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Integrating the economic production model with deteriorating raw material over multi-production cycles

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This paper considers an Economic Production Quantity (EPQ) model with deteriorating raw material and investigates the impact of deterioration on the production process. The EPQ base case with no deterioration is presented where raw material is ordered for multiple production cycles. We present the differential equations to calculate the on-hand inventory of raw material and present closed-forms expressions for the required order of raw material to result in a desired amount of effective raw material per order cycle. Closed-form expression for the total profit per unit time is obtained and we solve for the optimal production quantity of finished product per production cycle and the order quantity of raw material. We present numerical examples where we compare our model to a system which ignores the impact of deterioration and results in shorter production cycles due to an insufficient amount of effective raw material.

Keywords: production modelling; inventory management; raw material; deterioration

1. Introduction

Many real world inventory systems are subject to decay or deterioration and ignoring the effects of deterioration can result in misleading analysis of system cost and performance. Deteriorating inventory has been extensively addressed in the literature and Raafat (1991) presented a very thorough literature review on applications of deteriorating inventory. Raafat (1991) classified deteriorating items as either having a fixed shelf life time or continuously decaying over time. The work presented in this paper considers the latter where items are subject to exponential decay as first introduced by Ghare and Schrader (1963). We also refer to Goyal and Giri (2001) to a more recent survey of the literature on inventory systems with deterioration. As stated by the extensive surveys of Raafat (1991), and Goyal and Giri (2001), decaying products are applicable to a very wide range of inventory commodities from volatile liquids, food products, electronic components, etc.

In this work, we study the impact of deterioration on an Economic Production Quantity (EPQ) model of a single product where a raw material order is consumed over multiple production runs. The coordination of the production runs and procurement process of raw material is an example of integrated inventory systems and we refer the reader to Goyal and Deshmukh (1992) for a review and a classification of integrated supply chain systems. The early work of Goyal (1977) initiated this stream of literature where different types of raw material are ordered from an outside supplier where a raw material order is utilised over a single production run. In such a model and as illustrated by Goyal (1977), the procurement of raw material and the length of the production run need to be treated jointly as they are closely interrelated. Park (1983) considers deteriorating raw material where procurement occured at the start of every production run. A closed-form expression for the optimal manufacturing order quantity and optimal cycle time between production runs are obtained to minimise the total cost of procuring and the cost of holding the raw material and finished goods. Park (1983) acknowledged that in practice raw material is procured for several production runs, but for such a problem the mathematical analysis becomes complex. In this work, we consider the case where raw materials can be procured for multiple production runs. Raw material in the form of volatile liquids, dairy products, fruits, meat among others are usually unpackaged and more susceptible to deterioration than the finished product which is usually packaged with a relatively long shelf life. Placing large orders of raw material further motivates investigating the effects of raw material deterioration as well as assuming that the finished product is consumed before it deteriorates or expires. Although accounting for deterioration of raw material increases the complexity of the model, we justify this added complexity by presenting a measure of regret for ignoring deterioration. This measure of regret or penalty can be obtained by investigating the behaviour of a system that does include deterioration in the decision process and results in shorter order cycles.

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We refer to Widyadana, Cardenas-Barron, and Wee (2011) as well as Chung and Cardenas-Barron (2013) for a simplified approach to solve deteriorating inventory problems and to Cardenas-Barron (2001) for simplified approaches to derive the economic order quantity (EOQ) equations algebraically. The work in Chung and Cardenas-Barron (2012) presented an analytic solution procedure for the EPQ and EOQ models with fixed backorder costs which guaranteed optimal solutions. Raafat (1985) also considered raw material for a single production cycle and extended Park (1983) by accounting for the deterioration of the finished product. Balkhi (1999) extended the work of Raafat (1985) by adding a time dependency component to demand and deterioration. Yang and Wee (2003) extended the previous work of Raafat (1985) to one producer multiple buyers. Rau,Wu, and Wee (2003) considered an integrated inventory model consisting of a supplier, producer and a buyer under the effect of deterioration of raw material and finished goods. The authors in Rau,Wu, and Wee (2003) illustrated numerically that the total cost of the integrated system is lower than the sum of the costs if each of the three entities makes its decisions independently. Goyal and Gunasekaran (1995) further studied the effect of pricing of the product and advertising frequency of the demand of the finished product on the EOQ of raw material and optimal production quantity.

As stated earlier, a material in its raw and unpackaged state is more exposed to deterioration than the final packaged product. A well established and investigated occurrence of deteriorating raw material is postharvest fruits and vegetables within the food industry. This involves the state of the product at postharvest until the final packaged or canned product is formed. The significance of the impact of spoilage on postharvest fruit and vegetables is emphasised in Kantor et al. (1997) where the authors state that 18.9 billion pounds of fresh fruit and vegetables are lost annually and this represents 19.6% of US losses in edible foods. These losses are even more significant in developing countries where estimates can exceed 50%, Kader (2005). What constitutes decay of agricultural products includes several quality control measures such as loss in edibility, nutritional quality (caloric value among others) and the suitability of the product for consumer acceptance (Kader 2005).

When placing large orders of raw material in the form of postharvest, the effects of spoilage should be integrated as part of the production model. In this work, we also investigate the penalty of ignoring deterioration of raw material on the profit function. In the case of fruits and vegetables, obtaining a measure of the deterioration rate is a result of several factors. Such factors include the dehydration rate of the product which is dependent on the humidity and temperature of the storage environment (Tadhg and Sun 2001). The storage environment plays a significant role when determining the deterioration rate within the production model. Although there are several studies in the literature that investigate deterioration of postharvest vegetation and present storage as well as handling guidelines to control and reduce the spoilage (see study by the Food and Agriculture Organization of the United Nations by Barbosa-Canovas et al. (2003)), the impact of spoilage cannot be eliminated. The cost of improving the handling techniques and environmental conditions by providing the proper temperature and humidity would help decrease the deterioration rate and should be accounted for as part of the holding cost of raw material. In this work, the finished product is assumed to be in a packaged state where applications include vacuum packaging, drying and canned foods among others. In most cases, the finished product has a fixed shelf life represented by an expiry date.

