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A Valency Criterion for Harmonic Mappings

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Abstract
Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping of the closed unit disk $\overline{\mathbb{D}}$ with a Blashke product dilatation $B_m = g'/h'$ of order m. The aim of this paper is to prove that if h' has $p - 1$ zeros, counting multiplicity, in $\mathbb{D}$ and no zeros on $\partial\mathbb{D}$, and that

$$\operatorname{Re}\left\{1 + \mathrm{e}^{\mathrm{it}} \frac{h''(\mathrm{e}^{\mathrm{it}})}{h'(\mathrm{e}^{\mathrm{it}})}\right\} > -\frac{1}{2} \sum_{k=1}^{m} \frac{1 - |a_k|}{1 + |a_k|},$$

where $a_1, \ldots, a_m$ are the zeros of B_m, then f is $(m + p - 1)$-valent. The proof deploys a surface-theoretic technique based on an effective "pasting" procedure. This is an improvement of an earlier result of Bshouty et al. (Proc Am Math Soc 146:1113–1121, 2018) which asserts that if f is a sense-preserving harmonic mapping on $\mathbb{D}$, with dilatation z^m that satisfies the inequality

$$\operatorname{Re}\left\{1 + z \frac{h''(z)}{h'(z)}\right\} > -\frac{m}{2}, \quad z \in \mathbb{D},$$

then f is $(m + p)$-valent.

Keywords Multivalent function · Close-to-convex function · Planar harmonic mapping · Riemann surface · Covering surface

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1 Introduction

A planar harmonic mapping of a region Ω is a complex-valued function of the form

$$f(z) = u(z) + iv(z),$$

where $z = x + iy$ and u and v are (real) harmonic functions of Ω. Every such mapping can be written as

$$f = h + \overline{g},$$

where h and g are analytic functions of Ω which are single valued if Ω is simply connected and possibly multiple valued otherwise; in fact, this representation is unique up to an additive constant. The (second complex) dilatation of f is defined as the function $\omega = g'/h'$ which is meromorphic in Ω unless it is identically infinite. The mapping f is sense preserving (sense reversing) in Ω if, and only if, $|\omega| < 1$ ($|\omega| > 1$) in Ω or its Jacobian $J_f = \left|h'\right|^2 - \left|g'\right|^2$ is positive (negative) in deleted neighborhoods of Ω. Moreover, f is locally one to one in Ω if, and only if, J_f is non-vanishing there.

A simply connected subdomain Ω of $\mathbb{C}$ is called *close to convex* if its complement $\mathbb{C}\backslash\Omega$ is the union of closed half lines with pairwise disjoint interiors. A one-to-one analytic or harmonic mapping of the open unit disc $\mathbb{D}$ is called a *close-to-convex univalent* function if its image set $f(\mathbb{D})$ is close to convex.

In [3], the authors settled a conjecture of Mocanu [14] by proving the following result:

Theorem A *Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$, with $h'(0) \neq 0$ that satisfies $g'(z) = zh'(z)$ and*

$$\operatorname{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{1}{2}; \quad z \in \mathbb{D}. \tag{1}$$

Then, f is a univalent close-to-convex mapping.

Recently, Bshouty et al. [4] proved a more general result, Theorem B, which settled negatively a conjecture of Hayami [6]. To state this result, we need the following four definitions.

Definition 1 A sense-preserving harmonic mapping f of $\mathbb{D}$ is said to belong to the class $\mathrm{VH}_k(p)$, where p is a positive integer and k is a real number at least 2, if f has $p - 1$ critical points in $\mathbb{D}$, counting multiplicity, and

$$\limsup_{r\to 1^-} \int_0^{2\pi} \left|\frac{\mathrm{d}}{\mathrm{d}t} \arg \frac{\partial}{\partial t} f(\mathrm{re}^{\mathrm{it}})\right| \mathrm{d}t \leq pk\pi. \tag{2}$$

The classes $\mathrm{VH}_k(p)$, for all p and k, constitute all multivalent harmonic mappings of bounded boundary rotation.

The subclass of $\mathrm{VH}_k(p)$ consisting only of analytic functions of $\mathbb{D}$ is denoted by $V_k(p)$; particularly, $V_k(p)$ is a proper subclass of $\mathrm{VH}_k(p)$. The classes $V_k(p)$ were first introduced and investigated by Leach [8] and further studied by Lyzzaik [11]. The class of functions $f \in V_k(1)$ normalized by $f(0) = 0$ and $f'(0) = 1$, denoted simply by V_k, was introduced by Paatero [16] who showed that V_4 consists only of univalent functions. Much later, Brannan [2] showed that V_4 is properly contained in the class of univalent analytic close-to-convex functions. However, every class V_k, with $k > 4$, contains analytic functions of $\mathbb{D}$ that are not univalent.

Definition 2 Let P be a complex polynomial, and let γ be a ray in $\mathbb{C}$. A simple unbounded curve l is called a P-ray if $P : l \to \gamma$ is a homeomorphism.

Evidently, if P is a linear transformation, then a P-ray is a ray in $\mathbb{C}$.

Definition 3 A sense-preserving harmonic mapping $f : \mathbb{D} \to \mathbb{C}$ is called p-valent if it takes every value at most p times, counting multiplicity.

Definition 4 We say that a curve α admits a harmonic cusp at a point $z_0 \in \alpha$ if there exist a parametrization $z = z(t), a < t < b$, for α and a real $t_0 \in (a, b)$ such that $\arg z'(t)$ determines a continuously decreasing function in a deleted neighborhood of t_0 with a simple discontinuity of jump π at t_0.

The statement of [4, Thm. B] is as follows.

Theorem B *Let $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,\ \ p \geq 1$, and $g(z)$ be analytic functions of $\mathbb{D}$ satisfying $h'(z)/z^{p-1} \neq 0$, $g'(z) = z^m h'(z)$, $m = 1, 2, \ldots$, and the inequality*

$$\mathrm{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{m}{2}; \quad z \in \mathbb{D}. \tag{3}$$

Then the harmonic mapping $f = h + \overline{g}$ satisfies the following properties:

(a) $f \in \mathrm{VH}_{2(1+m/p)}(p)$;
(b) *$f = P \circ \phi$, where P is a polynomial of degree at most m if $p = 1$ and $p + m$ otherwise, and ϕ is a homeomorphism of $\mathbb{D}$ into $\mathbb{C}$ such that the zeros of P' lie in $\overline{\phi(\mathbb{D})}$ and $\mathbb{C}\backslash\phi(\mathbb{D})$ is non-empty and is a union of P-rays of disjoint interiors starting from $\partial\phi(\mathbb{D})$;*
(c) *f is m-valent if $p = 1$ and $(p + m)$-valent otherwise.*

Furthermore, if h is analytic on $\overline{\mathbb{D}}$ and h' is non-vanishing on $\partial\mathbb{D}$, then $f(\partial\mathbb{D})$ subdivides into $2p + m$ concave curves and comprises the same number of harmonic cusps whose vertices are the points of the subdivision.

Remark 1 A stronger version of [4, Thm. B] does hold: the proof of the technical [4, Lem. 1], hence the proof of Theorem B, follows almost verbatim for the case when the critical points of h' are scattered in $\mathbb{D}$, instead of having one critical point of order $p - 1$ at the origin, provided that the sum of their orders is exactly $p - 1$.

As an immediate application of Theorem B, if $m = p = 1$, then $f \in \mathrm{VH}_4$ and f is a univalent close-to-convex function—a stronger result than Theorem A; see [4, Cor. 1]. Note that the following conjecture was made in [4].

Conjecture A *The harmonic mappings f satisfying the assumptions of Theorem B are $(m+p-1)$-valent.*

The main result of this paper, Theorem 1, provides a positive answer to this conjecture in the case when weaker hypotheses than Theorem B hold regarding the dilatation and critical points of f, but with the proviso that f is harmonic on $\overline{\mathbb{D}}$. Of primary interest is the method of proof deployed, namely a surface-theoretic technique based on a "pasting" procedure which is more economical than the "cutting and pasting" procedure that the second author had used in [11].

Theorem 1 *Let f be a sense-preserving harmonic mapping of the closed unit disk $\overline{\mathbb{D}}$ of the form $f = h + \overline{g}$, where h and g are analytic functions of $\overline{\mathbb{D}}$ that satisfies the following properties:*

(a) *h' has $p-1$ zeros, counting multiplicity, in $\mathbb{D}$ and no zeros on the unit circle $\mathbb{T}$;*
(b) *the dilatation of f is the finite Blaschke product*

$$\omega(z) = \prod_{k=1}^{m} \frac{z-a_k}{1-\overline{a_k}z},$$

where $|a_k| < 1$ for every $k = 1, 2, \ldots$;

(c)

$$\mathrm{Re}\left\{1 + \mathrm{e}^{\mathrm{it}} \frac{h''(\mathrm{e}^{\mathrm{it}})}{h'(\mathrm{e}^{\mathrm{it}})}\right\} > -\frac{1}{2} \sum_{k=1}^{m} \frac{1-|a_k|}{1+|a_k|}.$$

Then the following hold:

(A) *$f(\mathbb{T})$ has a subdivision of $2p+m$ concave curves that need not be simple (i.e., they may self-intersect);*
(B) *$f(\mathbb{T})$ has $2p+m$ harmonic cusps; any two adjacent concave curves of the subdivision form a harmonic cusp;*
(C) *$f = P \circ \phi$, where P is a polynomial of degree at most $m+p-1$, with the zeros of P' lying in $\overline{\phi(\mathbb{D})}$, and ϕ is a homeomorphism on $\overline{\mathbb{D}}$ which is locally quasiconformal in $\mathbb{D}$ with dilatation $(1+|\omega|)/(1-|\omega|)$; thus f is at most $(m+p-1)$-valent on $\overline{\mathbb{D}}$;*
(D) *$\mathbb{C}\backslash\phi(\mathbb{D})$ is a union of P-rays with mutually disjoint interiors such that each starts at $\partial\phi(\mathbb{D})$.*

Remark 2 We contend that Theorem 1 holds with the more general dilatation $\lambda\,\omega$, where λ is a unitary constant and ω is as in the theorem. To see this, assume that f satisfies the assumptions of Theorem 1 with the exception that its dilatation is $\lambda\,\omega$. Then $\lambda^{-1/2} f$ also satisfies the assumptions of Theorem 1. It follows that our contention is true.

