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Decay estimates for the arithmetic means of coefficients connected with composition operators

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Abstract
Let $f \in L^\infty(T)$ with $\|f\|_\infty \leq 1$. If $f(0) \neq 0$, $n, k \in \mathbb{Z}$, and $b_{n,n-k} = \int_E f(x)^n e^{-2\pi i(n-k)x} dx$, $E = \{x \in T : |f(x)| = 1\}$, we prove that the arithmetic means $\frac{1}{N}\sum_{n=M}^{M+N} |b_{n,n-k}|^2$ decay like $\{\log N \log_2 N \cdots \log_q N\}^{-1}$ as $N \to \infty$, uniformly in $k \in \mathbb{Z}$.

Keywords Toeplitz · Fourier coefficient · Decay estimates

Mathematics Subject Classification Primary 42B05; Secondary 42B33

1 Introduction

Let ϕ be a function holomorphic in the open unit disk of the complex plane and satisfying the two conditions

$$\phi(0) \neq 0, \quad |\phi(z)| < 1, \quad (z \in U).$$

The composition operator $C_\phi : H(U) \to H(U)$, defined via ϕ by $C_\phi f = f \circ \phi$, restricts, by Littlewood's subordination principle, to a bounded linear operator on the Hardy space $H^2(U)$, and we can associate to it an infinite matrix $(c_{\alpha,\beta})$ defined by

$$c_{\alpha,\beta} = \int_{\partial U} \phi(z)^\alpha z^{-\beta} \frac{dz}{2\pi i z}.$$

The question of convergence of the sequence $\{c_{n,n-k}\}$, and in particular, existence of the following limit

$$\lim_{n\to\infty} c_{n,n-k} = 0, \quad (k \in Z),$$

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has been the subject of various articles. This is because, if true, the above limit result would imply that every composition operator C_ϕ is weakly asymptotically Toeplitz, or WAT. The exact definition of a WAT operator need not concern us here.

Now if $E = \{z \in \partial U : |\phi(z)| = 1\}$, and we put

$$b_{n,n-k} = \int_E \phi(z)^n z^{k-n} \frac{dz}{2\pi i z},$$

then it is easy to see that $c_{n,n-k} \to 0$, if and only if $b_{n,n-k} \to 0$, and so the conjecture of Nazarov and Shapiro [2], that every composition operator C_ϕ is WAT, is equivalent to the statement that

$$\lim_{n\to\infty} b_{n,n-k} = 0, \quad (k \in \mathbb{Z}). \tag{1}$$

A more general conjecture was proposed by Shayya [5], who formulated a higher dimensional version of the WAT conjecture in the setting of functions defined on the $d-$dimensional torus and having non-vanishing Fourier transform at 0.

At present, it is not known if (1) is true, even when $k = 0$. So, as might be expected, a number of results addressed the question of convergence of the arithmetic means of the sequence $\{b_{n,n-k}\}$. In [4], it was shown that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |b_{n,n-k}|^2 = 0, \quad (k \in \mathbb{Z}),$$

and in [1], it was shown that

$$\sum_{n=1}^{\infty} \frac{|c_{n,n-k}|^2}{n} < \infty, \quad (k \geq 0),$$

from which follows two results. The first, through a simple summation by parts argument, gives that the arithmetic means of $|c_{n,n-k}|^2$ tend to 0. The second gives that $\liminf_{N\to\infty} \frac{\log N}{N} \sum_{n=1}^{N} |c_{n,n-k}|^2 = 0$.

In [5], Shayya considered only the main diagonal elements $\{b_{n,n}\}$ and proved the existence of a constant C, essentially dependent on $\phi(0)$, such that

$$\frac{1}{N} \sum_{n=M}^{M+N} |b_{n,n}|^2 \leq \frac{C}{\log \frac{N}{\pi}}, \quad (M \geq 1, N \geq 2).$$

This result, which is not implied by (1), is of interest in its own right as it provides a quantitative version of previous statements about the arithmetic means. Its proof in [5] employs a positive Borel measure μ such that $|b_{n,n}|^2 = \int_{-1/2}^{1/2} e^{-2\pi n i\theta} d\mu(\theta)$, along with the convergence of the series $\sum_{n=1}^{\infty} \frac{|b_{n,n}|^2}{n}$. However, such a measure is only available for the main diagonal elements $\{b_{n,n}\}$, i.e. when $k = 0$. This left unanswered the question of whether, for any $k \in \mathbb{Z}$, an analogous result holds for the arithmetic

means of the sequence $\{b_{n,n-k}\}$. It also called for an investigation into the uniformity in $k \in \mathbb{Z}$, of the resulting estimates of those means.