This remainder of the paper is organised as follows. The notation used for the system parameters are presented in Section 2. The base model with no deterioration is considered in Section 3.1, where we investigate the concavity of the profit function. In Section 3.2, we present the model with deteriorating raw material and present closed-form expressions of the profit function in Section 3.3. Section 4 considers the impact of ignoring deterioration by implementing the base case decision on a system with deteriorating raw material. Numerical examples which are presented in Sections 5 and 6 conclude the paper.

2. Notation for EPQ with deteriorating raw material

Here, we present the notation used for the EPQ model which integrates the finished product and raw material inventories. A production cycle has a duration of t_0 and a new order of raw material is placed every n production cycles. The production cycle is given by $t_0 = \dfrac{y}{\beta}$, where y is the amount of finished product produced and consumed per production cycle and β is the demand rate of finished product. We refer to the amount of raw material consumed by the production process as the effective raw material. The amount of effective raw material required over n production cycles is ny. Let $f(t)$ be the amount of effective raw material over time. Define $g(t)$ to be the total available amount of raw material where $g(t)$ accounts for the effective raw material as well as the raw material which will be lost to deterioration. The amount of effective raw material at time 0 is $f(0) = ny$, and the actual amount of raw material to be ordered every n production cycles is $g(0)$. We present below a summary of the notation used,

σ	Deterioration proportion of raw material per unit time
n	Number of production cycles to be covered by one order of raw material (units)
y	Amount of finished product to be produced per production cycle (units)
$f(t)$	Effective raw material inventory where $f(0) = ny$ (units)

$g(t)$ Actual raw material inventory (units)
α Consumption rate of raw material during production, also production rate of final product (see comment below) (unit/time unit)
β Consumption rate of final product (unit/time unit)
t_1 Production time (time unit)
t_2 Idle time (time unit)
t_0 Duration of a production cycle ($t_1 + t_2$)

The following notation is used to describe the cost parameters,

C_p Production cost of one unit of final product ($/unit)
C_r Purchase cost of one unit of raw material ($/unit)
s Selling price of one unit of final product ($/unit)
K_p Set-up cost of a production cycle ($/production run)
K_r Set-up cost to order raw material ($/raw material order)
h_r Holding cost of raw material per item per unit time ($/unit/time unit)
h_p Holding cost of final product per item per unit time ($/unit/time unit)

Comment: Here, we note that a unit of raw material inventory is defined in relation with a unit of finished item inventory. A unit of raw material is the amount consumed in the production of one unit of finished product. The raw material parameters h_r, C_r and K_r as well as the inventory level, $g(t)$, are obtained accordingly. Accounting for different units of raw material can be obtained by simply multiplying raw material parameters and raw material inventory by a constant.

3. The integrated EPQ model

The model presented allows for multiple batches of raw material to be ordered for consecutive production cycles while taking into account the deteriorating nature of the raw material. Placing orders for raw material from outside suppliers justifies procuring higher inventory levels and possibly storing raw material for multiple production cycles. Since deterioration is proportional to the on hand inventory, procuring high inventory levels can result in a significant proportion of unusable raw material. The objective of the model is to determine the optimal length of the production cycle as well as the order amount of raw material required for n production cycles. In this work, the following assumptions are made (1) The finished material leaves the inventory before the shelf life expires, (2) The replenishment of raw material is instantaneous, (3) The demand rate is deterministic and constant, (4) Production rate of the finished product is constant, (5) The analysis assumes an infinite time horizon and (6) We define one unit of raw material as the amount required to produce one item of the finished product.

3.1 *Base model* ($\sigma = 0$)

We first consider the integrated EPQ model where raw material is assumed to be non-deteriorating. We refer to Figure 1 for an illustration of the model. Every production cycle produces y items of finished product and the raw material is ordered for n production cycles. The profit function per unit time is the ratio of the profit per cycle and cycle time, nt_0. The compenents of the profit function per unit time for the base case with no deterioration, $TPPU_b(y, n)$, are

- The selling price per unit time, $s\,\beta$.
- The holding cost of finished product and raw material per unit time, $\frac{h_p}{2} y \left(1 - \frac{\beta}{\alpha}\right) + \frac{h_r}{2} y \left(n - 1 + \frac{\beta}{\alpha}\right)$.
- The order cost of finished product and raw material per unit time, $\frac{\beta K_p}{y} + \frac{\beta K_r}{yn}$.
- The production cost of the finished product and purchase cost of raw material per unit time, $\left(C_p + C_r\right)\beta$.

The resulting profit function per unit time for the base case with no deterioration,

$$TPPU_b(y,n) = s\,\beta - \left[\left(C_p + C_r\right)\beta + \frac{\beta K_p}{y} + \frac{\beta K_r}{yn} + \frac{h_p}{2} y \left(1 - \frac{\beta}{\alpha}\right) + \frac{h_r}{2} y \left(n - 1 + \frac{\beta}{\alpha}\right)\right]. \tag{1}$$

Taking the second-order condition of (1) w.r.t. y, we obtain

$$\frac{\partial TPPU_b^2}{\partial^2 y} = -\left(\frac{2K_r\beta}{ny^3} + \frac{2K_p\beta}{y^3}\right) < 0,$$

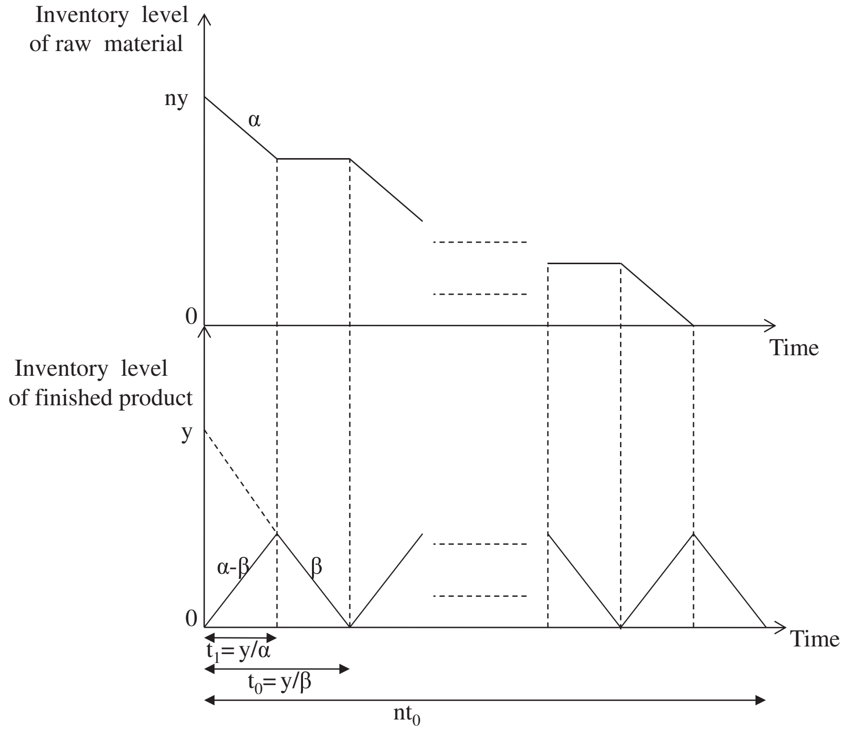

Figure 1. Behaviour of the inventory level of raw material and finished product.

