbb{P}(|\lambda| \leq 1/2)$$

As $\mathbb{P}(|\lambda| \leq 1/2) \lesssim a + \frac{logn}{n^{1/3}}$, then if $a = o(\frac{1}{\log n})$ we obtain $\mathbb{P}(|\lambda| \leq 1/2 = o(\frac{1}{\log n})$. Consequently, by Fubini-Tonelli theorem

$$\int_{r_1}^{r_2} \mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{max(|r-|\lambda|, n^{-10})} dr = o(1)$$

Hence, and by Markov's Inequality, we get

$$\mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{max(|r-|\lambda|, n^{-10})} = o(1)$$

for all $r \in [r_1, r_2]$ outside a set of measure o(1). Since $f$ has at most $n$ zeros, one has that $|r-|\lambda| \geq n^{-10}$ for all $r \in [r_1, r_2]$ outside a set of measure $o(1)$. So, we guarantee that for $n$ large enough, there exists $r \in [r_1, r_2]$, depending on $n$, such that

$$\mathbb{E}\frac{1_{|\lambda|\leq 1/2}}{|r-|\lambda||} = o(1)$$

Therefore, for $n$ large enough, and with the above choice of $r$, we conclude that

$$\frac{1}{n}\frac{f'}{f}(re^{i\theta}) = s_{\lambda(\infty)}(re^{i\theta}) + o(1)$$

uniformly in $\theta \in [0, 2\pi]$. By Rouche's theorem, we obtain that $m$ is negative, which is absurd.

[22] Note that $r_1$ and $|w|$ are independent of n

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