The organization of the paper is as follows. In Sect. 2, we introduce the requisite notions and notation. In Sect. 3, we prove the main result of the paper, namely Theorem 1. The last Sect. 4 is allocated for the corollaries of Theorem 1.

2 Notation and Preliminaries

This section is devoted for the requisite notions and preliminary results.

We need the following definitions regarding the local behavior of harmonic mappings which may be found in [12].

Definition 5 Let G be a Jordan domain of the extended complex plane $\mathbb{C}^{\infty}$, and let f be a complex-valued function of G.

(a) Let $z_0 \in G$. We write $f_{z_0} \sim z^r$, where $r = 1, 2, \ldots$, if there exist an open neighborhood U of z_0 and sense-preserving homeomorphisms $h_1 : U \to \mathbb{D}$ and $h_2 : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and $h_2 \circ f \circ h_1^{-1}(w) = w^r$ for all $w \in \mathbb{D}$. Also, we write $f_{z_0} \sim \overline{z}^r$ if $h_2 \circ f \circ h_1^{-1}(w) = \overline{w}^r$ for all $w \in \mathbb{D}$.
(b) Let $z_0 \in \partial G$, and let f be also defined on an open arc of ∂G containing z_0. We write $f_{z_0} \sim z^r\ (G)$, where $r = 1, 2, \ldots$, if there exist an open neighborhood U of z_0 and sense-preserving homeomorphisms $h_1 : U \cap \overline{G} \to \mathbb{D}^+$, where $\mathbb{D}^+$ is the closed upper-half unit disc, and $h_2 : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and $h_2 \circ f \circ h_1^{-1}(w) = w^r$ for all $w \in \mathbb{D}^+$. Also, we write $f_{z_0} \sim \overline{z}^r\ (G)$ if $h_2 \circ f \circ h_1^{-1}(w) = \overline{w}^r$ for all $w \in \mathbb{D}^+$.
(c) Again, let $z_0 \in G$. We write $f_{z_0} \sim z^r, \overline{z}^s$, where $r, s = 1, 2, \ldots$, if there exists a cross-cut of G through z_0 dividing G into two Jordan domains G_1 and G_2 such that $f_{z_0} \sim z^r\ (G_1)$ and $f_{z_0} \sim \overline{z}^s\ (G_2)$. Note that G_1 and G_2 are unique since f must be sense preserving in G_1 and sense reversing in G_2. That is, the covering properties of f near z_0 resemble those of the mappings $z \to z^r$, $\operatorname{Im} z \geq 0$, and $z \to \overline{z}^s$, $\operatorname{Im} z \leq 0$, near the origin.

Note that definitions (a) and (b) are independent of the open neighborhood U of z_0. In this paper, we shall only use the notation $f_{z_0} \sim z, \overline{z}^3$ and $f_{z_0} \sim z^3, \overline{z}$.

We assume the reader's familiarity with some surface topology concepts as presented in Ahlfors and Sario [1, Ch. I, Sects. 1–3], along with the associated notation and terminology. Note that all the surfaces that appear in what follows are orientable.

Definition 6 A bordered surface $\overline{F}$ is a connected Hausdorff space that admits an open covering by sets which can be mapped homeomorphically onto relatively open subsets of the closed upper half-plane.

One can distinguish between two mutually disjoint subsets of $\overline{F}$: the *interior* of $\overline{F}$, denoted by F, which is the set of points which are never mapped to the real axis under the homeomorphisms, and the *border* of F, denoted by $B(F)$, which is the set of points which are always mapped to the real axis under the homeomorphisms.

We shall be dealing with a special kind of bordered surfaces. Let $\overline{F}$ be a bordered surface, $\mathcal{W}$ be a surface, and $f : \overline{F} \to \mathcal{W}$ be a continuous function. We call the ordered pair $(\overline{F}, f)$ a *bordered covering surface* of $\mathcal{W}$, with *projection map* f, if (F, f) is a covering surface of $\mathcal{W}$ and f is a local homeomorphism on $B(F)$, that is, if there exists a relatively open neighborhood of every point in $B(F)$ that maps under f homeomorphically into $\mathcal{W}$.

Suppose now that $(\overline{F}, f)$ is a bordered covering surface of $\mathcal{W}$. Then f is a light (pre-image of every point is totally disconnected), open, continuous mapping of $\overline{F}$ into

$\mathcal{W}$ which is locally homeomorphic on $B(\overline{F})$. Evidently, f affects a bordered covering surface $(\overline{\mathcal{F}}, \varphi)$ of $\mathcal{W}$, where $\overline{\mathcal{F}}$ is the set of ordered pairs $\{(z, f(z)),\ z \in \overline{F}\}$ endowed via f the topological structure of $\overline{F}$ and $\varphi : \overline{\mathcal{F}} \to \mathcal{W}$ is the projection map defined by $\varphi((z, f(z))) = f(z)$. In this case, we call $(\mathcal{F}, \varphi_{|\mathcal{F}})$ the *image surface* of f and $(\overline{\mathcal{F}}, \varphi)$ the *image bordered surface* of f. In case $\overline{F}$ is contained in $\mathcal{W}$ and f is the identity map, then $(\overline{\mathcal{F}}, \varphi)$ is called *the bordered image surface of the identity map on* $\overline{F}$.

We shall also be using the concept of complete coverings.

Definition 7 A covering surface $(\tilde{F}, f)$ of a surface F is called complete if every point in F has a compact neighborhood V such that each component of $f^{-1}(V)$ is compact.

Complete coverings $(\tilde{F}, f)$ of a surface F have two important properties. First, at every point $\tilde{\zeta} \in \tilde{F}$ there exists a unique positive integer N, dependent on $\tilde{\zeta}$, such that f behaves near $\tilde{\zeta}$ as the mapping $z \to z^N$ near the origin; that is, if $\zeta = f(\tilde{\zeta})$, then there exist open neighborhoods $\tilde{U}$ of $\tilde{\zeta}$ and U of ζ, and homeomorphisms $h_1 : \tilde{U} \to \mathbb{D}$ and $h_2 : U \to \mathbb{D}$ such that $h_1(\tilde{\zeta}) = h_2(\zeta) = 0$ and $h_2 \circ f \circ h_1^{-1}(z) = z^N$. If $N \geq 2$, then N is called the *multiplicity* of f at $\tilde{\zeta}$, and $\tilde{\zeta}$ is called a *branch point* of f of *order* $N - 1$. The second property of $(\tilde{F}, f)$ is that it covers all the points of F equally the same number of times, counting multiplicity.

We will make use of the Riemann–Hurwitz relation between closed surfaces without boundary [13, pp. 33–34]. Let $\tilde{F}$ and F be such surfaces, $(\tilde{F}, f)$ be a complete covering of F taking on every value in F exactly k times, counting multiplicity, and β_f be the total order of branch points of f. Then,

$$\beta_f = 2[(g(\tilde{F}) - 1) - k(g(F) - 1)],$$

where $g(.)$ denotes genus. If $\tilde{F}$ and F are simply connected, then $g(\tilde{F}) = g(F) = 0$ and consequently $\beta_f = 2k - 2$.

Once Theorem 1 is obtained, we will establish a connection between harmonic mappings satisfying the assumptions of Theorem 1 and multivalent analytic close-to-convex functions. To do so, we need the following three definitions.

Definition 8 Let $S_a(p)$ be the class of functions f analytic in $\mathbb{D}$ with p zeros there, counting multiplicity, and such that $\mathrm{Re}\{zf'/f\} > 0$ for all z in some annulus $\{z : \rho < |z| < 1\}$. Then $S_a(p)$ is the class of p-valent annular starlike functions.

For this class see Hummel [7]. Note that the subclass of $S_a(1)$ consisting of functions f normalized by $f(0) = 0$ and $f'(0) = 1$ is the class S of *normalized univalent starlike functions*.

Definition 9 Let $K(p)$ be the class of functions F analytic in $\mathbb{D}$, with $F(0) = 0$, such that there exists a function $f \in S_a(p)$ for which $\mathrm{Re}\{zF'/f\} > 0$ for all z in some annulus $\{z : \rho < |z| < 1\}$. Then $K(p)$ is the class of p-valently close-to-convex functions.

The class $K(p)$ was introduced by Livingston [9]. Note that $K(1)$ coincides with the class of close-to-convex univalent functions of Kaplan [17, p. 51], and that a function

F of $\mathbb{D}$ satisfying $F(0) = 0$ belongs to $K(1)$ if, and only if, $F(\mathbb{D})$ is a close-to-convex domain.

The class $K(p)$ is known not to be closed under the topology of uniform convergence on compact subsets of $\mathbb{D}$; see Styer [15].

Definition 10 A function $F \neq 0$ analytic in $\mathbb{D}$, with $F(0) = 0$, is said to belong to $K_w(p)$ if there exist functions $F_n \in K(p)$, $n = 1, 2, \ldots,$ such that $F_n \to F$ uniformly on compact subsets of $\mathbb{D}$. Then $K_w(p)$ is the class of weakly p-valently close-to-convex functions.