so the profit expression in (1) is concave in y for a given value of n. Taking the first-order conditions with respect to y, we obtain the following expression for the optimal production quantity,

$$y_b^*(n) = \sqrt{\frac{2\,\alpha\,\beta\left(K_p + \dfrac{K_r}{n}\right)}{h_p\,(\alpha-\beta) + h_r\,(\alpha\,(n-1)+\beta)}}. \tag{2}$$

In order to find the optimal number of cycles, n_b^*, to maximise $TPPU_b(y_b^*(n), n)$, a simple and efficient sequential search for $n_b = 1, \ldots, n_m$ can be implemented where $[1, n_m]$ is the range of the sequential search. The size of the search interval can be determined as a function of the maximum allowable shipment amount of raw material. If g_m is the maximum shipment amount, then an approach is to set n_m to be the smallest positive integer n that satisfies $ny_b^*(n) > g_m$. In the case where an upper limit on the shipment amount is not provided, then an efficient approach is to consider the upper limit on the order amount g_m as a multiple, κ, of the independent EOQ problem with demand rate, holding cost and order cost approximated by β, h_r and K_r, respectively. A resulting heuristic is to set $\kappa = 1$,

Step 1: Set n_m to be the smallest n that satisfies $ny_b^*(n) > \left(\sqrt{2\,K_r\,\beta/h_r}\right)\kappa$.
Step 2: Set $\kappa = \kappa + 1$ and set n_m to be the smallest n that satisfies $ny_b^*(n) > \left(\sqrt{2\,K_r\,\beta/h_r}\right)\kappa$.
Step 3: If a new optimal profit is acheived, then go to Step 2. Otherwise, complete the search.

3.2 *Model with deteriorating raw material*

Here, we consider the case where raw material exhibits deterioration ($\sigma > 0$). The total on hand or actual raw material inventory at time t, $g(t)$, is the combination of the effective inventory at time t, $f(t)$, and the amount of inventory which decays by time $nt_0 - t_2 = (n-1)\,t_0 + t_1$. The differential equation of the actual inventory with respect to t,

$$g'(t) = f'(t) - \sigma\, g(t), \tag{3}$$

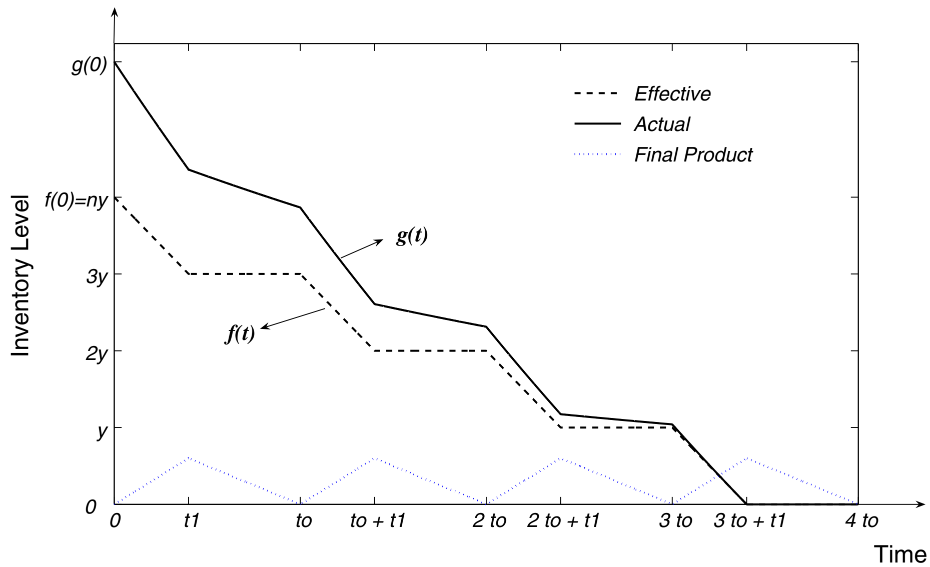

Figure 2. Model with deteriorating raw material and finished product.

where $g(0)$ is the initial order size of raw material. From Equation (3), we can express $g(t)$ by,

$$g(t) = \left(\int_0^t f'(\tau)\, e^{\sigma\tau}\, d\tau + g(0) \right) e^{-\sigma t}. \tag{4}$$

The change in the effective inventory is a function of the demand rate α, which results in the following differential equation with respect to t for $i = 1, \ldots, n$,

$$f'(t) = \begin{cases} -\alpha, & \text{if } \ (i-1)t_0 \le (i-1)t_0 + t_1 \\ 0, & \text{if } \ (i-1)\,t_0 + t_1 \le i\,t_0, \end{cases} \tag{5}$$

where $f(0) = ny$. We refer to Figure 2 for an example where raw material is ordered for four production cycles ($n = 4$).

