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Joint reserve stock and just-in-time inventory under regular preventive maintenance and random disruptions

Lama Moussawi-Haidar[a], Hoda Daou[a] and Khalil Khalil[b]

[a]Suliman Olayan School of Business, American University of Beirut, Beirut, Lebanon; [b]Maroun Semaan Faculty of Engineering, Department of Industrial Engineering, American University of Beirut, Beirut, Lebanon

ABSTRACT

We study a repairable production-inventory system with a constant demand rate subject to random disruptions, which supplies an assembly line according to a just-in-time configuration. Preventive maintenance (PM) actions are regularly performed on the production unit at fixed time instants. We assume that the time between breakdowns is constant and the PM and corrective actions have random durations that follow a general probability distribution. We propose the following policy to cope with production interruptions: A buffer stock is built up at the maximum production rate at the beginning of each PM cycle. At the same time, a reserve stock is maintained to hedge against random disruptions and shortages during planned maintenance. The joint convexity of the cost function is established and optimality conditions for the optimal policy parameters are derived. We perform numerical and sensitivity analyses under the exponential and Asymmetric Laplace distributions. The results suggest that: (i) The proposed policy performs better than traditional models that consider these two strategies separately; (ii) As the shortage cost increases, it is optimal to increase the reserve level; (iii) Higher reserve stock levels allow to hedge against the increased duration and rate of preventive maintenance and random disruptions.

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Inventory; preventive maintenance; random disruptions; reserve stock; buffer stock

1. Introduction

In the past decade, academics and practitioners in operations management have focussed on process unreliability and its impact on the operation of a manufacturing organisation, being a challenging issue facing many production systems (Moinzadeh and Aggarwal 1997, nov). The effect of unreliability on production systems has become increasingly pronounced with the Just-in-Time (JIT) manufacturing systems, which advocates minimising inventory and lean production processes. The JIT manufacturing philosophy has led to reduced levels of buffer inventories and increased levels of automation, making effective maintenance policies more important than ever.

Normal machine deterioration or unexpected external shocks can provoke machine breakdowns, which are important causes of a variability increase in process times and flows of production systems. To reduce the likelihood of machine breakdowns, preventive maintenance (also known as planned maintenance) that involves replacement of parts or tune-up of the machine can be scheduled to be performed periodically. PM can also improve production quality and yield by reducing machine breakdowns (Zequeira, Prida, and Valdés 2004). Research on preventive maintenance (PM) shows that implementing PM strategies for failure prone manufacturing systems is an effective way to extend their lives and reduce operating costs (Chelbi and Ait-Kadi 2004; Chelbi and Rezg 2006). Nevertheless, PM actions may render unavailable the production facility for the duration of the PM action. The unavailability of one machine may provoke the disruption of the entire production system. One way to avoid economic losses due to production unavailability caused by PM actions is to build up a buffer inventory in advance, when extra production capacity is available, to satisfy the demand during the unavailability period of the manufacturing facility. A buffer inventory is a temporary stock built just before the PM action, to be utilised when PM is performed.

Many authors have examined the conditions of building buffer stocks to guarantee continuous supply of the production unit during PM. A just-in-time inventory buffer has been considered by a wide variety of researchers (Salameh and Ghattas 2001; der Duyn

CONTACT Lama Moussawi-Haidar lm34@aub.edu.lb Suliman Olayan School of Business, American University of Beirut, Beirut 1107 2020, Lebanon

Schouten and Vanneste 1995; Chelbi and Ait-Kadi 2004; Chakrabarty, Roy, and Chaudhuri 2018; Iqbal, Huq, and Bhutta 2018). An implicit assumption common to preventive maintenance models discussed in the inventory literature is the continuous availability of supply. However, there are cases where this assumption may not hold. Since the product/raw material is usually purchased from another company, supply of the product may sometimes be interrupted due to random disruptions such as the supplier's equipment breakdowns, labour strikes or other randomly occurring events. Practitioners and researchers have concluded that the just-in-time philosophy popular in recent decades increases the vulnerability of supply chains to disruptions, since a tightly optimised lean design which performs well under normal situations, leaves little room for error when circumstances change drastically. For just-in-time manufacturers, losses from unpredictable events such as machine breakdowns, order cancellations, unscheduled maintenance, and labour strikes can be severe. For example, a strike at an auto parts may result in disruptions of supply to a car manufacturer such as GM or Toyota (Snyder et al. 2016; Parlar 1997). Other examples of the negative impact of process disruptions on the supply chain's operations can be found in the automobile and semi-conductor manufacturing industries (Groenevelt, Pintelon, and Seidmann 1992; Moinzadeh and Aggarwal 1997, nov; Chelbi and Ait-Kadi 2004). In these examples, when the outside supply can be cut off at random times for durations of random length, the product may be unavailable. Well-planned PM operations are usually much less costly than the costs associated with a sudden machine breakdown.

PM is not the only choice of remedy in the presence of machine breakdowns. A manufacturing organisation may maintain safety/reserve stocks of finished goods to hedge against random disruptions caused by both natural disasters and intentional and unintentional human actions (Snyder et al. 2016). Supply interruptions resulting from unpredictable events can produce adverse effects on the production/inventory system (Li, Xu, and Hayya 2004). The economic impact of disruptions can be enormous. In a series of empirical studies, Hendricks and Singhal (2003, 2005a, 2005b) find that companies that experienced even minor disruptions faced significant decline in sales growth, stock returns, and shareholder wealth, and these effects tend to linger for at least two years after the disruption. Thus it is extremely important to use mechanisms to hedge against disruptions. Research on random disruptions proposed safety stock policies as a way to hedge against random disruptions (Groenevelt, Pintelon, and Seidmann 1992; Moinzadeh and Aggarwal 1997, nov; Das and Sarkar 1999; Akella and Kumar 1986; Gharbi, Kenne, and Beit 2007; Cheung and Hausman 1997; Parlar 1997; Dohi, Okamura, and Osaki 2001).

Most of the past research on maintenance policies ignored the impact of preventive maintenance with random disruptions. This paper investigates the joint implementation of a buffer stock and safety/reserve stock in an unreliable production environment. A buffer stock is inventory built up temporarily prior to conducting preventive maintenance, with the aim to satisfy demand during corrective actions, while a reserve stock is safety inventory maintained continuously, with the aim to hedge against random interruptions.

This research presents a preventive maintenance model for a manufacturing system with a constant demand and subject to random disruptions, that operates according to a just-in-time configuration. Similarly to (Chelbi and Ait-Kadi 2004; Meller and Kim 1996; Cheung and Hausman 1997; Chelbi and Ait-Kadi 2004; Radhoui, Rezg, and Chelbi 2009), and in compliance with the machine manufacturer recommendations, preventive maintenance (PM) actions are regularly performed on the production unit at fixed time instants, aiming at reducing the likelihood of breakdowns. This policy is referred to as the Block Replacement Policy (BRP) and adopted in several research works in the literature.

The rest of the paper is organised as follows: The next section presents a review of the literature. In Section 3, the mathematical model is developed, expressing the total expected cost per unit time, and the joint convexity is established. The fourth section is dedicated to the numerical and sensitivity analysis. Managerial insights are presented in Section 4.6. A conclusion is provided in Section 5, with indications about possible extensions of the current model.