The class $K_w(p)$ was introduced by Styer [15]. Note that $K_w(1) = K(1)$ and $K(p)$ is a proper subset of $K_w(p)$ otherwise.

3 Proof of Theorem 1

The proof of Theorem 1 requires a sequence of five lemmas.

Lemma 1 *Under the assumptions of* Theorem 1, *$f(\mathbb{T})$ subdivides into $2p+m$ concave curves such that any two adjacent ones are edges of a harmonic cusp.*

Proof First, we show that $f(\mathbb{T})$ comprises $2p + m$ concave curves, not necessarily simple, with the same number of cusps. The dilatation of f on $\mathbb{T}$ may be written as $\omega(\mathrm{e}^{\mathrm{it}}) = \mathrm{e}^{i\psi(t)}$, $t \in \mathbb{R}$, where ψ is differentiable. Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{\mathrm{it}}) &= i\mathrm{e}^{\mathrm{it}} h'(\mathrm{e}^{\mathrm{it}}) + \overline{i\mathrm{e}^{\mathrm{it}} g'(\mathrm{e}^{\mathrm{it}})} \\
&= i\mathrm{e}^{\mathrm{it}} h'(\mathrm{e}^{\mathrm{it}}) + \overline{i\mathrm{e}^{\mathrm{it}} \omega(\mathrm{e}^{\mathrm{it}}) h'(\mathrm{e}^{\mathrm{it}})} \\
&= i\mathrm{e}^{\mathrm{it}} h'(\mathrm{e}^{\mathrm{it}}) + \overline{i\mathrm{e}^{i(t+\psi(t))} h'(\mathrm{e}^{\mathrm{it}})} \\
&= i\mathrm{e}^{-i\psi(t)/2} \left\{ \mathrm{e}^{i(t+\psi(t)/2)} h'(\mathrm{e}^{\mathrm{it}}) - \overline{\mathrm{e}^{i(t+\psi(t)/2)} h'(\mathrm{e}^{\mathrm{it}})} \right\} \\
&= -2\mathrm{e}^{-i\psi(t)/2} \operatorname{Im} \left\{ \mathrm{e}^{i(t+\psi(t)/2)} h'(\mathrm{e}^{\mathrm{it}}) \right\} \\
&= -2\mathrm{e}^{-i\psi(t)/2} \operatorname{Im} \Phi(t),
\end{aligned} \tag{4}$$

where $\Phi(t) = \mathrm{e}^{i(t+\psi(t)/2)} h'(\mathrm{e}^{\mathrm{it}})$. Observe that $-\psi(t)/2$ and, consequently, the argument of the tangent vector to $f(\mathbb{T})$ increases and decreases steadily, respectively, over an interval $[0, 2\pi + \epsilon)$ for sufficiently small $\epsilon > 0$ except at those values t where $\operatorname{Im} \Phi(t)$ changes sign; note that for every such value t, the point $f(\mathrm{e}^{\mathrm{it}})$ is the vertex of a harmonic cusp of $f(\mathbb{T})$.

We show that the number of harmonic cusps of $f(\mathbb{T})$ is $2p + m$. We write:

$$\begin{aligned}
\arg \Phi(t) &= t + \frac{1}{2} \arg \omega(t) + \arg h'(\mathrm{e}^{\mathrm{it}}) \\
&= t + \frac{1}{2} \operatorname{Im} \{\log \omega(t)\} + \operatorname{Im} \left\{ \log h'(\mathrm{e}^{\mathrm{it}}) \right\} \\
&= t + \frac{1}{2} \operatorname{Im} \left\{ \sum_{k=1}^{m} \log \left(\frac{\mathrm{e}^{\mathrm{it}} - a_k}{1 - \overline{a}_k \mathrm{e}^{\mathrm{it}}} \right) \right\} + \operatorname{Im} \left\{ \log h'(\mathrm{e}^{\mathrm{it}}) \right\}.
\end{aligned}$$

By differentiating $\arg\Phi(t)$ and invoking condition (c) of Theorem 1, we obtain:

$$\begin{aligned}\frac{d}{dt}\arg\Phi(t) &= \frac{1}{2}\sum_{k=1}^{m}\frac{1-|a_k|^2}{|e^{it}-a_k|^2} + \mathrm{Re}\left\{1+e^{it}\frac{h''(e^{it})}{h'(e^{it})}\right\}\\ &\geq \frac{1}{2}\sum_{k=1}^{m}\frac{1-|a_k|}{1+|a_k|} + \mathrm{Re}\left\{1+e^{it}\frac{h''(e^{it})}{h'(e^{it})}\right\} > 0. \end{aligned} \tag{5}$$

Thus, $\arg\Phi$ is a strictly increasing function on $[0, 2\pi+\epsilon)$.

Next, we show that the size of the range of $\arg\Phi(t)$ over $[0, 2\pi]$ is exactly $(2p+m)\pi$. Because of condition (a) of Theorem 1, we write

$$h'(z) = \prod_{j=1}^{p-1}(z-b_j)H(z),$$

where each $|b_j| < 1$, the values b_j may not be distinct, and H is a non-vanishing analytic function in $\overline{\mathbb{D}}$. Hence,

$$1+z\frac{h''(z)}{h'(z)} = 1+\sum_{j=1}^{p-1}\frac{z}{z-b_j} + z\frac{H'(z)}{H(z)}.$$

If $z = \mathrm{e}^{\mathrm{it}}$, then

$$\left\{1+\mathrm{e}^{\mathrm{it}}\frac{h''(\mathrm{e}^{\mathrm{it}})}{h'(\mathrm{e}^{\mathrm{it}})}\right\}\,\mathrm{d}t = -i\left\{\frac{1}{z}+\sum_{j=1}^{p-1}\frac{1}{z-b_j}+\frac{H'(z)}{H(z)}\right\}\mathrm{d}z.$$

Assuming $\mathbb{T}$ to be positively directed, it follows that:

$$\begin{aligned}\int_0^{2\pi}\mathrm{Re}\left\{1+\mathrm{e}^{\mathrm{it}}\frac{h''(\mathrm{e}^{\mathrm{it}})}{h'(\mathrm{e}^{\mathrm{it}})}\right\}\mathrm{d}t &= \mathrm{Im}\int_{\mathbb{T}}\frac{\mathrm{d}z}{z} + \mathrm{Im}\sum_{j=0}^{p-1}\int_{\mathbb{T}}\frac{\mathrm{d}z}{z-b_j}\\ &\quad + \mathrm{Im}\int_{\mathbb{T}}\frac{H'(z)}{H(z)}\mathrm{d}z\\ &= 2\pi+2(p-1)\pi = 2p\pi. \end{aligned} \tag{6}$$

On the other hand, we have:

$$\begin{aligned}\int_0^{2\pi}\left\{\sum_{k=1}^{m}\frac{1-|a_k|^2}{|\mathrm{e}^{\mathrm{it}}-a_k|^2}\right\}\mathrm{d}t &= -i\sum_{k=1}^{m}\int_{\mathbb{T}}\left(\frac{1}{z-a_k}+\frac{\overline{a_k}}{1-\overline{a_k}z}\right)\mathrm{d}z\\ &= -i\sum_{k=1}^{m}(2\pi i+0) = 2\pi m. \end{aligned} \tag{7}$$

Thus, by (5), (6), and (7), we conclude that:

$$\int_0^{2\pi} d\arg\Phi(t) = \int_0^{2\pi}\left[\frac{1}{2}\sum_{k=1}^{m}\frac{1-|a_k|^2}{|e^{it}-a_k|^2} + \mathrm{Re}\left\{1+e^{it}\frac{h''(e^{it})}{h'(e^{it})}\right\}\right] dt$$
$$= (2p+m)\pi. \tag{8}$$

Therefore, $\arg\Phi$ increases by $(2p+m)\pi+\delta$ for an arbitrarily small $\delta > 0$ over an interval $[0, 2\pi+\epsilon)$ for sufficiently small ϵ. Equivalently, over the last interval, Im $\Phi(t)$ changes sign at $2p+m$ values. This concludes the proof of Lemma 1. □

Remark 3 The concave curves of the subdivision of $f(\mathbb{T})$, with end points being the vertices of the harmonic cusps, need not be simple.

In view of Lemma 1, let t_j, $1 \le j \le 2p+m$, where $t_1 < t_2 < \cdots < t_{2p+m} < t_1 + 2\pi$, be the values at which Im $\Phi(t)$ changes sign. Also, let $z_j = e^{it_j}$ and $w_j = f(z_j)$.