Notice that $f(nt_0 - t_2) = g(nt_0 - t_2) = 0$, which is also illustrated in Figure 2. Also, from Equations (4) and (5) the values of $g(it_0)$ and $g(it_0 - t_2)$ for $i = 1, \ldots, n$,

$$g(it_0) = \left(\frac{-\alpha}{\sigma} \sum_{j=1}^{n} \left(e^{\sigma[(j-1)t_0 + t_1]} - e^{\sigma(j-1)t_0} \right) + g(0) \right) e^{-\sigma i t_0}, \tag{6}$$

and

$$g(it_0 - t_2) = \left(\frac{-\alpha}{\sigma} \sum_{j=1}^{n} \left(e^{\sigma[(j-1)t_0 + t_1]} - e^{\sigma(j-1)t_0} \right) + g(0) \right) e^{-\sigma(it_0 - t_2)}, \tag{7}$$

and note that $it_0 - t_2 = (i-1)t_0 + t_1$. The next step is to determine the actual order amount of raw material, $g(0)$, to allow for an effective inventory of $f(0) = ny$ over n production cycles. We present a closed-form expression of $g(0)$ as a function of the decision variables n and y,

$$g(0) = \frac{\alpha}{\sigma} (e^{\sigma n t_0} - 1)\,(e^{\sigma t_1} - 1)\,(e^{\sigma t_0} - 1)^{-1}. \tag{8}$$

Proof Consider Equation (7) for $i = n$ and using $g(nt_0 - t_2) = 0$,

$$\Rightarrow \left(\frac{-\alpha}{\sigma} \sum_{j=1}^{n} \left(\mathrm{e}^{\sigma[(j-1)t_0+t_1]} - \mathrm{e}^{\sigma(j-1)t_0} \right) + g(0) \right) \mathrm{e}^{-\sigma(nt_0-t_2)} = 0,$$

$$\Rightarrow g(0) = \frac{\alpha}{\sigma} \left(\sum_{j=1}^{n} \left(\mathrm{e}^{\sigma[(j-1)t_0+t_1]} - \mathrm{e}^{\sigma(j-1)t_0} \right) \right),$$

$$\Rightarrow g(0) = \frac{\alpha}{\sigma} \left[\left(\frac{\mathrm{e}^{\sigma(nt_0+t_1)} - \mathrm{e}^{\sigma t_1}}{\mathrm{e}^{\sigma t_0} - 1} \right) - \left(\frac{\mathrm{e}^{\sigma nt_0} - 1}{\mathrm{e}^{\sigma t_0} - 1} \right) \right],$$

$$\Rightarrow g(0) = \frac{\alpha}{\sigma} (\mathrm{e}^{\sigma nt_0} - 1)(\mathrm{e}^{\sigma t_1} - 1)(\mathrm{e}^{\sigma t_0} - 1)^{-1}$$

□

Confirming that the following equality holds, $\lim_{\sigma\to 0} g(0) = ny$, can serve as a verification of Equation (8). Applying l'Hopital's rule to Equation (8) twice,

$$\lim_{\sigma\to 0} g(0) = \lim_{\sigma\to 0} \frac{\alpha\left[\left(nt_0 e^{\sigma nt_0}\right)\left(e^{\sigma t_1} - 1\right) + t_1 e^{\sigma t_1}\left(e^{\sigma nt_0} - 1\right)\right]}{\left(e^{\sigma t_0} - 1\right) + \sigma t_0 e^{\sigma t_0}}$$

$$= \lim_{\sigma\to 0} \frac{\alpha\left[nt_0 e^{\sigma(nt_0+t_1)} - nt_0 e^{\sigma nt_0} + t_1 e^{\sigma(t_1+nt_0)} - t_1 e^{\sigma t_1}\right]}{e^{\sigma t_0} - 1 + \sigma t_0 e^{\sigma t_0}}$$

$$= \lim_{\sigma\to 0} \frac{\alpha\left[nt_0(nt_0 + t_1)\, e^{\sigma(nt_0+t_1)} - (nt_0)^2 e^{\sigma nt_0} + t_1(t_1 + nt_0)\, e^{\sigma(t_1+nt_0)} - t_1^2 e^{\sigma t_1}\right]}{2t_0 e^{\sigma t_0} + \sigma t_0^2 e^{\sigma t_0}}$$

$$= \frac{\alpha\left[nt_0(nt_0 + t_1) - (nt_0)^2 + t_1(t_1 + nt_0) - t_1^2\right]}{2t_0 + \sigma t_0^2}$$

$$= \alpha n t_1 = ny.$$

The result of Equation (8) facilitates the planning process by providing the exact amount of raw material inventory to allow for n complete cycles of length t_0. A numerical approach to calculate the average amount of raw inventory per unit time, G, is to compute the ratio of the numerical integration of $g(t)$ from 0 to $(nt_0 - t_2)$ and the order cycle duration nt_0.

To illustrate the impact of deterioration on the required effective raw material levels, consider the example where $y = 30$, $n = 4$, $t_1 = y/\alpha = 15$ ($\alpha = 2$), $t_0 = 35$ ($\beta = 6/7$) and $t_2 = t_0 - t_1 = 20$. For a deterioration rate of 1%, $\sigma = 0.01$, Figure 3 compares the policy of ordering raw material for two production cycles, $n = 2$, with ordering raw material for four production cycles $n = 4$. For the case of $n = 4$, the effective order amount is $f(0) = ny = 120$ and using Equation (8) and numerically integrating $g(t)$, the actual order amount is $g(0) = 235.97$ and the average actual inventory is $G = 82.84$. For the case of $n = 2$, the effective order amount is $f(0) = ny = 60$ and the actual order amount is $g(0) = 78.30$ and the average actual inventory is $G = 26.14$. Ordering over two cycles reduced the percentage of unused raw material, $(g(0) - f(0)) \times 100/g(0)$, from 49% to 23%. Ordering over two production cycles, $n = 2$, reduced the average raw material inventory by 68% when compared to the case where $n = 4$. The next step is to present closed-form expression for the total profit and investigate the optimal solution as a function of the cost parameters. A closed-form expression for the average amount of actual inventory, G,