2. Literature review

Preventive maintenance is an essential factor in production systems that aims to reduce the likelihood of failure resulting from deterioration and age (Yao et al. 2005; Rezg, Dellagi, and Chelbi 2008). It has been a significant topic of research in the field of Production and Operations Management (Manikas et al. 2020). Zheng et al. (2020) indicate that production control in manufacturing companies is the process most often investigated inn the field of Operations Management. Different types of maintenance models have been considered in the literature on production control. Preventive maintenance models fall into different types of maintenance policies (Wang 2002); Examples include age dependent (Gan et al. 2013), inventory level dependent

(Kaican Kang 2018) and condition based (Xiang, Cassady, and Pohl 2012) policies. Joint policies are also considered, taking into account multiple factors simultaneously such as system's age and inventory level (Yao et al. 2005).

Implementing maintenance in a production environment results in a shutdown of the production process for the duration of the maintenance. During such pause, a buffer stock is needed to adjust the market demand (Sana 2012) and make sure normal operations will not be interrupted (Salameh and Ghattas 2001). Models considering preventive maintenance of systems with safety stock have been extensively developed in the literature. The hedging point policy (HPP) concept (Akella and Kumar 1986) involves identifying the optimal inventory level, while the machine is operational to respond to customer demands when the machine is down. HPP is first proven for the case of one-machine one-product type manufacturing system with constant demand rate (Akella and Kumar 1986), then extended to consider stochastic demand (Pal, Sana, and Chaudhuri 2013), two-product manufacturing system (Assid, Gharbi, and Hajji 2015) and multi-components with a shared pool of spares (Keizer, Teunter, and Veldman 2017). The average consumption during the repair periods and planned preventive maintenance are also considered by Chelbi and Ait-Kadi (2004), where authors propose to build a buffer stock to cover at least the average demand during repair periods following breakdowns. Bahria et al. (2019) consider quality of produced lots and integrate it into the joint control of production and maintenance. Guo and Gu (2020) investigate the joint decision making of production and maintenance policies in mixed models assembly systems with delayed differentiation, and formulate the problem as a Markov Decision Process. The optimal decisions are when to perform maintenance on each machine and which product variant should be selected to produce. Hajej et al. (2020) study the optimal integrated production and maintenance strategy for a wind farm to determine the optimal plan of energy production characterised by the number of working wind turbines and their production rates along with the optimal preventive maintenance policy. Hajej, Rezg, and Gharbi (2020) study the case of returned used products in a lease contract with and without maintenance monitoring option, and its influence on the production and maintenance strategies for a remanufacturing system. Hajej and Rezg (2020) consider an integrated lot sizing model for multi-machines producing a single item under preventive maintenance. These aforementioned papers study preventive maintenance policies, but ignore random interruptions, which is considered in the current paper.

Unreliable production environment can cause random breakdowns and failures of the production process and thus buffer stock can help palliate against such interruptions. Random machine breakdowns are considered by Cheung and Hausman (1997) in the Economic Manufacturing Quantity (EMQ) model for the purpose to develop optimal strategies for preventive maintenance and safety stocks. This model assumes the failure rate of the machine to be increasing in time with no specific restriction regarding the distribution of time to machine breakdown. It is extended by Dohi, Okamura, and Osaki (2001) by taking more restrictive assumptions with the lifetime of the production obeying an exponential distribution; a similar assumption is taken by Groenevelt, Pintelon, and Seidmann (1992) in their production control policy under general randomly distributed repair times. Joint implementation of preventive maintenance and safety stocks are also considered by Gharbi, Kenne, and Beit (2007), where the authors take into account backorders (i.e. unmet demands during the repair period are not lost) and with possible failure at any age of the machine. Unscheduled shutdown of the production process can also be triggered by random supply interruptions. Snyder et al. (2016) present a survey of supply chain disruptions, focussing on different models of disruptions, impact of disruptions and mitigation strategies. Obaidat and Liao (2020) develop an integrated model for sampling and proactive maintenance, which utilises the occurrence of a false alarm as a proactive maintenance opportunity in addition to corrective and scheduled maintenance. Rebaiaia and Aitkadi (2020) proposes an approach for comparing numerically three maintenance strategies, with minimal repairs at failure, and consider preventive and corrective maintenance actions. Similarly to this stream of research, our model studies both preventive and corrective maintenance actions; However, our model considers random interruptions caused not only by random breakdowns as assumed in the literature, but by any other randomly occurring interruptions, and studies the impact of jointly holding a reserve stock to withstand random interruptions, as well as buffer stock during preventive maintenance actions. To the best of our knowledge, none of the existing models have proposed using reserve stock to hedge against random interruptions including machine breakdowns.

A JIT inventory buffer has been considered by a wide variety of researchers (Salameh and Ghattas 2001; der Duyn Schouten and Vanneste 1995; Chelbi and Ait-Kadi 2004; Bevilacqua, Ciarapica, and Sanctis 2017). Lean production is investigated in an unreliable environment and a survey is conducted to highlight various contingency factors such as production systems, order

types and product types for possible failure of the system (Demeter and Matyusz 2011). Salameh and Ghattas (2001) develop a model that determines the optimal JIT buffer level by minimising holding and shortage costs. Zequeira, Prida, and Valdés (2004) extend this work by considering the issue of imperfect production and the dependency of the duration of the maintenance action on the state of the production facility. The same authors apply another extension to the model by considering several aspects of the operation of production systems whose behaviour is random (Zequeira, Valdes, and Berenguer 2008). They propose two machines and one buffer stock in between, and a PM policy based on the age of the machine and buffer size. Just-for-Peak inventory, for multiple machines in electricity production, is also considered and applied similarly to JIT buffer in the work of (Fernandez, Li, and Sun 2013). Another extension to the work of Salameh and Ghattas (2001) assumes more realistic assumptions by developing an age-dependent optimisation model for the joint implementation of preventive maintenance and safety stocks (Kenné, Gharbi, and Beit 2007). Souiden, Nyounguè, and Rezg (2017) propose another extension to the work of Salameh and Ghattas (2001) by taking into consideration different parts in a production system such as production workstation, transport and buffer stock and tries to minimise the total costs associated with the production and maintenance. Similarly to Salameh and Ghattas (2001), this work proposes a preventive maintenance model for a manufacturing system that operates according to a just-in-time configuration; However, we introduce buffer reserve stock to reduce shortage during restoration actions and cope with random disruptions. A detailed description of the proposed model is presented in the next section.

3. Model

This section describes an analytical model for the proposed policy, that takes into account the probability distributions of PM and repair durations, as well as the renewal processes associated with the PM and random disruption cycles of the production unit.

3.1. Model overview

This paper proposes a preventive maintenance model for a manufacturing system with a constant demand, subject to random disruptions, that operates according to a just-in-time configuration. Similarly to (Chelbi and Ait-Kadi 2004; Meller and Kim 1996; Cheung and Hausman 1997; Chelbi and Ait-Kadi 2004; Radhoui, Rezg, and Chelbi 2009), and in compliance with the machine manufacturer recommendations, preventive maintenance (PM) actions are regularly performed on the production unit at fixed time instants, aiming at reducing the likelihood of breakdowns. The machine could stop producing either following random disruption such as machine breakdowns, or to undergo regular PM actions scheduled at regular time instants. This works assumes that (1) the unitary maintenance costs are known and constant (Chelbi and Ait-Kadi 2004), (2) the times between breakdowns and PM actions are constant, and (3) the PM and corrective actions have random durations that follow a general probability distribution, as in Groenevelt, Pintelon, and Seidmann (1992).