Lemma 2 *Under the assumptions of* Theorem 1, *f is locally univalent at every point in $\mathbb{T}$; in particular, $f_{z_j} \sim z, \bar{z}^3$ for every z_j, $1 \le j \le 2p+m$.*

Proof Since h' is non-vanishing on $\mathbb{T}$, f is locally univalent at every point of $\mathbb{T}$ except possibly at the points z_j where the local behavior of f is either $f_{z_j} \sim z, \bar{z}^3$ (and f is locally 1-1 at z_j) or $f_{z_j} \sim z^3, \bar{z}$ (and f is locally 2–1 at z_j); see [12, Thm. 5.1]. Thus, the size of the interior angle of $f(\mathbb{D})$ at every w_j is either zero or 2π, and consequently the size of the respective exterior angle is either π or $-\pi$. Denote by ν the number of points w_j at which the interior angles of $f(\mathbb{D})$ have measure zero, and by μ the number of the remaining points w_j. Obviously, $\nu+\mu = 2p+m$. On the other hand, since $|\omega| < 1$ and h' has $p-1$ zeros, counting multiplicity, in $\mathbb{D}$, h' and g' share the same zeros, counting multiplicities. Note that every such zero is also a critical point of f whose order is exactly the multiplicity of zero minus 1; see [12, Thm. 4.1(c)]. Hence, by (4) and (8),

$$\int_0^{2\pi} \arg\, df(e^{it}) = -\frac{1}{2}\left(\psi(2\pi+0)-\psi(0)\right) + \int_0^{2\pi} d\arg\,\Phi(t)$$
$$= -m\pi + (2p+m)\pi = 2p\pi,$$

and the net variation of arg df, with account for the jumps at the points w_j, is exactly $2p\pi$. That is, $\nu\pi - \mu\pi - m\pi = 2p\pi$, or $\nu - \mu = 2p+m$. Hence, $\nu = 2p+m$ and $\mu = 0$, and the size of the angle of $f(\mathbb{D})$ at every point w_j is exactly zero. Therefore, f is locally univalent at every z_j and the proof of Lemma 2 is complete. □

Let $\alpha_j \subset \mathbb{T}$, $1 \le j \le 2p+m$, be the positively directed closed arc with initial point z_j and terminal point z_{j+1}, with $z_{2p+m+1} = z_1$, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_{2p+m}\}$ be the subdivision of $\mathbb{T}$ induced by these arcs. Denote by $\Delta_{\alpha_j}\arg\, df$ the net variation of arg df on α_j. Note that either $\Delta_{\alpha_j}\arg\, df \ge -\pi$ for every j or otherwise. It is enough to consider the latter case; then $\Delta_{\alpha_j}\arg\, df < -\pi$ for some values j. We call α_j a *Type I arc* if $\Delta_{\alpha_j}\arg\, df \ge -\pi$, and a *Type II arc* if otherwise; see Figs. 1

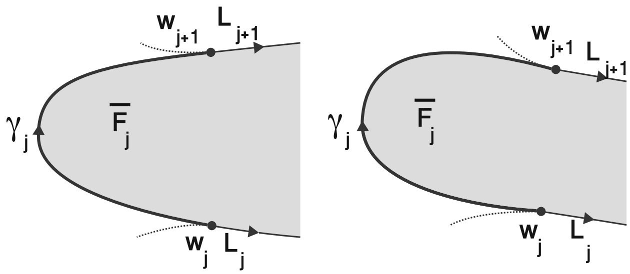

Fig. 1 Type I arc γ_j and associated region $\overline{F}_j$

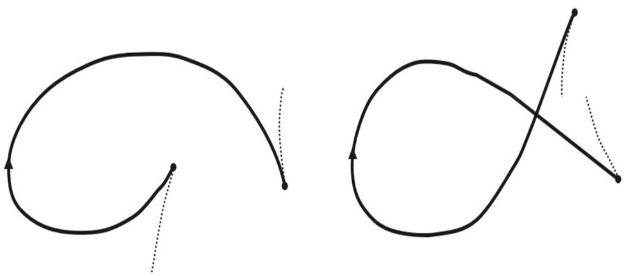

Fig. 2 Type II arc γ_j

and 2. Suppose α_j is a Type II arc; then we subdivide α_j into positively directed closed subarcs $\alpha_{j,i},\ 1 \le i \le r_j + 1$, where $r_j = 1, 2, \dots,$ that satisfy the following properties:

(i) $\Delta_{\alpha_{j,i}} \arg \mathrm{d}f = -\pi$ for $1 \le i \le r_j$ and $\Delta_{\alpha_{j,r_j+1}} \arg \mathrm{d}f \ge -\pi$;
(ii) α_j and $\alpha_{j,1}$ have the same initial point;
(iii) α_j and α_{j,r_j+1} have the same terminal point;
(iv) the initial point of every $\alpha_{j,i+1}$ coincides with the terminal point of $\alpha_{j,i}$.

Lemma 3 *Suppose α_j is a Type II arc. Then,*

(a) *r_j is the largest integer less than $(1/\pi)\left|\Delta_{\alpha_j} \arg f\right|$, and*
(b) *$\sum r_j \le m - 1$, where the sum is taken over all values j for which α_j is of Type II.*

Proof The proof of (a) is immediate. As for (b), note that

$$r_j = -\left\lfloor 1 + (1/\pi)\Delta_{\alpha_j} \arg f \right\rfloor.$$

By summing up over all the values j for which α_j is a Type II arc, we obtain:

$$\begin{aligned}\sum r_j &= -\sum \left\lfloor 1 + (1/\pi)\Delta_{\alpha_j} \arg f \right\rfloor \\ &\le -\left\lfloor 1 + \frac{1}{\pi}\sum \Delta_{\alpha_j} \arg f \right\rfloor \\ &\le m - 1.\end{aligned}$$

The first inequality follows by induction on j and using the inequality $\lfloor 1+a+b \rfloor \le \lfloor 1+a \rfloor + \lfloor 1+b \rfloor$ for real values a and b, and the second inequality is straightforward. This ends the proof. □

The method of proof for Theorem 1 is the well-known "pasting" technique applied particularly for bordered covering surfaces. For completeness, we formulate this technique explicitly as follows.

Lemma 4 *Suppose the following:*

(a) $(\overline{F}_1, f_1)$ *and* $(\overline{F}_2, f_2)$ *are two bordered covering surfaces of a surface* $\mathcal{W}$.
(b) $\tilde{\gamma}_1$ *and* $\tilde{\gamma}_2$ *are closed subarcs of* $B(\overline{F}_1)$ *and* $B(\overline{F}_2)$, *respectively.*
(c) $\kappa = f_2^{-1} \circ f_1 : \tilde{\gamma}_1 \to \tilde{\gamma}_2$ *is a homeomorphism.*
(d) *For points* $\zeta_j \in \tilde{\gamma}_j,\ j = 1, 2$, *where* $f_j(\zeta_j) = \xi \in \mathcal{W}$, *there exist relative open neighborhoods* U_j *of* ζ_j *in* $\overline{F}_j$ *such that* $f_1(U_1) \cup f_2(U_2)$ *is an open neighborhood of* ξ *and* $f_1(U_1)$ *and* $f_2(U_2)$ *have disjoint interiors.*

Then there exists a bordered covering surface $(\overline{F}, f)$ *of* $\mathcal{W}$ *such that* $\overline{F}_1,\ \overline{F}_2 \subset \overline{F}$, F_1 *and* F_2 *are disjoint in* F, *and* $f_1 = f_{|\overline{F_1}}$ *and* $f_2 = f_{|\overline{F_2}}$.

We call $(\overline{F}, f)$ the *crosswise identification* of $(\overline{F}_1, f_1)$ and $(\overline{F}_2, f_2)$ along $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

The proof of Lemma 4 is straightforward.

It is immediate from Lemma 2 that f yields a bordered image surface, denoted by $(\overline{\mathcal{Y}}, \tau)$. Let $\tilde{f} : \overline{\mathbb{D}} \to \overline{\mathcal{Y}}$ be the lift of f defined by $\tilde{f}(z) = (z, f(z))$ for all $z \in \overline{\mathbb{D}}$; then $f = \tau \circ \tilde{f}$ in $\overline{\mathbb{D}}$. Define the paths $\gamma(\theta) = f(\mathrm{e}^{i\theta})$ and $\tilde{\gamma}(\theta) = \tilde{f}(\mathrm{e}^{i\theta})$, where $\theta \in [0, 2\pi]$. Then $\tilde{\gamma} = B(\overline{\mathcal{Y}})$ is a lift of γ under τ. By the definition of f, $(\overline{\mathcal{Y}}, \tau)$ admits branch points with total order $p-1$ in $\mathcal{Y}$ and no branch points in $\tilde{\gamma}$.

In what follows, the lift under τ of a subarc $\sigma = f_{|[\theta_1,\theta_2]}$ of γ will be the subarc $\tilde{\sigma} = \tilde{f}_{|[\theta_1,\theta_2]}$ of $\tilde{\gamma}$. Also, a connected subset of a covering of $\mathbb{C}^\infty$ will be called a *ray* if it maps under the projection map homeomorphically to a ray in $\mathbb{C}^\infty$.

Lemma 5 *There exists a* q*-fold covering surface* $(\mathcal{X}, \pi)$ *of* $\mathbb{C}^\infty$ *that satisfies the following properties:*

(1) $\mathcal{X}$ *is simply connected with* $q \le m + p - 1$;
(2) $\overline{\mathcal{Y}} \subset \mathcal{X}$ *and* $\tau = \pi_{|\overline{\mathcal{Y}}}$;
(3) $\mathcal{X} \backslash \mathcal{Y} = \bigcup_{l \in \mathcal{L}} l$, *where* $\mathcal{L}$ *is a collection of rays with mutually disjoint interiors such that each starts at* $B(\mathcal{Y})$.

To prove this lemma, we use the geometry of γ to construct a bordered covering surface of $\mathbb{C}^\infty$ whose border is a lift of γ, then paste this surface and $\overline{\mathcal{Y}}$ crosswise along their borders to obtain the desired covering surface $\mathcal{X}$.

Proof Let $\gamma_j = f(\alpha_j),\ 1 \le j \le 2p + m$. Recall the points w_j; then each γ_j is a subarc of γ with initial point w_j and terminal point w_{j+1}, with $w_{2p+m+1} = w_1$, and $\gamma = \gamma_1\gamma_2 \ldots \gamma_{2p+m}$. Let L_j be the closed ray in $\mathbb{C}^\infty$ with initial point w_j and direction opposite to that of the tangent vector of γ_j at w_j, and let L_{j+1}, also in $\mathbb{C}^\infty$, be the closed ray with initial point w_{j+1} and direction same as that of the tangent vector of γ_j at w_{j+1}.