$$G = \frac{g(0) - ny}{\sigma\, nt_0}. \tag{9}$$

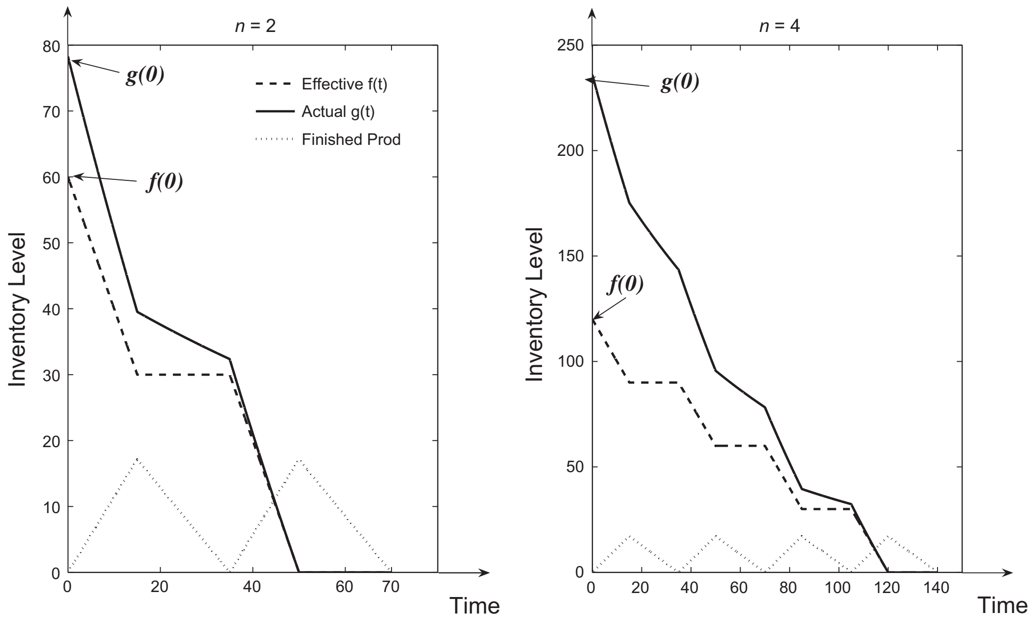

Figure 3. Model comparison – raw material ordered for two and four production cycles.

Proof Starting from Equation (3),

$$\begin{aligned} g'(t) &= f'(t) - \sigma\, g(t), \\ &\Rightarrow \int_0^{nt_0} g'(\tau)\, d\tau = \int_0^{nt_0} f'(\tau)\, d\tau - \sigma \int_0^{nt_0} g(\tau)\, d\tau, \\ &\Rightarrow g(t)\,\big|_{t=0}^{t=nt_0} = f(t)\,\big|_{t=0}^{t=nt_0} - \sigma\, G\, t_0\, n, \\ &\Rightarrow -g(0) = -f(0) - \sigma\, G\, t_0\, n = -ny - \sigma\, G\, t_0\, n, \\ &\Rightarrow G = \frac{g(0) - ny}{\sigma\, n\, t_0}. \end{aligned}$$

□

Note that applying Equation (9) to the previous example results in $G = 26.14$ and $G = 82.84$ for $n = 2$ and $n = 4$, respectively, which is consistent with the results obtained via the numerical integration of $g(t)$ from 0 to $(nt_0 - t_2)$ and dividing by the cycle length nt_0.

3.3 *Profit function*

The total profit per cycle accounts for the total sales per cycle as well as the holding and set-up cost of the raw material and finished product. Using the notation presented in Section 2, the Total Profit per Production Cycle (TPPC),

$$\begin{aligned} TPPC(n, y) = s\, ny \\ - \left[C_r\, g(0) + C_p\, ny + h_r\, G\, n\, t_0 + h_p\, n\, t_0\, y(\alpha - \beta)/(2\,\alpha) + K_p\, n + K_r\right]. \end{aligned} \tag{10}$$

Total Profit per Unit Time (TPPU),

$$\begin{aligned} TPPU(n, y) &= \frac{TPPC(n, y)}{n\, t_0} \\ &= s\,\beta - \Bigg[C_p\,\beta + h_p\, y(\alpha - \beta)/(2\,\alpha) + \beta\, K_p/y + \beta\, K_r/(yn) \\ &\qquad + g(0)\left(\frac{C_r}{n\, t_0} + \frac{h_r}{\sigma\ n\, t_0}\right) - \frac{h_r\,\beta}{\sigma}\Bigg] \end{aligned}$$

$$= \left(s\beta - C_p\beta + h_r\frac{\beta}{\sigma}\right) - \left[h_p\frac{y(1-\beta/\alpha)}{2} + \frac{K_p\beta}{y} + \frac{K_r\beta}{ny}\right] - \frac{g(0)}{ny}\beta\left(C_r + h_r/\sigma\right). \quad (11)$$

Lemma 1 *For a given n, the total profit function per unit time, given in (11) is concave in y.*

Proof Taking the second derivative of the terms within the second brackets in (11), we obtain $2K_p\beta/y^3 + 2K_r\beta/(ny^3) > 0$, for all y, so this term is convex. Thus, the expression in (11) is concave in y for a given n. It remains to show that the expression $\frac{g(0)}{ny}\beta\ (C_r + h_r/\sigma)$ in (11) is convex in y. Consider $g(0)$ in (9) which is the summation from $j = 1 \cdots n$ of the expression $g(j) := \left(e^{\sigma[(j-1)t_0+t_1]} - e^{\sigma(j-1)t_0}\right)$ (see Equation 9). Next we show that $g(j)$ is convex, thus $g(0)$ is convex, being the summation of convex expressions. First, we rewrite $g(j)$ as follows,

$$\begin{aligned} g(j) &= e^{\sigma[(j-1)t_0+t_1]} - e^{\sigma(j-1)t_0} \\ &= e^{\sigma[(j-1)y/\beta+y/\alpha]} - e^{\sigma(j-1)y/\beta} \\ &= e^{\sigma(j-1)y/\beta}e^{y/\alpha} - e^{\sigma(j-1)y/\beta} \\ &= e^{\sigma(j-1)y/\beta}(e^{y/\alpha} - 1). \end{aligned} \quad (12)$$

Using (12), the first and second derivatives of $g(j)$ are as follows,

$$\begin{aligned} \frac{\partial}{\partial y}g(j) &= \frac{\sigma\left(j-1\right)}{\beta}e^{\sigma\left(j-1\right)y/\beta}\left(e^{y/\alpha} - 1\right) + \frac{1}{\alpha}e^{y/\alpha}e^{\sigma\left(j-1\right)y/\beta}, \\ \frac{\partial^2}{\partial y^2}g(j) &= \frac{\sigma\left(j-1\right)}{\beta}\left\{\frac{\sigma(j-1)}{\beta}e^{\sigma\left(j-1\right)y/\beta}(e^{y/\alpha} - 1) + \frac{1}{\alpha}e^{\sigma\left(j-1\right)y/\beta}e^{y/\alpha}\right\} \\ &\quad + \left(\frac{1}{\alpha}\right)\left\{\left(\frac{1}{\alpha} + \frac{\sigma\left(j-1\right)}{\beta}\right)e^{\sigma\left(j-1\right)y/\beta}e^{y/\alpha}\right\}. \end{aligned} \quad (13)$$