Stock-outs result when the PM duration exceeds the buffer and reserve stock supply time, or when the random disruption time exceeds the reserve supply time. According to the hedging point policy (HPP), introduced by Akella and Kumar (1986) and adopted by several research papers (Dhouib, Gharbi, and Aziza 2012; Gharbi, Kenne, and Beit 2007), a reserve stock is built up at a maximum production rate in order to palliate for interruptions. Once the reserve stock is built, the production rate is reduced in order to just satisfy the demand rate. This reserve stock is put in place to reduce shortage during restoration actions and cope with random disruptions. Particularly, the critical reserve stock level R^* in our model is characterised as follows: If the current inventory level is less than the optimal level R^*, an instantaneous replenishment brings the level back to R^*; if exactly equal, one should produce exactly enough to meet demand, i.e. production rate is equal to the demand rate; if the current inventory level exceeds the optimal, one should not produce at all. An instantaneous replenishment model is adopted; such model assumes an infinite production rate to bring the inventory level back to its level R. Akella and Kumar (1986) have shown that the HPP policy is optimal for systems prone to failures and subject to corrective maintenance.

The following inventory policy is proposed to cope with production stoppage during PM and random interruptions: Prior to each PM cycle, an inventory buffer stock Z^* is built up at the maximum production rate α to satisfy the demand during the PM periods. Furthermore, a reserve stock R is maintained, following Akella and Kumar (1986)'s hedging point policy (HPP), to cope with random disruptions such as machine breakdowns or supply interruptions, and/or shortages during planned maintenance when the PM duration exceeds the buffer supply time.

In what follows, expressions for the expected stock-out time, average on-hand inventory, expected cycle time, and expected total cost per unit time, are developed.

3.2. Analytical model

Before the start of PM actions at time T_M, a buffer stock of level Z is built at the maximum production rate α units per unit time, to ensure the continuous supply of the assembly line and cover at least demand with rate β during the PM periods. The finite replenishment starts Z/α units of time before the beginning of T_M, so that a buffer of level Z is built up. The regular preventive maintenance starts and lasts for a period of T_M units of time. T_M is a random variable with a probability distribution function $f(.)$, and depends on how long it takes to enhance the condition of the production unit to an acceptable level that prevents any random failures, and maintains the same quality of output as before. During T_M, demand is met from the JIT buffer at the rate β per unit per unit time. Three cases are observed as illustrated in Figure 1: (a) The PM duration is less than the buffer supply time, Z/β. In this case, demand is satisfied from the buffer

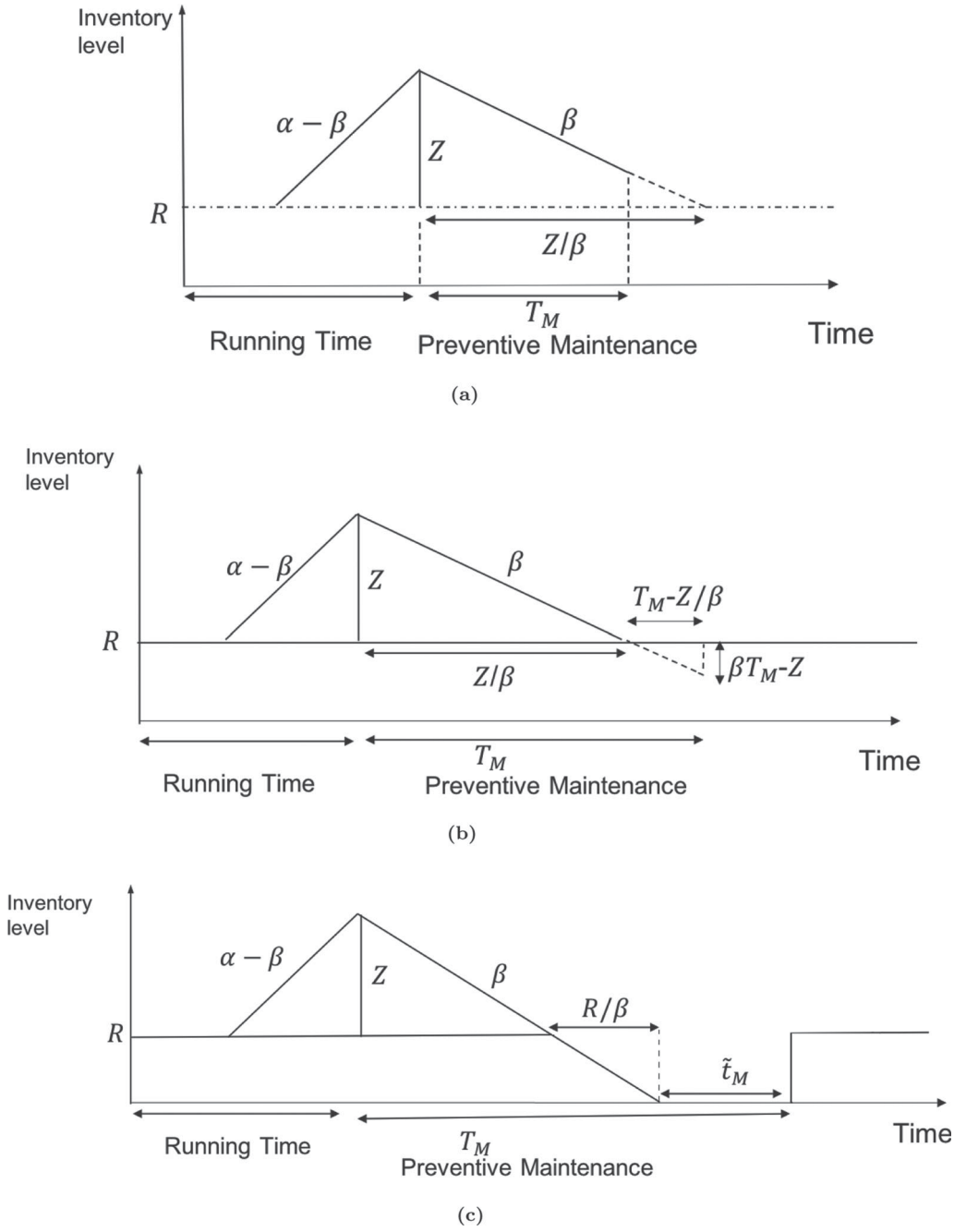

Figure 1. Just in time buffer inventory during preventive maintenance under three different cases for T_M. (a) $T_M < \frac{Z}{\beta}$. (b) $\frac{Z}{\beta} < T_M < \frac{Z+R}{\beta}$. (c) $\frac{Z+R}{\beta} < T_M$.

inventory and no shortage occurs; (b) The PM duration is greater than the buffer supply time, but less than the reserve supply time. In this case, demand is satisfied from the reserve stock, and no shortage occurs. Here, the hedging point policy (HPP) is adopted to bring the reserve stock back to its optimal level R; (c) The PM duration is greater than the reserve supply time. In this case, shortage occurs. At the end of the PM period, the production process resumes.