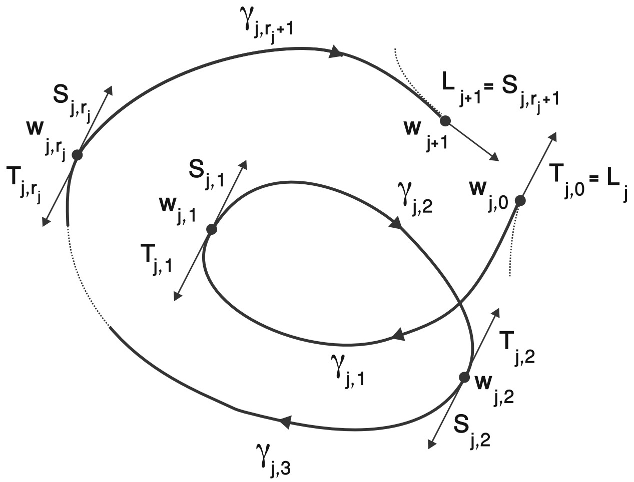

Fig. 3 Subdivision $\{\gamma_{j,i},\ 1 \le i \le r_j + 1\}$ of Type II arc γ_j

Associated with every arc γ_j, we construct a simply connected, bordered, covering surface whose border comprises lifts of L_j, L_{j+1}, and γ_j. We consider two cases:

Case I γ_j is a Type I arc.

In this case, the arc composition $L_j^{-1}\gamma_j L_{j+1}$ is an unbounded simple curve that divides $\mathbb{C}^\infty$ into two simply connected domains; let F_j be the one lying on the right-hand side of γ_j; see Fig. 1. Note that $\overline{F}_j$ is also a simply connected region in $\mathbb{C}^\infty$ that can be ruled with mutually disjoint closed rays, except for ∞, each starting at γ_j. Denote by $(\overline{\mathcal{F}}_j, \rho_j)$ the bordered image surface of the identity on $\overline{F}_j$. Then $B(\overline{\mathcal{F}}_j) = \tilde{L}_j^{-1}\hat{\gamma}_j\tilde{L}_{j+1}$, where $\tilde{L}_j$, $\hat{\gamma}_j$, and $\tilde{L}_{j+1}$ are the lifts in $\overline{\mathcal{F}}_j$ under ρ_j of L_j, γ_j, and L_{j+1}, respectively.

Case II γ_j is a Type II arc.

Recall the subdivision $\{\alpha_{j,i},\ 1 \le i \le r_j + 1\}$ of α_j, and define the subarcs $\gamma_{j,i} = f(\alpha_{j,i})$. Obviously, each $\gamma_{j,i}$ is a simple arc and $\gamma_j = \gamma_{j,1}\gamma_{j,2}\ldots\gamma_{j,r_j+1}$. Denote by $w_{j,i}$, $1 \le i \le r_j$, the terminal point of $\gamma_{j,i}$; note that γ_j is smooth at every point $w_{j,i}$. Denote by $S_{j,i}$ and $T_{j,i}$ the closed rays in $\mathbb{C}^\infty$ starting at $w_{j,i}$ and directed in the direction and opposite direction of the tangent vector of γ_j at $w_{j,i}$, respectively. For convenience, we also denote by $T_{j,0}$ and S_{j,r_j+1} the rays L_j and L_{j+1}, respectively; see Fig. 3. Observe the following:

(a) The rays $T_{j,i-1}, S_{j,i}$, $1 \le i \le r_j$, have the same directions, and the rays T_{j,r_j} and S_{j,r_j+1} do not meet and need not be parallel.

(b) Every arc composition $T_{j,i-1}^{-1}\gamma_{j,i}S_{j,i}$, $1 \le i \le r_j$, is an unbounded simple curve that divides $\mathbb{C}^\infty$ into two simply connected domains. Let $U_{j,i}$ be the one lying on the right-hand side of $\gamma_{j,i}$. Note that $\overline{U}_{j,i}$ is also a simply connected region in $\mathbb{C}^\infty$ that can be ruled with rays having the same direction as $S_{j,i}$ and starting at $\gamma_{j,i}$ [11, Prop. 6.4]; see Fig. 4.

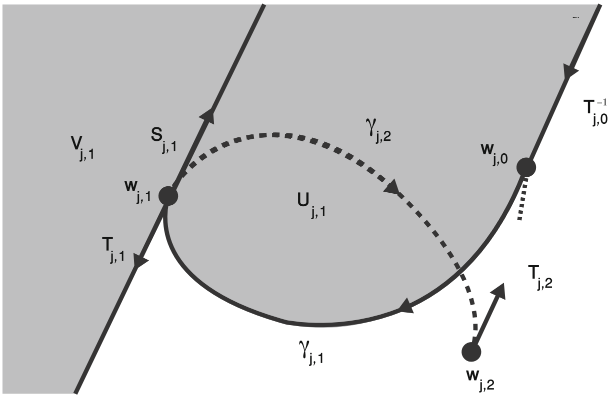

Fig. 4 Bordered covering surface $(\overline{\mathcal{F}}_{j,1}, \rho_{j,1})$

(c) Every arc composition $S_{j,i}^{-1}T_{j,i},\ 1 \le i \le r_j$, is a directed straight line with opposite direction to the tangent vector of γ_j at $w_{j,i}$; see Fig. 3. Let $V_{j,i}$ be the open half plane lying on the right-hand side of $S_{j,i}^{-1}T_{j,i}$; it is immediate that every closed half-plane $\overline{V}_{j,i}$ can be ruled with rays starting at $w_{j,i}$ otherwise having mutually disjoint interiors; see Figs. 4 and 5.

(d) Let $\overline{W}_{j,i} = \overline{U}_{j,i} \bigcup \overline{V}_{j,i},\ 1 \le i \le r_j$. Obviously, the interior $W_{j,i}$ of $\overline{W}_{j,i}$ is a Jordan domain in $\mathbb{C}^\infty$ whose boundary is the arc composition $T_{j,i-1}^{-1}\gamma_{j,i}T_{j,i}$ and which lies on the right-hand side of $T_{j,i-1}^{-1}\gamma_{j,i}T_{j,i}$; see Figs. 4 and 5.

(e) The arc composition $T_{j,r_j}^{-1}\gamma_{j,r_j+1}S_{j,r_j+1}$ is, as above in (b), an unbounded simple curve that divides $\mathbb{C}^\infty$ into two simply connected domains. Let W_{j,r_j+1} be the one lying on the right-hand side of γ_{j,r_j+1}. Note that $\overline{W}_{j,r_j+1}$ is also simply connected in $\mathbb{C}^\infty$ and can be ruled with mutually disjoint rays, except for ∞, starting at γ_{j,r_j+1}; see Fig. 3.

Now, we construct a bordered covering surface of $\mathbb{C}^\infty$ that embeds every bordered image surface $\overline{\mathcal{W}}_{j,i},\ 1 \le i \le r_j + 1$, of the identity map on $\overline{W}_{j,i}$ in such a way that the interiors of any two of the embedded surfaces are mutually disjoint. To do so, we consider the bordered covering surface $(\overline{\mathcal{F}}_{j,1}, \rho_{j,1})$ obtained by crosswise identification of the bordered image surfaces $\overline{\mathcal{W}}_{j,1}$ and $\overline{\mathcal{W}}_{j,2}$ along the respective lifts of the ray $T_{j,1}$; see Figs. 4 and 5. Then, we consider the bordered covering surface $(\overline{\mathcal{F}}_{j,2}, \rho_{j,2})$ obtained by crosswise identification of the bordered image surfaces $(\overline{\mathcal{F}}_{j,1}, \rho_{j,1})$ and $\overline{\mathcal{W}}_{j,3}$ along the lifts of the ray $T_{j,2}$ in $\overline{\mathcal{W}}_{j,2}$ as embedded in $(\overline{\mathcal{F}}_{j,1}, \rho_{j,1})$, and $\overline{\mathcal{W}}_{j,3}$; see Figs. 3 and 5. By repeating this procedure r_j times, we obtain the desired bordered covering surface $(\overline{\mathcal{F}}_{j,r_j}, \rho_{j,r_j})$ of $\mathbb{C}^\infty$ which, for convenience and with slight abuse of notation, is denoted by $(\overline{\mathcal{F}}_j, \rho_j)$.

The bordered coverings $(\overline{\mathcal{F}}_j, \rho_j)$ obtained for both cases, Case I and Case II, satisfy the following properties:

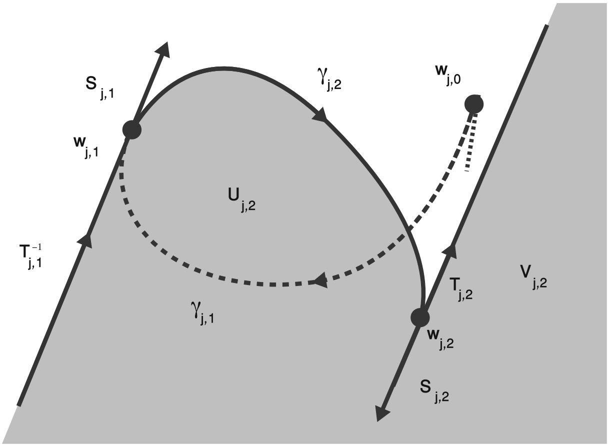

Fig. 5 Bordered covering surface $(\overline{\mathcal{F}}_{j,2}, \rho_{j,2})$

(a) Every $(\overline{\mathcal{F}}_j, \rho_j)$ is simply connected: this is straightforward for Case I; as for Case II, it follows by induction using Van Kampen's theorem successively to infer that every bordered covering $\overline{\mathcal{F}}_{j,i},\ 1 \le i \le r_j$, is simply connected.
(b) Every $B(\overline{\mathcal{F}}_j)$ comprises the arc composition $\tilde{L}_j^{-1}\hat{\gamma}_j\tilde{L}_{j+1}$, where $\tilde{L}_j$ and $\tilde{L}_{j+1}$ are the lifts of L_j and L_{j+1} in $\overline{\mathcal{F}}_j$ for Case I and in $\overline{\mathcal{W}}_{j,1}$ and $\overline{\mathcal{W}}_{j,r_j+1}$, respectively, for Case II, and $\hat{\gamma}_j = \hat{\gamma}_{j,1}\hat{\gamma}_{j,2}\ldots\hat{\gamma}_{j,r_j+1}$, where $\hat{\gamma}_{j,i},\ 1 \le i \le r_j+1$, is the lift of $\gamma_{j,i}$ in $\overline{\mathcal{W}}_{j,i}$.
(c) $(\overline{\mathcal{F}}_j, \rho_j)$ has no branch points.
(d) With $\hat{w}_{j,i},\ 1 \le i \le r_j$, denoting the terminal point of the subarc $\hat{\gamma}_{j,i}$ of $\hat{\gamma}_j$, ρ_j is locally 2-valent at every $\hat{w}_{j,i}$ and locally one-to-one at every other point $\hat{w} \in \hat{\gamma}_j$.