Note that the second derivative of $g(j)$ in (13) is positive for $j \geq 1$, so $g(j)$ is convex for $j \geq 1$. This implies that $g(0) = \sum_{j=1}^{n} g(j)$ is convex. Using Avriel (2003), $g(0)/ny$ is convex, being the ratio of a convex and a linear function. As a result, the total profit function in (11) is concave in y for a given n. □

4. Model ignoring deterioration

The previous section presents the mathematical model for the integrated system exhibiting deterioration. The impact of deterioration on the profit function is investigated in the numerical examples of Section 5. Accounting for deterioration increases the mathematical and computational complexity of the model as well as the complexity involved in the system analysis. This can be seen by comparing the base case model ($\sigma = 0$) of Section 3.1 with the model of Section 3.2 ($\sigma > 0$). As a justification for using the more complicated model which accounts for deterioration, we consider the case which ignores deterioration and utilises the optimal decisions of the base case model, y_b^* and n_b^*. The model presented in this section describes the behaviour of the system when deterioration is exhibited but ignored. This allows the calculation of the savings that result from including deterioration in the analysis which also serves as a measure of the regret or penalty of not accounting for deterioration.

Ignoring the effects of deterioration on the raw material results in a shortage of raw material before completing the n_b^* production cycles of duration $n_b^*\, y_b^*/\beta$. Denote n_e to be the number of complete production cycles before running out raw material. Let $\omega(t)$ be the available quantity of raw material when deterioration is ignored and the base model optimal decisions, y_b^* and n_b^*, are utilised. If $t_{0,id}^* = \frac{y_b^*}{\beta}$ then for $i = 1, \ldots, n_e$,

$$w(i\, t_{0,id}^*) = \left(n_b^*\, y_b^* - \frac{\alpha}{\sigma}\left(e^{\sigma\, i\, y_b^*/\beta} - 1\right)\left(e^{\sigma\, y_b^*/\alpha} - 1\right)\left(e^{\sigma\, y_b^*/\beta} - 1\right)^{-1}\right)e^{-\sigma\, i\, y_b^*/\beta}. \quad (14)$$

The number of complete production cycles before running out of raw material, n_e, can be calculated by finding the largest i in Equation (14) that satisfies $w(it_{0,id}^*) \geq 0$. The residual amount of raw material at the end of the n_e production cycle, r, is given below and is utilised during the final residual cycle,

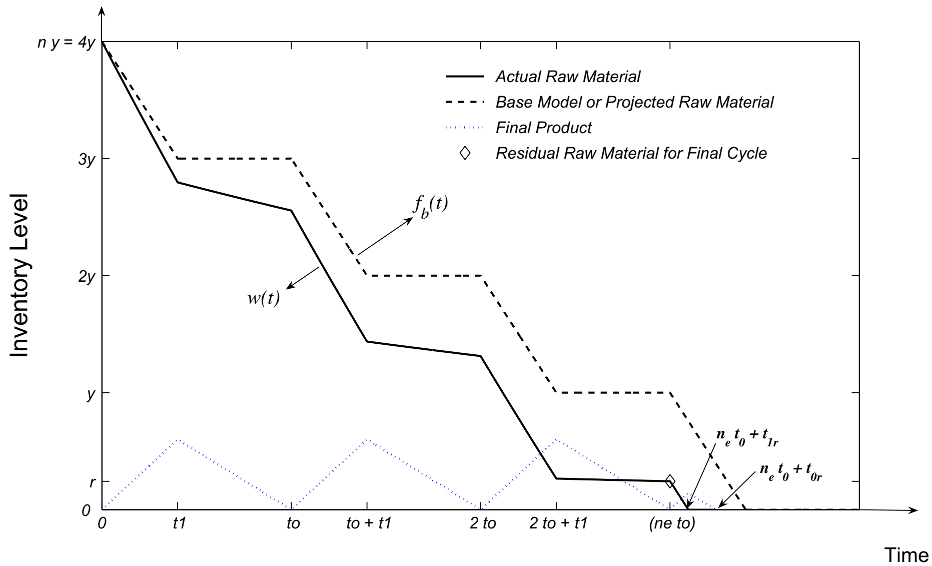

Figure 4. Implementing base model parameters on a system with deteriorating raw material.

$$r = w(n_e\, t^*_{0,id}). \tag{15}$$

Let $t_{1,r}$ be the time the system is producing in the final residual cycle where the residual raw material r is utilised. The amount of raw material during the final cycle, w_r using the residual raw material r can be calculated by the following differential equation,

$$w_r(t)' = -(\alpha + \sigma\, w_r(t)) \quad \text{where } w_r(0) = r.$$

which results in

$$w_r(t) = \frac{-\alpha}{\sigma} + \left(r + \frac{\alpha}{\sigma}\right) \mathrm{e}^{-\sigma\, t}. \tag{16}$$

The length of the production time t_{1r} is calculated by solving $w_r(t_{1r}) = 0$ in Equation (16),

$$t_{1r} = \frac{1}{\sigma} \ln\left(\frac{\alpha + \sigma\, r}{\alpha}\right).$$

The amount of finished product produced during the residual cycle is $y_r = \alpha\, t_{1r}$ and the length of the residual cycle is $t_{0,r} = y_r/\beta$. Figure 4 is an illustration of the effects of deterioration on a system where deterioration is ignored and the base model decisions are implemented.

Total Profit during a production cycle of length $(n_e t^*_{0,id} + t_{0,r})$ when deterioration is ignored,

$$\begin{aligned} TPPC_{id} = s\ &\left(n_e\, y^*_b + y_r\right) \\ &- [C_r\ \left(n^*_b\, y^*_b\right) + C_p\, n_e\, y^*_b + C_p\, y_r + h_r\, G_{id} \left(n_e\, t^*_{0,id} + t_{0,r}\right) \\ &+ h_p\, n_e\, t^*_{0,id}\, y^*_b\, (\alpha - \beta)/(2\,\alpha) + h_p\, t_{0,r}\, y_r\, (\alpha - \beta)/(2\,\alpha) \\ &+ K_p\, (n_e + 1) + K_r] \end{aligned} \tag{17}$$

where $G_{id} = \int_0^{n_e\, \tau^*_{0,id} + t_{0,r}} \omega(\tau)\, d\tau$.