Once the production process is down due to random disruptions, it can be restored to full production capability within a random amount of time of length T_R assumed to follow a probability distribution function $g(\cdot)$. While the system is being restored, excess demand is satisfied from a reserve/safety inventory of level R. As illustrated in Figure 2, two cases are observed: (a) The random breakdown with duration T_R is less than the reserve supply time R/β. In this case, no shortage occurs and demand is satisfied from the reserve stock; (b) The random breakdown duration T_R is greater than the reserve supply time R/β. In this case, shortage of duration $\tilde{t_R}$ occurs before an instantaneous replenishment brings the inventory level back to R.

The following assumptions have been used in developing the model:

Assumptions

- PM actions are performed at regular instants as in Chelbi and Ait-Kadi (2004).
- PM and repair duration probability distributions are known.
- Failures are detected instantaneously.

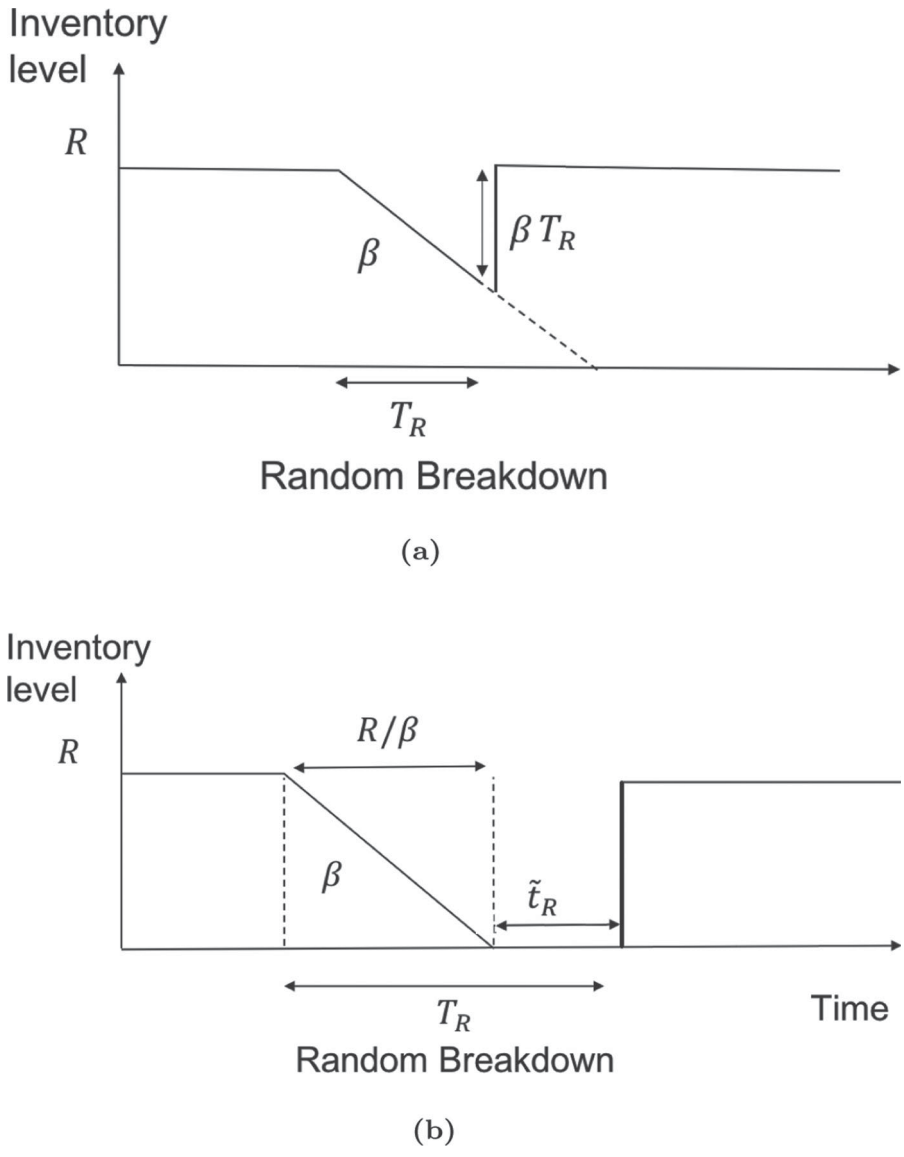

Figure 2. Reserve Stock during random breakdowns under two different cases for T_R. (a) $T_R < \frac{R}{\beta}$. (b) $\frac{R}{\beta} < T_R$.

- The PM duration is a random variable and follows a general distribution function.
- The duration of a breakdown/corrective action is a random variable.
- The time between breakdowns is constant (constant failure rate; Groenevelt, Pintelon, and Seidmann 1992).
- PM actions restore the equipment to a state as good as new.
- Maintenance and corrective unitary costs are known and constant.
- The preventive maintenance instants are planned in advance and cannot be changed.

The objective is to jointly determine the optimal levels of just-in-time buffer and reserve stock to protect against two types of interruptions: (1) scheduled preventive maintenance and (2) random disruption due to a breakdown of the production unit and/or supply interruption.

The following notation is used throughout the remainder of the paper:

R Reserve stock level, decision variable.
y Amount produced for preventive maintenance per cycle.
Z Level of just-in-time buffer at the beginning of preventive maintenance (PM) per PM cycle.
T_M Duration of the preventive maintenance per cycle, random variable with probability density function $f(T_M)$ and cumulative density function $F(t) = \int_0^t f(x)dx$. In the numerical examples, $f(T_M)$ is taken to follow an exponential distribution with mean λ_M.
T_R Duration of the random breakdown per cycle, random variable with probability density function $g(T_R)$ and cumulative density function $G(t) = \int_0^t g(x)dx$. In the numerical examples, $g(T_R)$ is taken to follow an exponential distribution with mean λ_R.
$\tilde{t}_M$ Stockout time due to preventive maintenance per cycle.
$\tilde{t}_R$ Stockout time due to random breakdown.
α Production rate (units/unit time).
β Consumption rate (units/unit time).
h_M Holding cost of preventive maintenance just-in-time buffer per unit per unit time.
h_R Holding cost of reserve stock per unit per unit time, $h_M > h_R$.
ρ Shortage cost per unit time.
γ Average time between random breakdowns per year.
ω Time between beginning of scheduled preventive maintenance per year.
n The number of preventive maintenance per unit time, $n = 1/\omega$.
λ The scale parameter of the Asymmetric Laplace (AL) distribution.
κ The asymmetry parameter of the AL distribution.
m The location parameter of the AL distribution.
K Setup cost of preparing the production system for preventive maintenance ($/preventive maintenance).