The main contribution of the present paper is to obtain decay estimates for the arithmetic means of all sequences $\{b_{n,n-k}\}$ uniformly in $k \in \mathbb{Z}$. Our method may be of interest because it is short, novel, and completely elementary. We are able to sharpen Shayya's result, but more importantly, we show that the decay of those arithmetic means is much faster than has been obtained so far. In addition, the special case where ϕ is an inner function, may be of independent interest.

Our method extends immediately to the case of functions defined on the torus T^d, as considered in [5]. But, for the sake of simplicity, we shall restrict ourselves to functions defined on the unit circle T.

2 Preliminaries and statement of results

Let ϕ be a function holomorphic in the open unit disk of the complex plane and satisfying the two conditions

$$\phi(0) \neq 0, |\phi(z)| < 1, \quad (z \in U).$$

We can extend ϕ to be defined almost everywhere on ∂U by defining $\phi(e^{2\pi i\theta}) = \lim_{r\to 1-} \phi(re^{2\pi i\theta})$, and then we will have that $|\phi(e^{2\pi i\theta})| \leq 1$. Now if $r \in (0, 1)$, then the function $\log|r\phi(0)\overline{\phi(w)}|$ is subharmonic on the open unit disk, while the function $\log|z\bar{w} - r\phi(z)\overline{\phi(w)}|$ is subharmonic in each variable separately on the open bi-disk. Take ρ_1 so that $r < \rho_1 < 1$. It then follows by two applications of the mean-value inequality for subharmonic functions that, for $\rho_1 < \rho < 1$,

$$\begin{aligned}\log r|\phi(0)|^2 &\leq \int_{-1/2}^{1/2} \log|r\phi(0)\bar{\phi}(\rho e^{2\pi i\omega})|d\omega \\ &\leq \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \log|\rho^2 e^{2\pi i(\theta-\omega)} - r\phi(\rho e^{2\pi i\theta})\bar{\phi}(\rho e^{2\pi i\omega})|d\theta d\omega.\end{aligned}$$

Since $\log(\rho_1 - r) \leq \log|\rho^2 e^{2\pi i(\theta-\omega)} - r\phi(\rho e^{2\pi i\omega})\bar{\phi}(\rho e^{2\pi i\theta})| \leq \log(1 + r)$, we may let $\rho \to 1-$, and apply Dominated Convergence, to obtain the key inequality of this paper, namely

$$\log r|\phi(0)|^2 \leq \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \log|e^{2\pi i(\theta-\omega)} - r\phi(e^{2\pi i\theta})\bar{\phi}(e^{2\pi i\omega})|d\theta d\omega.$$

In particular, for a fixed $r \in (0, 1)$, if E is any measurable subset of $(-1/2, 1/2]$, and we put $F(\theta, \omega) = e^{2\pi i(\theta-\omega)} - r\phi(e^{2\pi i\theta})\bar{\phi}(e^{2\pi i\omega})$, then

$$\begin{aligned}\int_E\int_E |\log|F(\theta,\omega)||d\theta d\omega &\leq \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |\log|F(\theta,\omega)||d\theta d\omega \\ &= 2\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \log^+|F(\theta,\omega)|d\theta d\omega\end{aligned}$$

$$
\begin{aligned}
&- \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \log |F(\theta, \omega)| d\theta d\omega \\
&\leq 2 \log(1 + r) - \log r |\phi(0)|^2 \\
&\leq \log \frac{4}{r|\phi(0)|^2}, \quad (0 < r < 1). \qquad (2)
\end{aligned}
$$

This inequality will be used in the course of proof of our main result.

Theorem 1 *Let ϕ be a function holomorphic in the open unit disk of the complex plane and satisfying the two conditions*

$$\phi(0) \neq 0, |\phi(z)| < 1, \quad (z \in U).$$

For $n, k \in \mathbb{Z}$, define

$$b_{n,n-k} = \int_E \phi^n(e^{2\pi i\theta}) e^{-2\pi i(n-k)\theta} d\theta,$$

where $E = \{\theta \in (-1/2, 1/2] : |\phi(e^{2\pi i\theta})| = 1\}$. Put $C = \log \frac{16}{|\phi(0)|^2}$. Then, for each $p \in \mathbb{Z}$,

$$\sum_{\substack{m=-\infty \\ m \neq p}}^{\infty} \frac{|b_{m,m-k}|^2}{|m - p|} \leq \log \frac{16}{|\phi(0)|^2}, \quad (k \in \mathbb{Z}). \qquad (3)$$