We conclude that there exist bordered covering surfaces $(\overline{\mathcal{F}}_j, \rho_j),\ 1 \le j \le 2p+m$, with borders $\tilde{L}_j^{-1}\hat{\gamma}_j\tilde{L}_{j+1}$, with $L_{2p+m+1} = L_1$ and $\tilde{L}_{2p+m+1} = \tilde{L}_1$ as mentioned above. Note that at every point of $\tilde{w} \in \tilde{L}_j^{-1}\hat{\gamma}_j\tilde{L}_{j+1}$, there exists a relative local neighborhood $N_{\tilde{w}}$ of $\overline{\mathcal{F}}_j$ that projects homeomorphically into $\mathbb{C}^\infty$ such that $\rho_j(N_{\tilde{w}} \cap \mathcal{F}_j)$ lies on the right-hand side of $\rho_j(N_{\tilde{w}}) \cap L_j^{-1}\gamma_j L_{j+1}$; see Fig. 3.

We proceed to construct a bordered covering surface $(\overline{\mathcal{Z}}, \rho)$ of $\mathbb{C}^\infty$ which embeds the bordered covering surfaces $\overline{\mathcal{F}}_j,\ 1 \le j \le 2p+m$, so that their interiors are mutually disjoint. First, we consider the crosswise identification of the bordered covering surfaces $(\overline{\mathcal{F}}_1, \rho_1)$ and $(\overline{\mathcal{F}}_2, \rho_2)$ along the respective lifts of the ray L_2 in $B(\overline{\mathcal{F}}_1)$ and $B(\overline{\mathcal{F}}_2)$ to obtain a bordered covering surface $\overline{\mathcal{Z}}_1$ of $\mathbb{C}^\infty$. Then, we consider the crosswise identification of this surface and $(\overline{\mathcal{F}}_3, \rho_3)$ along the respective lifts of the ray L_3 in $B(\overline{\mathcal{F}}_2)$ (as embedded in $\overline{\mathcal{Z}}_1$) and $B(\overline{\mathcal{F}}_3)$ to obtain a bordered covering surface $\overline{\mathcal{Z}}_2$ of $\mathbb{C}^\infty$. By repeating this procedure $2p+m-2$ times, we obtain a bordered covering surface $(\overline{\mathcal{Z}}_{2p+m-2}, \rho_{2p+m-2})$ of $\mathbb{C}^\infty$. Observe now that $B(\overline{\mathcal{Z}}_{2p+m-2})$ and $B(\overline{\mathcal{F}}_{2p+m})$ contain the lift of the closed arc composition $L_{2p+m}L_1^{-1}$, containing ∞, in $\mathbb{C}^\infty$ on which property (d) of Lemma 4 holds except possibly over ∞. In view of this, we go one step further and consider the crosswise identification of the latter two surfaces along the mentioned lift of $L_{2p+m}L_1^{-1}$ to obtain the desired bordered covering surface $(\overline{\mathcal{Z}}, \rho)$. We can easily conclude the following:

(i) $\overline{\mathcal{Z}}$ is simply connected; the proof is similar to that of the above property (a) of $(\overline{\mathcal{F}}_j, \rho_j)$.
(ii) $(\overline{\mathcal{Z}}, \rho)$ has exactly one point over ∞, possibly a branch point.
(iii) $B(\overline{\mathcal{Z}})$ is the arc composition $\hat{\gamma} = \hat{\gamma}_1\hat{\gamma}_2 \ldots \hat{\gamma}_{2p+m}$, where each $\hat{\gamma}_j$, $1 \le j \le 2p+m$, is as defined above in property (b).
(iv) In view of the above property (d), ρ is locally 2-valent at every $\hat{w}_{j,i}$ and locally one to one at every other point $\hat{w} \in \hat{\gamma}_j$.
(v) $\overline{\mathcal{Z}} = \bigcup_{l\in\mathcal{L}} l$, where $\mathcal{L} = \{l\}$ is a collection of rays with mutually disjoint interiors each starting from $B(\mathcal{Z})$.

Denote by $\kappa : \tilde{\gamma} \to \hat{\gamma}$ the homeomorphism defined by $\kappa = \rho_{|\hat{\gamma}}^{-1} \circ \tau_{|\tilde{\gamma}}$ and $\kappa(\tilde{\gamma}_j) = \hat{\gamma}_j$ for all j, $1 \le j \le 2p+m$. Now, we identify the covering surfaces $(\overline{\mathcal{Y}}, \tau)$ and $(\overline{\mathcal{Z}}, \rho)$ along their borders $\tilde{\gamma}$ and $\hat{\gamma}$ via the identification map κ. We contend that this process results in a simply connected, complete, covering surface $(\mathcal{X}, \pi)$ of $\mathbb{C}^\infty$.

Let $w \in \gamma$, $\tilde{w} \in \tilde{\gamma}$ be its lift under τ, and $\hat{w} = \kappa(\tilde{w})$. Let $\mathcal{O}_{\tilde{w}}$ and $\mathcal{O}_{\hat{w}}$ be relative neighborhoods of $\tilde{w}$ and $\hat{w}$ in $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Z}}$, respectively, with $\tilde{\sigma} = \mathcal{O}_{\tilde{w}} \cap \tilde{\gamma}$ and $\hat{\sigma} = \mathcal{O}_{\hat{w}} \cap \hat{\gamma}$ that are mapped homeomorphically into $\mathbb{C}$ under τ and ρ. Here, we consider two cases:

(i) $w \ne w_{j,i}$ $1 \le i \le r_j$ for all possible values j. In this case, we may choose $\mathcal{O}_{\tilde{w}}$ and $\mathcal{O}_{\hat{w}}$ such that the identification κ on $\mathcal{O}_{\tilde{w}} \bigcup \mathcal{O}_{\hat{w}}$ yields an open neighborhood $\mathcal{O}$ of $\mathcal{X}$ that maps under π homeomorphically to a topological disc in $\mathbb{C}$ containing w.
(ii) w equals some $w_{j,i}$. In this case, in view of the pasting procedure leading to the coverings $\overline{\mathcal{F}}_j$ in Case II, we may choose $\mathcal{O}_{\tilde{w}}$ and $\mathcal{O}_{\hat{w}}$ such that the above identification κ on $\mathcal{O}_{\tilde{w}} \bigcup \mathcal{O}_{\hat{w}}$ yields an open neighborhood $\mathcal{O}$ of $\mathcal{X}$ such that $(\mathcal{O}, \Pi)$ is a two-sheeted covering of a topological disc containing w and $(\mathcal{O}, \Pi)$ having a single branch point of order 1 over w.

It follows that every point $w \in \mathbb{C}^\infty$ has a compact neighborhood whose preimage under π consists of compact components in $\mathcal{X}$. This proves that $\mathcal{X}$ is a crosswise identification of $(\overline{\mathcal{Y}}, \tau)$ and $(\overline{\mathcal{Z}}, \rho)$ and that it is a complete covering surface of $\mathbb{C}^\infty$.

Moreover, $\mathcal{X}$ is simply connected being, by construction, a double of $\overline{\mathcal{Y}}$ since both $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Z}}$ are simply connected. Also, $(\mathcal{X}, \pi)$ has at most $m+p-1$ branch points on $\tilde{\gamma}$ (or $\hat{\gamma}$), as embedded in $\mathcal{X}$, each of order exactly one, branch points in $\mathcal{Y}$ of total order $p-1$ induced by the zeros of h', and a possible branch point over ∞ of order say $q-1$, where $q \ge 1$; see above property (ii). Thus, the total order of the branch points of $(\mathcal{X}, \pi)$ is at most $m+p+q-3$. Being complete, $\mathcal{X}$ covers all the points of $\mathbb{C}^\infty$ equally the same number, namely q, of times, counting multiplicity. Since $\mathcal{X}$ is simply connected, the Riemann–Hurwitz theorem implies that the sum of the orders of the branch points of $(\mathcal{X}, \pi)$ is exactly $2q-2$. Hence, $2q-2 \le m+p+q-3$, or $q \le m+p-1$.

Therefore, $\mathcal{X}$ covers every point of $\mathbb{C}^\infty$ q times, counting multiplicity, where $q \le m+p-1$, and $\mathcal{Y}$ is embedded in $\mathcal{X}$ such that $\mathcal{X} \backslash \mathcal{Y} = \bigcup_{l\in\mathcal{L}} l$, where $\mathcal{L} = \{l\}$ is a collection of rays with disjoint interiors starting at $\partial\mathcal{Y}$. This completes the proof of Lemma 5. □

We are now ready to prove Theorem 1.