The expected total cost per unit time can be expressed as a function of the reserve stock level R and number of just-in-time buffer units produced for preventive maintenance, y:

$$\begin{aligned} TCU(y,R) = {} & \text{setup cost of preparing the production system for preventive maintenance per unit time} \\ & + \text{holding cost of just-in-time buffer per unit time} \\ & + \text{holding cost of reserve stock} \\ & + \text{shortage cost due to preventive maintenance per unit time} \\ & + \text{shortage cost due to random breakdown per unit time} \end{aligned} \quad (1)$$

3.2.1. Shortage time due to preventive maintenance

Shortage time due to preventive maintenance $\tilde{t}_M$ can be expressed as:

$$\tilde{t}_M = \begin{cases} 0, & \text{if } T_M < \dfrac{Z+R}{\beta}, \\ T_M - \dfrac{Z+R}{\beta}, & \text{if } T_M \geq \dfrac{Z+R}{\beta} \end{cases} \quad (2)$$

The expected value of $\tilde{t}_M$ is:

$$E(\tilde{t}_M) = \int_{\frac{Z+R}{\beta}}^{\infty} \left[t - \frac{Z+R}{\beta} \right] f(t)\, \mathrm{d}t \quad (3)$$

The expected shortage time due to preventive maintenance, per unit time, is equal to $nE(\tilde{t}_M)$ where $n = \frac{1}{\omega}$ is the number of preventive maintenance per unit time.

3.2.2. Shortage time due to random breakdown can be expressed as:

$$\tilde{t}_R = \begin{cases} 0, & \text{if } T_R < \dfrac{R}{\beta}, \\ T_R - \dfrac{R}{\beta}, & \text{if } T_R \geq \dfrac{R}{\beta}. \end{cases} \quad (4)$$

Expected value of $\tilde{t}_R$ is:

$$E(\tilde{t}_R) = \int_{\frac{R}{\beta}}^{\infty} \left[t - \frac{R}{\beta} \right] g(t)\, \mathrm{d}t \tag{5}$$

The expected shortage time due to random breakdown, per unit time, is equal to $\frac{E(\tilde{t}_R)}{\gamma}$, where γ is the average time between random breakdowns per year.

3.2.3. Preventive maintenance buffer

Before the start of preventive maintenance, just-in time (*JIT*) buffer is produced at a rate α units per unit time with $\alpha > \beta$ and thus increasing the level of the buffer inventory at the rate $\alpha - \beta$ units per unit time.

$$\begin{aligned} A_{JIT} &= \frac{y^2}{2} \left[\frac{\alpha - \beta}{\alpha\beta} \right] \\ &= \frac{z^2}{2} \left[\frac{\alpha}{\beta(\alpha - \beta)} \right] \end{aligned} \tag{6}$$

The average buffer inventory is equal to nA_{JIT} per preventive maintenance and hence per unit time.

3.2.4. Drop in reserve stock: preventive maintenance and random breakdowns

The drop in buffer stock level S during preventive maintenance, defined as $A_M(T_M)$, can be expressed as:

$$A_M(T_M) = \begin{cases} 0, & \text{if } T_M \le \dfrac{Z}{\beta}, \\ \dfrac{\beta T_M - Z}{2} \left[T_M - \dfrac{Z}{\beta} \right], & \text{if } \dfrac{Z}{\beta} < T_M \le \dfrac{Z+R}{\beta}, \\ \dfrac{R}{2} \left[2\tilde{t}_M + \dfrac{R}{\beta} \right], & \text{if } T_M > \dfrac{Z+R}{\beta}. \end{cases} \tag{7}$$

Using Equation (2) for the value of $\tilde{t}_M$, $A_M(T_M)$ is now equal to:

$$A_M(T_M) = \begin{cases} 0, & \text{if } T_M \le \dfrac{Z}{\beta}, \\ \dfrac{(\beta T_M - Z)^2}{2\beta}, & \text{if } \dfrac{Z}{\beta} < T_M \le \dfrac{Z+R}{\beta}, \\ -\dfrac{R^2}{2\beta} - \dfrac{ZR}{\beta} + RT_M, & \text{if } T_M > \dfrac{Z+R}{\beta}. \end{cases} \tag{8}$$

The drop in buffer stock level S during random breakdown, defined as $A_R(T_R)$, can be expressed as:

$$A_R(T_R) = \begin{cases} \dfrac{\beta T_R^2}{2}, & \text{if } T_R \le \dfrac{R}{\beta}, \\ \dfrac{R}{2} \left[2\tilde{t}_R + \dfrac{R}{\beta} \right], & \text{if } T_R > \dfrac{R}{\beta}. \end{cases} \tag{9}$$

Substituting the value of $\tilde{t}_R$ with Equation (4), $A_R(T_R)$ is now equal to:

$$A_R(T_R) = \begin{cases} \dfrac{\beta T_R^2}{2}, & \text{if } T_R \le \dfrac{R}{\beta}, \\ T_R R - \dfrac{R^2}{2\beta}, & \text{if } T_R > \dfrac{R}{\beta}. \end{cases} \tag{10}$$

Using Equations (8) and (10), the expected drops in reserve stock during preventive maintenance and random breakdown, the expected values of A_M and A_R respectively, are equal to:

$$\begin{aligned} E(A_M) = &\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} \frac{(\beta t - Z)^2}{2\beta} f(t)\, \mathrm{d}t \\ &+ \int_{\frac{Z+R}{\beta}}^{\infty} \left[-\frac{R^2}{2\beta} - \frac{ZR}{\beta} + Rt \right] f(t)\, \mathrm{d}t \end{aligned} \tag{11}$$

The expected drop in reserve stock, due to preventive maintenance, per year is equal to $R - nE(A_M)$.

$$E(A_R) = \int_0^{\frac{R}{\beta}} \frac{\beta t^2}{2} g(t)\, \mathrm{d}t + \int_{\frac{R}{\beta}}^{\infty} \left(Rt - \frac{R^2}{2\beta} \right) g(t)\, \mathrm{d}t \tag{12}$$

with:

$$\int_0^{\frac{R}{\beta}} g(t)\, \mathrm{d}t = P\left(T_R \le \frac{R}{\beta} \right) = G\left(\frac{R}{\beta} \right), \quad \text{and} \tag{13}$$

$$\int_{\frac{R}{\beta}}^{\infty} g(t)\, \mathrm{d}T_R = P\left(T_R > \frac{R}{\beta} \right) = \bar{G}\left(\frac{R}{\beta} \right) \tag{14}$$

Equation (12) is now equal to:

$$E(A_R) = \frac{\beta t^2}{2} G\left(\frac{R}{\beta} \right) + \left(Rt - \frac{R^2}{2\beta} \right) \bar{G}\left(\frac{R}{\beta} \right) \tag{15}$$

Given γ as the average time between breakdowns per year, the expected drop in reserve stock, due to random breakdown, per year is equal to $R - \frac{E(A_R)}{\gamma}$.

Having defined all terms in Equation (1), this equation can be expressed as:

$$\begin{aligned} TCU(Z,R) = &nK + h_M nA_{JIT} \\ &+ h_R \left[R - nE(A_M) - \frac{E(A_R)}{\gamma} \right] \end{aligned}$$

$$+ \rho \left[nE(\tilde{t}_M) + \frac{E(\tilde{t}_R)}{\gamma} \right] \tag{16}$$

Lemma 3.1: *The total cost formula TCU(Z, R) in* (16) *is convex in Z* $\forall$ $R > 0$.

Lemma 3.2: *The total cost formula TCU(Z, R) in* (16) *is convex in R* $\forall$ $Z > 0$.

Lemma 3.3: *The total cost formula TCU(Z, R) in* (16) *is jointly convex in both Z and R.*

The proofs of Lemmas 3.1–3.3 are in the Appendix.