Proof It will be convenient to use the notation $f(x) = \phi(e^{2\pi i x})$. The definition of $b_{n,n-k}$ gives us that

$$|b_{n.n-k}|^2 = \int_E \int_E \left(f(x)\overline{f(y)} e^{-2\pi i(x-y)} \right)^n e^{2\pi i k(x-y)} dx dy,$$

and, for $p, k \in \mathbb{Z}$, we have, since $|f(x)| = 1$ for $x \in E$,

$$
\begin{aligned}
&|b_{n+p,n+p-k}|^2 + |b_{-n+p,-n+p-k}|^2 \\
&= 2 \int_E \int_E \left(f(x)\overline{f(y)} e^{-2\pi i(x-y)} \right)^p e^{2\pi i k(x-y)} \mathrm{Re} \left(f(x)\overline{f(y)} e^{-2\pi i(x-y)} \right)^n dx dy.
\end{aligned}
$$

Multiplying by r^n, $0 < r < 1$, dividing by n and summing over $n \geq 1$, we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{|b_{n+p,n+p-k}|^2 + |b_{-n+p,-n+p-k}|^2}{n} r^n \\
&= 2 \int_E \int_E \left(f(x)\overline{f(y)} e^{-2\pi i(x-y)} \right)^p e^{2\pi i k(x-y)} \log \frac{1}{|F(x, y)|} dx dy,
\end{aligned}
$$

where $F(x, y) = e^{2\pi i(x-y)} - rf(x)\overline{f(y)}$. In view of the estimate (2) on the integral of $|\log|F(x, y)||$ we get that

$$\sum_{n=1}^{\infty} \frac{|b_{n+p,n+p-k}|^2 + |b_{-n+p,-n+p-k}|^2}{n} r^n \le \log \frac{16}{r^2|\phi(0)|^4},$$

and letting $r \to 1-$, we obtain independently of k and p,

$$\sum_{n=1}^{\infty} \frac{|b_{n+p,n+p-k}|^2 + |b_{-n+p,-n+p-k}|^2}{n} \le C.$$

Reindexing we arrive at

$$\sum_{p\neq m=-\infty}^{\infty} \frac{|b_{m,m-k}|^2}{|m-p|} \le C, \quad (k \in \mathbb{Z}),$$

and this completes the proof of (3). □

Corollary 2 *If $b_{n,n-k}$ is the sequence defined in Theorem 1, $M \ge 1$, and $N \ge 1$, then*

$$\frac{1}{N+1} \sum_{m=M}^{M+N} |b_{m,m-k}|^2 \le \frac{C}{\log(N+1)}, \quad (k \in \mathbb{Z}). \tag{4}$$

Proof If $M \ge 1$, $N \ge 1$, then, for a given p, the partial sums in (3) for $M \le m \le M+N$, $m \ne p$, being also bounded by the same constant C uniformly in k, $p \in \mathbb{Z}$, we may sum them over p from M to $M+N$ to get, after an interchange in the order of summation

$$\sum_{m=M}^{M+N} \sum_{m\neq p=M}^{M+N} \frac{|b_{m,m-k}|^2}{|m-p|} \le \sum_{p=M}^{M+N} C = C(N+1), \quad (k \in \mathbb{Z}). \tag{5}$$

On the left-hand side of (5) we use the inequality $\sum_{m\neq p=M}^{M+N} \frac{1}{|m-p|} \ge \sum_{j=1}^{N} \frac{1}{j}$:

$$\left(\sum_{m=M}^{M+N} |b_{m,m-k}|^2\right)\left(\sum_{m\neq p=M}^{M+N} \frac{1}{|m-p|}\right)$$
$$\ge \left(\sum_{m=M}^{M+N} |b_{m,m-k}|^2\right)\left(\sum_{j=1}^{N} \frac{1}{j}\right) \ge \log(N+1)\left(\sum_{m=M}^{M+N} |b_{m,m-k}|^2\right),$$

so that

$$\log(N+1)\sum_{m=M}^{M+N}|b_{m,m-k}|^2 \le C(N+1), \quad (k\in\mathbb{Z}),$$

and the inequality (4) follows. □

Remark 3 For $k=0$, the inequality in (4), is the result of Shayya already mentioned in the introduction, but with a more precise constant.

The corollary tells us that the arithmetic means in (4) decay like $\log^{-1} N$. But in fact they can be shown to decay like $\left(\log N \log_2 N \cdots \log_q N\right)^{-1}$, where $\log_q N$ is the q-iterated logarithm. To avoid technical details, we shall limit ourselves to a proof of the following special case.