A closed-form expression for the average actual inventory of raw material when deterioration is ignored, G_{id}, is presented below,

$$G_{id} = \frac{n^*_b\, y^*_b - n_e\, y^*_b - \ y_r}{\sigma\left(n_e\, t^*_{0,id} + t_{0,r}\right)}. \tag{18}$$

Proof Let f_e be the effective raw inventory,

$$\begin{aligned} w'(t) &= f_e'(t) - \sigma\, w(t), \\ &\Rightarrow \int_0^{n_e\, t^*_{0,id} + t_{0,r}} w'(\tau)\, d\tau = \int_0^{n_e\, t^*_{0,id} + t_{0,r}} f_e'(\tau)\, d\tau - \sigma \int_0^{n_e\, t^*_{0,id} + t_{0,r}} w(\tau)\, d\tau, \\ &\Rightarrow -w(0) = -f_e(0) - \sigma\, G_{id}\, n_e\, t^*_{0,id} + t_{0,r} \\ &\Rightarrow -n_b^*\, y_b^* = -n_e\, y_b^* - \; y_r - \sigma\, G_{id}\, n_e\, t^*_{0,id} + t_{0,r} \\ &\Rightarrow G_{id} = \frac{n_b^*\, y_b^* - n_e\, y_b^* - \; y_r}{\sigma \left(n_e\, t^*_{0,id} + t_{0,r} \right)}. \end{aligned}$$

□

Total Profit per Unit Time when deterioration is ignored ($TPPU_{id}$),

$$TPPU_{id} = \frac{TPPC_{id}}{n_e\, t^*_{0,id} + t_{0,r}}. \tag{19}$$

5. Numerical examples

In this set of numerical examples, we consider the following system parameter values, $\beta = 4$, $\alpha = 10$, $C_r = 10$, $C_p = 15$, $s = 100$, $K_p = 50$, $K_r = 1500$, $h_r = 1$ and $h_p = 2$. The deterioration rate, σ, is varied in Table 1. The base case $TPPU_b$ is the optimal profit for the base case where $\sigma = 0$. For the base case, the resulting decision and profit are $n_b^* = 4$, $y_b^* = 27.19$ and $TPPU_b^* = 174.94$. Table 1 presents the impact of deterioration on the profit, Δ_b, which is calculated as,

$$\Delta_b = \frac{TPPU^* - TPPU_b^*}{TPPU_b^*} \times 100. \tag{20}$$

Accounting for deterioration results in the following savings when compared to using the base case decisions n_b^* and y_b^*,

$$\Delta_s = \frac{TPPU^* - TPPU_{id}}{TPPU_{id}} \times 100. \tag{21}$$

The Δ_s in Equation (21) represents the percentage savings of implementing the more complicated model, Section 3.2, over utilising the decisions obtained from the relatively straightforward base case system of Section 3.1. The value of Δ_s can also serve as a measure of regret or the penalty associated with not accounting for deterioration in the decision process. Higher values of Δ_s would justify the added complexity of accounting for deterioration of the model. For example in Table 1, the percentage savings Δ_s increase as the deterioration rate σ increases and reach 30.31% for $\sigma = 0.1$. Although the impact of deterioration reaches $\Delta_b = -23.64\%$ (reduction from 174.94 to 133.58) for $\sigma = 0.1$, the saving is $\Delta_s = 30.31\%$. If deterioration is ignored, the impact of deterioration would have been −41.41% (reduction from 174.94 to 102.50).

Table 1. Optimal solution for model and base case with penalty for ignoring deterioration – varying σ.

σ	n	Effective per production cycle y	Actual order size $g(0)$	$TPPU^*$	Δ_b (%)	$TPPU_{id}$	Δ_s(%)
0.01	3	32.19	106.60	166.64	−4.74	166.19	0.27
0.02	3	29.78	107.63	159.14	−9.03	157.39	1.11
0.03	2	41.29	103.87	153.04	−12.52	149.18	2.59
0.04	1	80.86	95.46	149.52	−14.53	143.91	3.90
0.05	1	79.05	96.95	146.96	−15.99	136.15	7.94
0.06	1	77.27	98.30	144.36	−17.48	128.59	12.27
0.07	1	75.51	99.50	141.72	−18.99	121.34	16.79
0.08	1	73.79	100.57	139.04	−20.52	114.53	21.40
0.09	1	72.11	101.51	136.32	−22.07	108.23	25.96
0.1	1	70.47	102.32	133.58	−23.64	102.50	30.31

Table 2. Optimal solution for model and base case with penalty for ignoring deterioration – varying raw material order cost K_r.

K_r	Effective order size (ny)	Actual order size $g(0)$	$TPPU^*$	Base order size $(n_b^* y_b^*)$	$TPPU_b$	Δ_b (%)	$TPPU_{id}$	Δ_s(%)
500	48.05	54.32	209.74	60.76	221.01	−5.10	201.26	4.21
750	57.61	66.76	190.82	77.46	207.05	−7.84	182.48	4.57
1000	65.65	77.71	174.60	87.56	194.93	−10.43	165.79	5.31
1250	72.71	87.68	160.14	100.43	184.50	−13.20	150.44	6.45
1500	79.05	96.95	146.96	108.75	174.94	−15.99	136.15	7.94
1750	84.85	105.69	134.76	119.52	166.13	−18.88	119.78	12.51
2000	90.22	114.00	123.34	126.77	158.01	−21.94	107.28	14.97
2250	95.22	121.96	112.55	133.63	150.33	−25.13	95.39	17.99
2500	99.93	129.63	102.31	140.15	143.03	−28.47	84.02	21.76

Table 3. Optimal solution for model and base case with penalty for ignoring deterioration – varying h_r.

h_r	Effective order size (ny)	Actual order Size $g(0)$	$TPPU^*$	Base case order $n_b^* y_b^*$	$TPPU_b$	Δ_b (%)	$TPPU_{id}$	Δ_s(%)
0.1	98.11	175.19	178.26	342.86	244	−26.94	162.20	9.90
0.5	83.31	124.26	158.86	153.45	203.55	−21.95	153.02	3.82
1	79.05	96.95	146.96	108.75	174.94	−15.99	136.15	7.94
1.5	74.57	90.38	138.21	88.08	154.67	−10.65	123.99	11.46
2	70.84	85.01	129.97	78.74	142.52	−8.80	129.58	0.30

As illustrated in Table 1, increasing the deterioration rate results in smaller order sizes $g(0)$. Smaller values of σ (0.01–0.03) result in raw material being ordered for multiple cycles with smaller cycle length t_0 compared to higher values of σ where raw material is ordered for a single production cycle.