The proposed model assumes the durations of preventive maintenance and random breakdowns to be independent and identically distributed (iid) random variables following the exponential or Asymmetric Laplace distributions, which ensures ergodicity and stationarity of the problem. In this case, the optimal policy parameters depend only on the current state variables (Szeidl 2002). In our model, we assume that a stationary process and iid random variables, which would ensure ergodicity and therefore the convexity result of the cost function. A natural modification is to consider general distributions, in which case ergodicity is not ensured and the convexity result may not hold anymore.

This paper proposes a novel inventory policy that advocates the joint consideration of just-in-time inventory buffer and a reserve stock policy in the presence of preventive maintenance with random disruptions. Traditionally, these two strategies have been considered separately. Thus, in the literature, two models have been studied separately: The first model considers optimising a just-in-time buffer inventory (Lemma 3.1 above), assuming a given level of reserve stock that the manufacturing organisation maintains to withstand random disruptions; The second model considers optimising the reserve/safety stock level, assuming a given level of just-in-time buffer to cope with planned maintenance (Lemma 3.2 above). This paper jointly optimises the buffer and reserve stock levels, Z and R (Lemma 3.3). The optimal levels of buffer and reserve inventory levels Z^* and R^* minimise the expected total inventory cost in (16), thus yielding a lower expected total cost than when considering the just-in-time inventory buffer or reserve stock each separately.

Comparative Analysis: Special Case $R = 0$

When setting $R = 0$, the model of Salameh and Ghattas (2001) is recovered as a special case of our model. In this case, the total cost function is reduced to the below equation:

$$TCU(Z) = nK + h_M n \frac{Z^2}{2} \left[\frac{\alpha}{\beta(\alpha - \beta)} \right]$$

$$+ \rho n \int_{\frac{Z}{\beta}}^{\infty} \left[t - \frac{Z}{\beta} \right] f(t)\, \mathrm{d}t \tag{17}$$

With reference to the paper of Salameh and Ghattas (2001), the total cost function obtained reduces to their work using the following relations:

- Z (Just in time reserve) is taken as S.
- K (Setup cost) is taken to be 0.
- n is equivalent to $E(\frac{1}{(T+t)})$.
- Stock-up rate: $\alpha - \beta$ is taken as α (including consumption rate). Thus, $h_M n \frac{z^2}{2} [\frac{\alpha}{\beta(\alpha-\beta)}]$ is equivalent to $h\frac{S^2}{2}[\frac{\alpha+\beta}{\alpha\beta}]$ in Salameh and Ghattas (2001).
- Shortage cost per unit time ρ is equivalent to shortage cost per unit short multiplied by demand rate $\rho\beta$.

Therefore, the total cost formula obtained in Equation (17) is equivalent to the following:

$$TCU(S) = h\frac{S^2}{2} \left[\frac{\alpha + \beta}{\alpha\beta} \right] + \frac{\rho\beta}{T + E(t)} \int_{\frac{S}{\beta}}^{\infty} \left[t - \frac{S}{\beta} \right] f(t)\, \mathrm{d}t \tag{18}$$

Section 4 presents numerical proofs that random disruptions such as supply interruptions have significant impact on the inventory system. Thus, it is better to model such an uncertainty than to ignore it. Our model yields a lower expected total cost rate when both Z and R are accounted for. Models that assume no reserve level exhibit higher expected total cost rate. Hence, manufacturing organisations can recover significant cost savings by incorporating joint decisions into its inventory planning. Next, numerical and sensitivity analysis are performed to illustrate the effect of important model parameters on the optimal decisions and cost rate.

4. Numerical analysis

This section presents sensitivity analysis of the optimal just-in-time buffer, Z, and reserve stock level, R, and the total cost rate, TCU, with respect to changing some key model parameters including the shortage cost, holding cost of JIT buffer and reserve stock, durations and rates of preventive maintenance and random breakdowns. A set of parameter values are considered, given in Table 1 below.

In the numerical examples, it is assumed that the duration of random breakdown, T_R follows the exponential distribution with average rate of random breakdown $\lambda_R = 1/10$. Similarly, it is assumed that the duration of preventive maintenance, T_M, follows the exponential distribution with average rate of PM $\lambda_M = 1/8$. For the set

Table 1. Base Problem Parameters.

Parameter	Value
α	5, 000 units/unit time
β	4, 500 units/unit time
h_R	\$2.2/unit/unit time
h_M	\$2.8/unit/unit time
ρ	\$30,000/unit time
$E(T_M)$	8 days/breakdown
$E(T_R)$	10 days/breakdown
γ	1 year
ω	0.5 year
K	\$250/cycle

Table 2. Optimal cost for different models of reserve and JIT buffer inventory.

	R	Z	TCU
Optimal Policy	67	201	**\$1,208**
Optimal $Z, R = 0$	0	177	\$1,735
Optimal $R, Z = 0$	237	0	\$1,246
$R = 0, Z = 0$	0	0	\$2,637

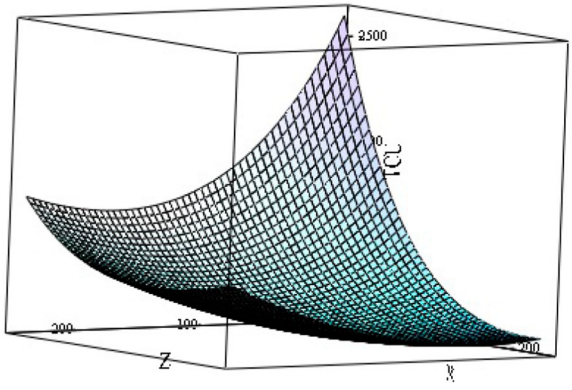

Figure 3. Joint convexity plot of the total cost rate with respect to Z and R.

of parameters in Table 1, the optimal values of the just-in-time buffer, reserve level R, and total cost are as follows: $Z^* = 67$ units, $R^* = 201$ units, and $TCU = 1{,}208$\$, as reported in the first row of Table 2. The convexity plot of the total cost rate with respect to Z and R is depicted in Figure 3.

Next, the performance of the proposed model is compared with those existing in the literature. Classical models in the literature of preventive maintenance and random disruptions have considered the JIT buffer and reserve stock separately. None of those models have studied the joint determination of the JIT buffer and reserve stock, a common practice of manufacturing organisations that are susceptible to disruptions. The results of this comparison are reported in Table 2.

Table 2 indicates that, if only just-in-time buffer inventory is maintained but not reserve stock ($R = 0$), the optimal JIT buffer stock $Z^* = 177$ units and optimal total cost increases to \$1, 735. On the other hand, if only reserve stock is maintained, but not just-in-time buffer, ($Z = 0$), the optimal reserve stock $R^* = 237$ and optimal cost also is \$1, 246. When neither inventory is kept, the optimal total cost is the highest and is equal to \$2, 637. The optimal cost is the lowest when our proposed inventory strategy is utilised, i.e. when both JIT buffer and reserve stock are used to satisfy the demand during preventive maintenance or when the duration of preventive maintenance exceeds the buffer supply time, and to hedge against random disruptions. The following section shows the effect of varying the shortage cost on the optimal values of the buffer inventory and reserve levels, and optimal cost.