Theorem 4 *Under the conditions of Theorem 1, we have*

$$\frac{1}{N+3}\sum_{m=M}^{M+N}|b_{m,m-k}|^2 \le \frac{C/(1-1/\log 3)}{\log(N+3)[\log\log(N+4)-\log\log 4]}, \quad (k\in\mathbb{Z}). \quad (6)$$

Proof We proceed exactly as in the proof of the Corollary, except that we start by dividing both sides of (3) by $\log(p-M+3)$ and then sum over p from M to $M+N$. In the resulting inequality the left-hand side is

$$\begin{aligned}
&\left(\sum_{m=M}^{M+N}|b_{m,m-k}|^2\right)\sum_{m\ne p=M}^{M+N}\frac{1}{|m-p|\log(p-M+3)}\\
&\ge\left(\sum_{m=M}^{M+N}|b_{m,m-k}|^2\right)\sum_{j=1}^{N}\frac{1}{j\log(j+3)}\\
&\ge\sum_{m=M}^{M+N}|b_{m,m-k}|^2\int_1^{N+1}\frac{dt}{t\log(t+3)}\\
&\ge\left(\sum_{m=M}^{M+N}|b_{m,m-k}|^2\right)(\log\log(N+4)-\log\log 4), \qquad (7)
\end{aligned}$$

while the right-hand side is

$$C\sum_{p=M}^{M+N}\frac{1}{\log(p-M+3)}=C\sum_{k=3}^{N+3}\frac{1}{\log k}<C\left(\frac{1}{\log 3}+\int_3^{N+3}\frac{dt}{\log t}\right). \quad (8)$$

An application of the Cauchy mean-value theorem [3], gives a $\xi\in(3,x)$ such that

$$\frac{\int_3^x\frac{dt}{\log t}}{\frac{x}{\log x}-\frac{3}{\log 3}}=\frac{1}{1-\frac{1}{\log\xi}}<\frac{1}{1-\frac{1}{\log 3}}.$$

This last inequality used in (8) gives

$$C\sum_{p=M}^{M+N}\frac{1}{\log(p-M+3)} < \frac{C}{\left(1-\frac{1}{\log 3}\right)}\cdot\frac{N+3}{\log(N+3)}. \tag{9}$$

Now (6) follows immediately from (7) and (9). □

Remark 5 For general q, we follow the same steps as in the proof of Theorem 4, but in place of the logarithm $\log(M-p+3)$, we first divide by a product L_q of iterated logarithms. Then, for well defined constants, α_q, and γ_q, we obtain

$$\frac{1}{N+\gamma_q}\sum_{m=M}^{M+N}|b_{m,m-k}|^2 \le \frac{C/(1-\alpha_q)}{L_q(N+\gamma_q)[\log_{q+1}(N+1+\gamma_q)-\log_{q+1}(1+\gamma_q)]}, \quad (k\in\mathbb{Z}).$$

Corollary 6 *Let ϕ be an inner function and suppose that $\phi(0)\neq 0$. Then the sequence*

$$c_{n,n-k}=\int_{\partial U}\phi^n(z)z^{k-n}\frac{dz}{2\pi i z}, \quad (n,k\in\mathbb{Z}),$$

satisfies all conclusions of Theorems 1, Corollary 2, and Theorem 4.

3 Connection with a functional equation

In the study of the diagonal elements $\{c_{n,n}\}$, there is a connection with a certain functional equation which appears to merit closer study. If ϕ is the function of Theorem 1, it was shown in [1] that the functional equation

$$\zeta(z)=z\phi(\zeta(z)),$$

has a unique solution in the open unit disk U. The solution ζ, is connected to $F(z)=\sum_{n=0}^{\infty}c_{n,n}z^n$, the generating function of the diagonal elements, by the equation $F(z)=z\frac{\zeta'(z)}{\zeta(z)}$. This function ζ has some interesting properties: ζ is one-to-one, $\zeta(0)=0$, $\zeta'(0)=\phi(0)$, so that $\frac{1}{\phi(0)}\zeta(z)$ is in the class S of normalized univalent functions in U. Also ζ omits all the zeros of ϕ, which (zeros) may very well be infinite in number. The inverse function $\zeta^{-1}$is well-defined and analytic on the open simply connected set $\zeta(U)$, and $\zeta^{-1}(z)=\frac{z}{\phi(z)}$, there. Thus $\log|\frac{\phi(z)}{z}|$ is the Green's function of $\zeta(U)$ with pole at 0. In some special cases, the function ζ can be used to compute explicitly the diagonal coefficients. For example, if $\phi(z)=\frac{1}{2-z}$, then $b_{n+1,n+1}=0$, $c_{n+1,n+1}=\frac{(2n)!}{(n!)^2 2^{2n+1}}\sim\frac{C}{\sqrt{n}}$ as $n\to\infty$. If on the other hand $\phi(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$ with $0<|\alpha|<1$, then $b_{n+1,n+1}=c_{n+1,n+1}=\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\binom{n+j}{j}|\alpha|^{2j}$.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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