Proof of Theorem 1 Lemma 1 yields at once (A) and (B) of Theorem 1. As for (C) and (D), we invoke Theorem 4B in [1, Ch. 2] to equip the covering surface $(\mathcal{X}, \pi)$ with the conformal structure that makes $\pi : \mathcal{X} \to \mathbb{C}^\infty$ analytic. Then, by the Uniformization Theorem [1, Ch. III, Theorem 11G], there exists a conformal map $\tilde{P}$ from $\mathbb{C}^\infty$ onto $\mathcal{X}$ such that $P = \pi \circ \tilde{P}$ is a polynomial of degree q. Let $\phi = \tilde{P}^{-1} \circ \tilde{f}$; then ϕ is a homeomorphism on $\overline{\mathbb{D}}$, $\tilde{f} = \tilde{P} \circ \phi$, and $\tilde{P}^{-1}(\overline{\mathcal{Z}}) = \mathbb{C}\backslash\phi(\mathbb{D})$. Consequently, $f = \pi \circ \tilde{f} = \pi \circ \tilde{P} \circ \phi = P \circ \phi$ on $\overline{\mathbb{D}}$ and, by (v), $\mathbb{C}\backslash\phi(\mathbb{D})$ is a union of P-rays with mutually disjoint interiors such that each starts at $\partial\phi(\mathbb{D})$. On the other hand, $f_z = P' \circ \phi\ \phi_z$ and $f_{\overline{z}} = P' \circ \phi\ \phi_{\overline{z}}$ in $\mathbb{D}$ which yields $|f_{\overline{z}}/f_z| = |\phi_{\overline{z}}/\phi_z|$; consequently, ϕ is locally quasiconformal with dilatation $(1 + |\omega|)/(1 - |\omega|)$ in $\mathbb{D}$. This concludes Theorem 1. □

4 Corollaries and Examples

By invoking [17, Thm. 2.12], Theorem A follows at once from Theorem 1 provided that f is harmonic on $\overline{D}$.

Corollary 1 *Under the assumptions of Theorem 1 and if $p = m = 1$, then f is a univalent close-to-convex mapping.*

Another immediate conclusion of Theorem 1 is:

Corollary 2 *Under the assumptions of* Theorem 1 *and if $m = 1$, then f is p-valent. This result is sharp.*

Proof By Theorem 1, f is p-valent. To prove that this result is sharp, observe that the harmonic mapping

$$f(z) = z^p + \frac{p}{p+1}\overline{z}^{p+1}$$

satisfies the hypotheses of Theorem 1, with f_z having $p - 1$ zeros at the origin and nowhere else in $\overline{\mathbb{D}}$ and f has dilatation z^p, f is p-valent in $\overline{\mathbb{D}}$ and is exactly p to 1 near the origin; see [12, Thm. 5.1]. This ends the proof. □

A stronger version of Conjecture A follows positively at once from Theorem 1 and where f is harmonic on $\overline{D}$.

Corollary 3 *Under the assumptions of* Theorem 1, *f is $(m + p - 1)$-valent.*

Our next conclusion requires extending [5, Def. 2.2].

Definition 11 Let f be a sense-preserving harmonic mapping of $\mathbb{D}$. An analytic associate of f is an analytic function F of $\mathbb{D}$ of the form $F = f \circ \psi$, where $\psi : \mathbb{D} \to \mathbb{D}$ is a homeomorphism.

Obviously, an analytic associate of f is an analytic function of $\mathbb{D}$ that shares with f the same image surface. It is immediate that if F_1 and F_2 are analytic associates of f, then $F_1 = F_2 \circ S$ for some automorphism S of $\mathbb{D}$.

It is convenient to establish the following definition:

Definition 12 A sense-preserving harmonic mapping f of $\mathbb{D}$ is called a p-valently harmonic close-to-convex function if $f(0) = 0$ and f has an analytic associate $F \in K_w(p)$. Let $HK_w(p)$ denote the class of all p-valently harmonic close-to-convex functions.

In view of these definitions, we have:

Corollary 4 *Under the assumptions of* Theorem 1, $f - f(0) = P \circ \phi \in HK_w(q)$, *where* $q \le m + p - 1$, *P is a polynomial of degree q and ϕ is a locally quasiconformal mapping of $\mathbb{D}$. Furthermore, q may be chosen so that the zeros of P' lie in $\overline{\phi(\mathbb{D})}$ and $\mathbb{C} \backslash \phi(\mathbb{D})$ is a union of P-rays with mutually disjoint interiors.*

Proof Let $\tilde{P}$ and ϕ be as defined in the proof of Theorem 1, and let A be a conformal map from $\mathbb{D}$ onto $\phi(\mathbb{D})$. Then, $F = P \circ A$ is an analytic function in $\mathbb{D}$. But, by Theorem 1, $f = P \circ \phi$ where P and ϕ are as stated in the theorem. Consequently, $F = P \circ \phi \circ \phi^{-1} \circ A = f \circ \psi$, where $\psi = \phi^{-1} \circ A : \mathbb{D} \to \mathbb{D}$ is a homeomorphism. Hence, F is an analytic associate of f and, by properties (C) and (D) of Theorem 1, the zeros of P' lie in $\overline{\psi(\mathbb{D})}$ and $\mathbb{C} \backslash \psi(\mathbb{D})$ is a union of P-rays with mutually disjoint interiors, respectively. By invoking [10, Thm. 4.1], we conclude that $F - F(0) \in K_w(q)$ for some $q \le m + p - 1$. Therefore, $f - f(0) \in HK_w(q)$ and the proof is complete. □

The following example shows that Theorem 1 is sharp when $p = 1$ and $m = 2$.

Example 1 Consider the harmonic mapping $f = P \circ \phi$, where $P(w) = w^3 + \rho^{-2}(1 - \overline{w})^3$, $\phi(z) = 1/(1 - \rho z)$, where $0 < \rho < 1$ and ρ is sufficiently close to 1. Then f has analytic part $h(z) = (1 - \rho z)^{-3}$ and co-analytic part $g(z) = \rho^{-2}(1 - (1 - \rho z)^{-1})^3$. It can be easily verified that f preserves symmetry about the real axis, has dilatation $\omega(z) = -z^2$, $h'(0) \ne 0$, and

$$\operatorname{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} = -1 + 2\operatorname{Re}\left\{\frac{1 + \rho z}{1 - \rho z}\right\} > -1$$

for all $z \in \mathbb{D}$. Moreover,

$$\frac{d}{dt} f(e^{it}) = 6i\rho e^{-it} \ \operatorname{Re}\left\{[\kappa_\rho(e^{it})]^2\right\},$$

where $\kappa_\rho(z) = \rho^{-1}\kappa(\rho z)$ is the dilation of the Koebe function. It is easy to see that $\operatorname{Re}\left\{[\kappa_\rho(\mathrm{e}^{\mathrm{it}})]^2\right\}$ changes sign exactly four times on $[-\pi, \pi + \epsilon[$ for an arbitrarily small $\epsilon > 0$, and that these changes may occur on any arbitrarily small interval $] - \delta, \delta[$ by choosing ρ sufficiently close to 1. Hence, for sufficiently small ρ, the curve $f(\mathbb{T})$

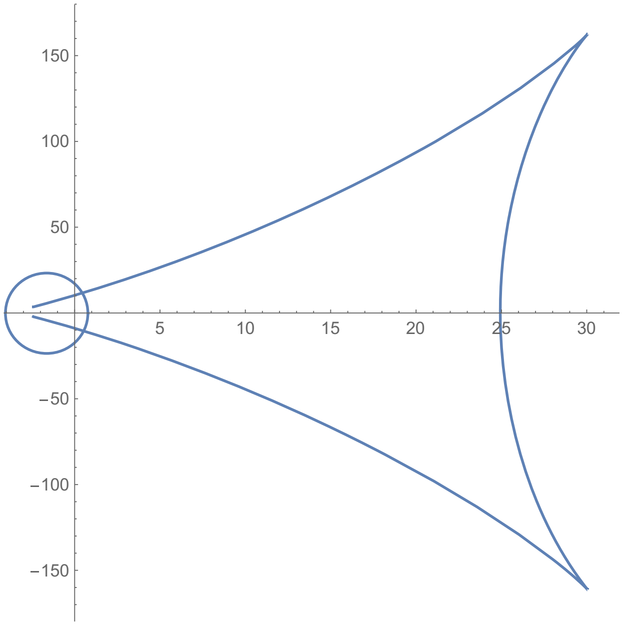

Fig. 6 Plot of $f(\mathrm{e}^{\mathrm{it}})$ over interval $[-\pi/8.5, \pi/8.5]$ with $\rho = 0.8$

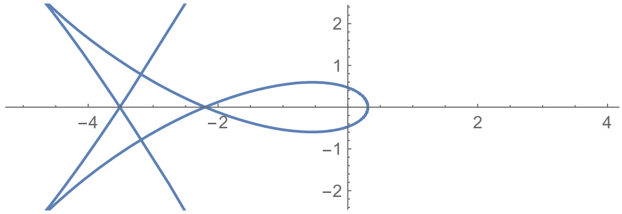

Fig. 7 Plot of $f(\mathrm{e}^{\mathrm{it}})$ over interval $[\pi/11, 21\pi/11]$ with $\rho = 0.8$

admits exactly four harmonic cusps symmetric about the real axis in an arbitrarily small subinterval of $\mathbb{T}$ containing 1.

Graphing $f(\mathbb{T})$ by *Mathematica*®, see Figs. 6 and 7, we conclude that f is 2-valent in $\mathbb{D}$; namely, f assumes every point of its image domain exactly once for those points lying inside the quadrilateral bounded by $f(\mathbb{T})$ and depicted in Fig. 7. Note that the circles appearing in Fig. 6 and $f(\mathbb{T})$ meet at exactly two points and are otherwise mutually disjoint; moreover, this circle encloses the graph of Fig. 7.