4.1. Varying the shortage cost

One of the main incentives to hold a reserve stock is to reduce shortage costs incurred due to random disruptions such as breakdowns and/or supply interruptions. Thus, sensitivity analysis is performed on the value of shortage cost per unit time. For different values of the shortage cost, the optimal values of the total cost, just-in-time buffer, and reserve supply are evaluated and shown in Figures 4 and 5. Figure 4 shows a direct relationship between the increase in shortage cost and the total cost of the system. As the shortage cost increases, the optimal cost increases. As for the levels of just-in-time reserve supply, Figure 5 indicates the following observations: For low shortage cost, R is minimum while Z is at its maximum level. Therefore, it is more feasible not to hold reserve supply for low values of the shortage cost. As the shortage cost increases, the level of R increases while that of Z decreases till it reaches a certain limit. Thus, holding a reserve stock is beneficial to reduce the effects of shortage costs incurred due to random interruptions. In this case, the benefit is realised by increasing the level of reserve stock while limiting the buffer inventory. In brief, as the shortage cost increases, manufacturing organisations should consider carrying more reserve inventory.

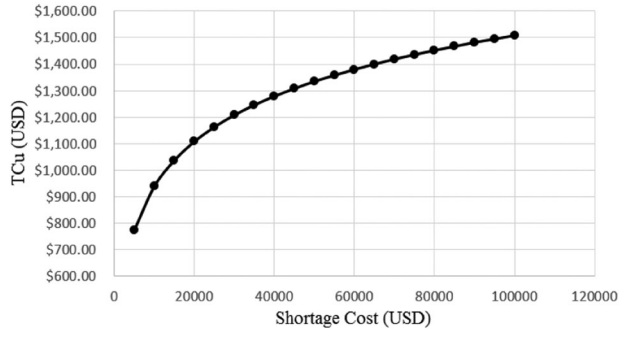

Figure 4. Sensitivity of TCU with respect to the shortage cost.

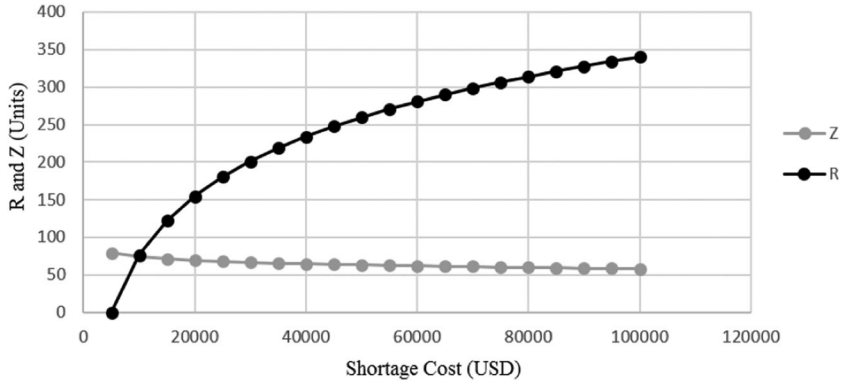

Figure 5. Sensitivity of R and Z with respect to the shortage cost.

4.2. Varying the holding costs of JIT buffer and reserve stock

In this subsection, sensitivity analysis is performed as the holding costs of the JIT buffer, h_M, and reserve stock, h_R, change. The results are shown in Figures 6 and 7. First, the values of h_R are varied from 2.8 to \$8.8/unit/unit time, by increments of 1, keeping $h_M = \$8.8$/unit/unit time constant. The optimal value of Z increases from $Z = 33$ when $h_R = 2.8$, to $Z = 91$ when $h_R = 8.8$, while the optimal R decreases from $R = 192$ for $h_R = 2.8$ to $R = 38$ for $h_R = 8.8$ (Figure 6). The optimal total cost increases from \$2,852 till \$3,443 when increasing h_R from 2.8 to 8.8, keeping h_M constant. In addition, Figure 6 shows that as h_R increases, the optimum level of reserve supply decreases while that of just-in-time buffer inventory increases. This result is affected by the high holding costs incurred with the increase in h_R. In this case, increasing

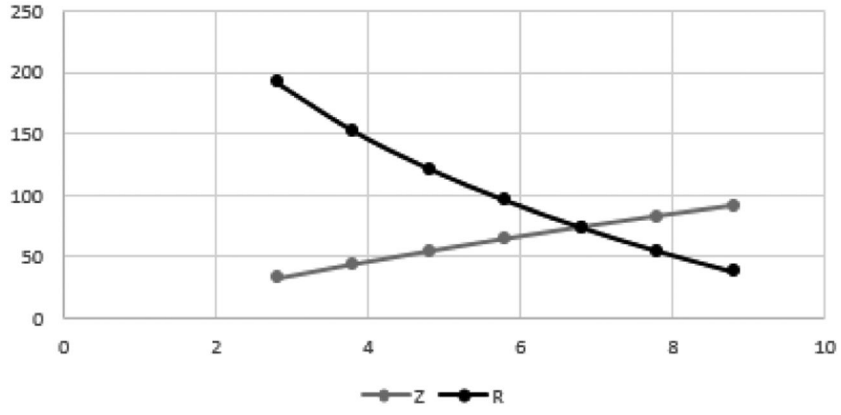

Figure 6. Sensitivity of R and Z with respect to h_R.

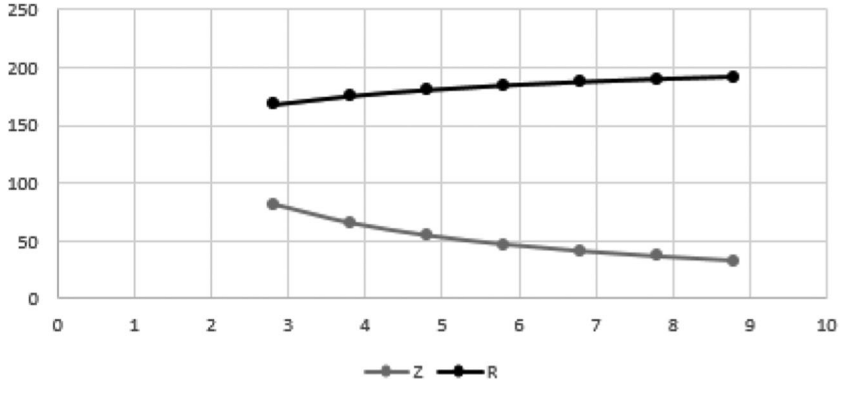

Figure 7. Sensitivity of R and Z with respect to h_M.

the just-in-time buffer inventory becomes more beneficial.

Next, the values of the holding cost of the JIT buffer, h_M, are varied from 2.8 to \$8.8/unit/unit time by increments of 1, keeping $h_R = 2.8$ constant. The optimal value Z decreases from $Z = 82$ when $h_M = 2.8$ to $Z = 33$ when $h_M = 8.8$, while the optimal R increases from $R = 168$ for $h_M = 2.8$ to $R = 192$ for $h_M = 8.8$ (Figure 7). As illustrated by Figure 7, as h_M increases, the optimal level of just in time inventory decreases while that of reserve supply increases asymptotically. Thus, increasing the reserve stock while reducing the JIT is optimal in this case. The optimal total cost increases as well from \$2, 816 till \$2, 852. The levels of Z and R are less sensitive to h_M and more sensitive to a change in the holding cost of reserve stock, h_R.