In Table 2, the deterioration rate is set to $\sigma = 0.05$ and the raw material set-up cost, K_r, is varied. The other system parameters are $\beta = 4$, $\alpha = 10$, $C_r = 10$, $C_p = 15$, $s = 100$, $K_p = 50$, $h_r = 1$ and $h_p = 2$. Increasing the raw material set-up cost increases the order size as expected. This increase is more significant for the base case due to the impact of deterioration on the larger amount of on hand raw material as can be seen for $K_r = 2500$, where $ny = 99.93$ compared to $n_b^* y_b^* = 140.15$. The impact of deterioration reaches $\Delta_b = -28.47\%$ (reduction from 143.03 to 102.31) for $\sigma = 0.1$, and the percentage saving is $\Delta_s = 21.76\%$. If deterioration is ignored, the impact of deterioration would have been 41.26% (reduction from 143.03 to 84.02).

Increasing the holding cost as presented in Table 3 decreases $TPPU^*$ and $TPPU_b^*$ as expected. The profit when ignoring deterioration is not continuously decreasing as can be seen for when $h_r = 1.5$ and $h_r = 2$ where $TPPU_{id}$ increases from 123.99 to 129.58. This is explained by the value of $n_b^* = 2$ when $h_r = 1.5$ which results in an $n_e = 1$ and a much smaller residual production cycle. When $h_r = 2$, then $n_b^* = 1$ which is more efficient and results in a higher $TPPU_{id}$. The impact of deterioration is significant for low h_r ($\Delta_b = -26.94\%$, $h_r = 0.1$) where it is more profitable to hold high amounts of raw material but this is countered by the property that the proportion lost to deterioration increases for higher inventory levels. As can be seen for in Table 3 for the case of $h_r = 2$, higher values of raw material holding cost result in negligible percentage savings, Δ_s. Although the value of Δ_s is negligible for higher holding costs, not including deterioration in the analysis results in a misleading overestimation on the profit as where $\Delta_b = -8.80\%$ for $h_r = 2$.

According to several studies on postharvest handling and storage, for example see Barbosa-Canovas et al. (2003), the impact of spoilage can be reduced by improving the storage environmental conditions which include temperature and humidity. A realistic motivation to varying the holding cost is to reduce the spoilage rate. A managerial incentive to invest in improving the existing storage environment can be justified by investigating the impact and sensitivity of increasing the raw material holding cost on the total profit. To clarify such an approach, consider the following example presented in Table 4 where the cost associated with improving σ is assumed as follows. The deterioration rate is $\sigma = 0.18$ when there is a minimal investment in the environmental conditions, $h_r = 0.6$. In Table 4, the improvement in σ is 0.01 for every increase in 0.1 in h_r for $\sigma > 0.1$. Also, improving the deterioration rate further below $\sigma = 0.1$ requires an additional 0.25 in holding cost h_r for every 0.01 improvement in σ. The example of Table 4 is presented for illustrative purposes in the case of justifying the managerial decisions of investing in the environmental condition of raw material storage. For example investing beyond

Table 4. Optimal solution for model and base case with penalty for ignoring deterioration – varying (σ, h_r).

(σ, h_r)	Effective order size (ny)	Actual order size $g(0)$	$TPPU^*$	Base case order $n_b^* y_b^*$	$TPPU_b$	$\Delta_b(\%)$	$TPPU_{id}$	$\Delta_s(\%)$
(0.02,3.4)	66.85	71.52	116.79	69.6	121.83	−4.13	116.73	0.05
S (0.04,2.9)	66.85	76.64	118.81	72.49	128.93	−7.85	118.59	0.19
(0.06,2.4)	66.85	82.24	121.09	75.77	136.34	−11.19	120.63	0.38
(0.08,1.9)	67.07	88.76	123.65	79.54	144.1	−14.19	122.90	0.61
(0.1,1.4)	66.85	95.12	126.53	93.19	158.35	−20.09	96.56	31.05
(0.12,1.2)	65.91	100.46	124.46	101.5	166.01	−25.03	85.14	46.19
(0.14,1)	64.40	104.55	122.35	108.75	174.94	−30.06	84.90	44.12
(0.16,0.8)	62.94	108.60	120.21	121.78	185.04	−35.04	70.77	69.85
(0.18,0.6)	61.53	112.62	118.05	143.37	196.77	−40.01	52.92	123.06

$h_r = 1.4$ does not improve TPPU, $(\sigma, h_r) = (0.1,1.4)$. Other managerial insights to the numerical examples is to provide an approach to identify the holding costs which would make the impact of adjusting the decision variables to account for deterioration negligible. For example, if we invest beyond $h_r = 1.9$, then the impact of deterioration on the decision process (order size of raw material) can be ignored (negligible Δ_s) although the impact of deterioration on profit is still significant (Δ_b). So in such a case, managerial insights for Δ_b represent the impact of deterioration which can not be eliminated. Whereas, Δ_s presents the percentage which a manager can control by adjusting the decision variables to improve on a system with deteriorating raw material.

6. Conclusion

This paper considers the integrated system of an EPQ model with raw material being ordered over multiple cycles. As illustrated in this work and in the numerical examples, ignoring deterioration by using the base case optimal decision parameters is not a practical approach and finding the optimal decision while accounting for deterioration leads to significant savings. Eventhough accounting for deterioration considerably complicates the mathematical model and analysis, the numerical examples motivate the use of the presented model by illustrating that the penalty for using the base case decisions is significant. A closed-form expression of the profit is presented and shown to be convex in y by conditioning on the number of production cycles. To facilitate the analysis and the planning process, closed-form expressions of the raw material order amount and profit function are obtained. This also provides a tool to easily calculate the actual amount of raw material to be ordered which accounts for the deteriorating raw material as well as the desired effective raw material over consecutive production cycles.

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