The following example shows that Theorem 1 is sharp when $p = m = 2$.

Example 2 Consider the harmonic mapping $f = P \circ \phi$, where $P(w) = (4w^5 - 5w^4) + \rho^{-2}\overline{(-4w^5 + 15w^4 - 20w^3 + 10w^2)}$, $\phi(z) = 1/(1-\rho z)$, where $0 < \rho < 1$ and ρ is sufficiently close to 1. It can be verified that f preserves symmetry about the real axis, has analytic part $h(z) = (5\rho z - 1)(1-\rho z)^{-5}$ and co-analytic part $g(z) = \rho^2 z^4(\rho z - 1)(1-\rho z)^{-5}$, $h'(z) = 20\rho^2 z(1-\rho z)^{-6}$, $g'(z) = -20\rho^2 z^3(1-\rho z)^{-6} = -z^2 h'(z)$, $h'(0) = 0$ and $h''(0) \neq 0$. Hence, f has dilatation $\omega(z) = -z^2$,

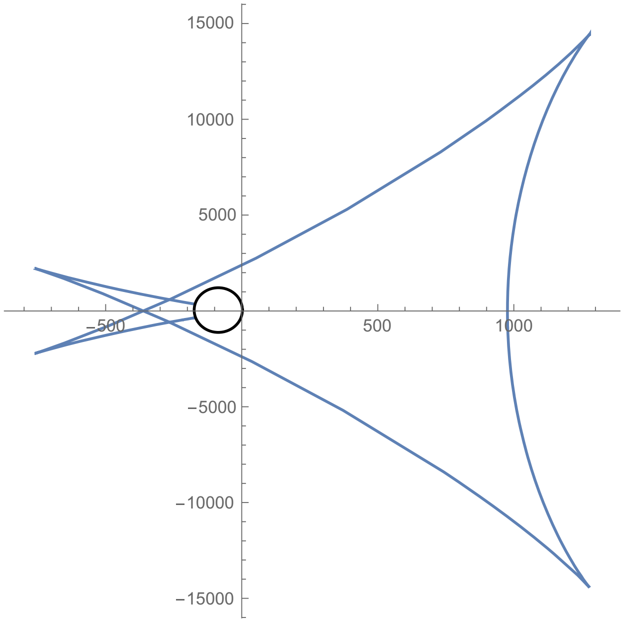

Fig. 8 Plot of $f(\mathrm{e^{it}})$ over interval $[-0.4, 0.4]$ with $\rho = 0.8$

$$\mathrm{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} = -1 + 2\,\mathrm{Re}\left\{\frac{1+\rho z}{1-\rho z}\right\} > -1$$

for all $z \in \mathbb{D}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e^{it}}) = 40i\rho^2 \mathrm{e^{-it}}\ \mathrm{Re}\left\{[\kappa_\rho(\mathrm{e^{it}})]^3\right\}.$$

Note that $\mathrm{Re}\left\{[\kappa_\rho(\mathrm{e^{it}})]^3\right\}$ changes sign exactly six times on $[-\pi, \pi + \epsilon[$ for an arbitrarily small $\epsilon > 0$, and that these changes may occur on any arbitrarily small interval $]-\delta, \delta[$ by choosing ρ sufficiently close to 1. Hence, $f(\mathbb{T})$ admits exactly six harmonic cusps in an arbitrarily small subinterval of $\mathbb{T}$ containing 1 provided that ρ is sufficiently small, and it is symmetric about the real axis.

We graph $f(\mathbb{T})$ by *Mathematica*® for $\rho = 0.8$. The graph of $f(\mathrm{e^{it}})$ over the interval $[-0.4, 0.4]$ starts from a point, denoted by a, in the upper half semicircle of Fig. 8 and terminates at a point, denoted by b, in the lower half semicircle of the figure. Note that this graph is symmetric about the real axis, the circle in Fig. 8, denoted by C, is mutually disjoint with $f(\mathbb{T})$ except for the points a and b, and that C encloses completely the rest of $f(\mathbb{T})$. Moreover, a closer look at C shows that it crosses the imaginary axis.

Figure 9 depicts the graph of $f(\mathrm{e^{it}})$ over the two-interval $[-0.61, -0.4] \cup [0.4, 0.61]$. As t varies from 0.4 to 0.61, $f(\mathrm{e^{it}})$ traverses the simple arc of the figure that lies in the lower half-plane, starts at the point b and terminates at a point, denoted by c, in the circle C' appearing in Fig. 9; this circle is tangent to both C and

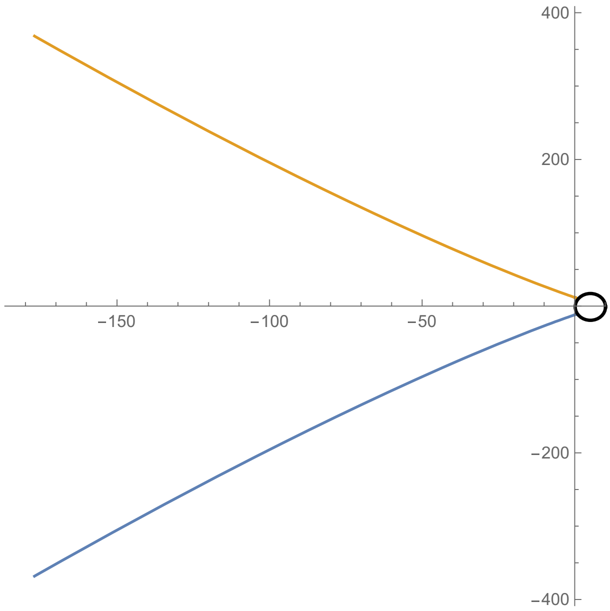

Fig. 9 Plot of $f(e^{it})$ over the two-interval $[-0.61, -0.4] \cup [0.4, 0.61]$ with $\rho = 0.8$

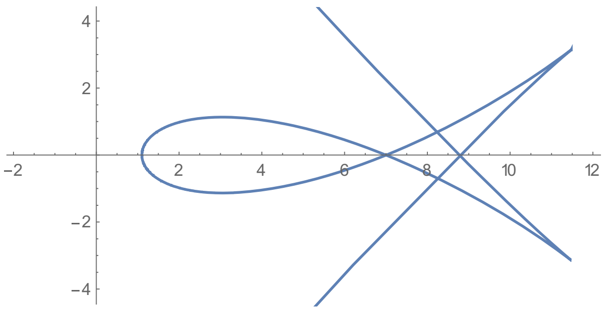

Fig. 10 Plot of $f(e^{it})$ over interval $[0.6, 2\pi - 0.6]$ with $\rho = 0.8$

the imaginary axis and lies otherwise in the right half-plane. By symmetry, the graph of $f(e^{it})$ over $[-0.61, -0.4]$ is the reflection of the graph of $f(e^{it})$ over $[0.4, 0.61]$; denote by d the reflection of c about the real axis.

Figure 10 depicts the graph of $f(e^{it})$ over the interval $[0.61, 2\pi - 0.61]$. As t varies from 0.61 to $2\pi - 0.61$, $f(e^{it})$ traverses the graph in the figure starting at the point c and terminating at the point d.

Adjoining the graphs shown in Figs. 8, 9 and 10 appropriately, we easily conclude that f assumes every point lying inside the quadrilateral bounded by $f(\mathbb{T})$ in Fig. 10 exactly three times and other points once or twice. Therefore, f is 3-valent in $\mathbb{D}$.

In view of the above examples where Theorem 1 is sharp, we have:

Conjecture 1 Theorem 1 *is sharp for all positive integers p and m.*

Remark 4 If Theorem 1 is sharp for some p for a harmonic mapping f whose dilatation is z^{m} and satisfies otherwise the assumptions of the theorem, then it is sharp for the same p for the harmonic mapping $f \circ \varphi$, where $\varphi(z) = (z-a)/(1-\overline{a}z)$ and $|a| < 1$, whose dilatation is $[(z-a)/(1-\overline{a}z)]^m$ and satisfies otherwise the assumptions of the theorem.

We close the paper with the following:

Conjecture 2 *Let f be a sense-preserving harmonic mapping of $\mathbb{D}$ of the form $f = h+\overline{g}$, where h and g are analytic functions of $\mathbb{D}$ that satisfies the following properties:*

(a) *h' has $p-1$ zeros, counting multiplicity, in $\mathbb{D}$;*
(b) *the dilatation of f is the finite Blaschke product*

$$\omega(z) = \prod_{k=1}^{m} \frac{z-a_k}{1-\overline{a_k}z}, \quad |a_k| < 1,\ k = 1, 2, \ldots;$$

(c)

$$\mathrm{Re}\left\{1 + \mathrm{re}^{\mathrm{it}} \frac{h''(\mathrm{re}^{\mathrm{it}})}{h'(\mathrm{re}^{\mathrm{it}})}\right\} > -\frac{1}{2}\sum_{k=1}^{m} \frac{1-|a_k|}{1+|a_k|}$$

for real values r sufficiently close to 1.

Then the following hold:

(i) *$f = P \circ \phi$, where P is a polynomial of degree at most $m+p-1$, with the zeros of P' lying in $\overline{\phi(\mathbb{D})}$, and ϕ is a locally quasiconformal homeomorphism on $\mathbb{D}$ with dilatation $(1+|\omega|)/(1-|\omega|)$, and consequently f is at most $(m+p-1)$-valent on $\mathbb{D}$;*
(ii) *$\mathbb{C}\backslash\phi(\mathbb{D})$ is a union of P-rays with mutually disjoint interiors such that each starts at $\partial\phi(\mathbb{D})$.*

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