4.3. Varying the expected durations of preventive maintenance and random breakdown

This section studies the impact of varying the uncertain durations of preventive maintenance and random breakdown on the optimal policy decisions and optimal cost. First, the values of the average duration of random breakdown, $E(T_R)$, are varied. For each value of $E(T_R)$, the optimal values of Z, R, and TCU are obtained (this corresponds to the 0% deviation row in Table 3 below). Then, the values of Z and R are increased (decreased) by increments of 10% from their optimal values, computing the corresponding cost each time. Table 3 illustrates the results when $E(T_R) = 5, 10$, and 15. It is observed that as the duration of random breakdown increases, the total cost increases, JIT buffer inventory level decreases, while the reserve stock level increases. As holding JIT buffer inventory is more expensive, it is optimal to reduce the JIT inventory while raising the level of reserve stock to offset the increasing uncertainty caused by longer interruption periods. Also, the more Z and R deviate from their optimal values, the higher the cost is.

Table 4 illustrates the impact of varying the expected duration of preventive maintenance. Three different cases are considered: $E(T_M) = 6$, 8, and 10. It is observed that as the duration of preventive maintenance period increases, the total cost increases, JIT buffer inventory level increases, and the reserve stock level increases. As holding JIT inventory is more expensive, it is optimal to increase the JIT inventory while increasing the reserve stock to cope with the increasing uncertainty caused by longer preventive maintenance periods. Also, the more Z and R deviate from their optimal values, the higher the cost is. In summary, higher levels of reserve stocks are needed to hedge against longer preventive maintenance durations and/or longer disruption periods.

Table 3. Variation of optimal Z, R, and total cost with respect to the average duration of breakdown $E(T_R)$.

	$E(T_R) = 5$			$E(T_R) = 10$			$E(T_R) = 15$		
% Deviation	Z	R	TCU	Z	R	TCU	Z	R	TCU
50%	155	188	1,120	101	302	1,310	64	398	1,537
40%	145	175	1,088	94	282	1,277	60	372	1,502
30%	135	163	1,061	87	262	1,249	55	345	1,472
20%	124	150	1,040	81	241	1,228	51	319	1,449
10%	114	138	1,026	74	221	1,213	47	292	1,434
0%	103	125	1,021	67	201	1,208	43	265	1,428
−10%	93	113	1,027	60	181	1,214	38	239	1,435
−20%	83	100	1,046	54	161	1,234	34	212	1,456
−30%	72	88	1,080	47	141	1,272	30	186	1,497
−40%	62	75	1,134	40	121	1,332	26	159	1,562
−50%	52	63	1,212	34	101	1,421	21	133	1,658

Table 4. Variation of optimal Z, R, and total cost with respect to the average duration of preventive maintenance $E(T_M)$.

	$E(T_M) = 6$			$E(T_M) = 8$			$E(T_M) = 10$		
% Deviation	Z	R	TCU	Z	R	TCU*	Z	R	TCU*
50%	71	269	1,217	101	302	1,310	126	336	1,419
40%	66	251	1,192	94	282	1,277	118	314	1,378
30%	62	233	1,170	87	262	1,249	109	291	1,344
20%	57	215	1,153	81	241	1,228	101	269	1,317
10%	52	197	1,142	74	221	1,213	92	246	1,299
0%	47	179	1,138	67	201	1,208	84	224	1,293
−10%	43	161	1,143	60	181	1,214	76	202	1,300
−20%	38	143	1,159	54	161	1,234	67	179	1,325
−30%	33	125	1,188	47	141	1,272	59	157	1,371
−40%	28	108	1,235	40	121	1,332	50	134	1,444
−50%	24	90	1,305	34	101	1,421	42	112	1,549

4.4. Varying the rates of preventive maintenance and breakdowns

The sensitivity analysis herein pertains to varying the rates of preventive maintenance and random breakdown. In the proposed model, these are assumed to be constant. In the initial set of parameters, the average time between preventive maintenance is taken as $\omega = 0.5$ year, which translates into 2 preventive maintenance per year, whereas it is assumed the time between random breakdowns γ to be 1, so there is 1 breakdown per year. Table 5 illustrates the results where ω assumes the values 0.25 (i.e. 4 PM per year), 0.5 (i.e. 2 PM per year), and 1.5 (0.6 PM per year). For each value of ω, the optimal values of Z, R, and TCU are obtained (this corresponds to the 0% deviation row in Table 5 below). Then, the values of Z and R are increased (decreased) by increments of 10% from their optimal values, computing the corresponding cost each time. It is observed that as the rate of PM per year increases, the levels of JIT buffer decrease, while the reserve stock increases, and the total cost increases. Also, when the more the deviation from the optimal decision is, the higher the cost will be.

Table 6 illustrates the impact of varying γ, the rate per year of random breakdowns which is assumed in the proposed model to be constant. In the initial set of parameters, the average time between breakdowns is assumed to be $\gamma = 1$ year, i.e. 1 random breakdown occurs over a period of a year. In Table 6, ω varies from 0.5 (2 breakdonws per year), 1 (1 breakdown per year), and 1.5 (0.6 breakdown per year). Values show that as the rate of random breakdowns per year increases, the level of JIT buffer decreases and reserve stock increases, and the total cost increases. Also, when the more the deviation from the optimal decision is, the higher the cost will be. These observations show that as the preventive maintenance and random breakdowns occur more frequently, it is optimal to reduce the JIT buffer and increase the reserve stock.

Finally, to better understand the sensitivity of the expected total cost per unit time with respect to the rates of random breakdown and preventive maintenance, the percentage deviation of the average total cost rate from the optimal is studied; this deviation is expressed as a percentage deviation from the optimal values of Z and R, for different values of ω and γ. A close examination of Figure 8 indicates that the sensitivity in the total cost rate grows as the rate of preventive maintenance and random breakdowns increase, i.e. when the system is less reliable and more prone to disruption. In the next subsection, the Asymmetric Laplace distribution is proposed as an appropriate probability distribution of the corrective and preventive durations.

Table 5. Variation of optimal Z, R, and total cost with respect to the average rate of preventive maintenance ω.

	$\omega = 0.25$			$\omega = 0.5$			$\omega = 1.5$		
% Deviation	$Z*$	$R*$	TCU*	$Z*$	$R*$	TCU*	$Z*$	$R*$	TCU*
50%	72	375	1,940	101	302	1,310	129	241	850
40%	67	350	1,896	94	282	1,277	120	225	830
30%	62	325	1,857	87	262	1,249	111	209	813
20%	58	300	1,826	81	241	1,228	103	193	800
10%	53	275	1,806	74	221	1,213	94	176	791
0%	48	250	1,798	67	201	1,208	86	160	788
−10%	43	225	1,807	60	181	1,214	77	144	792
−20%	38	200	1,838	54	161	1,234	69	128	803
−30%	34	175	1,899	47	141	1,272	60	112	823
−40%	29	150	1,997	40	121	1,332	51	96	853
−50%	24	125	2,146	34	101	1,421	43	80	897

Table 6. Variation of optimal Z, R, and total cost with respect to the average rate of random breakdowns γ.

	$\gamma = 0.5$			$\gamma = 1$			$\gamma = 1.5$		
% Deviation	$Z*$	$R*$	TCU*	$Z*$	$R*$	TCU*	$Z*$	$R*$	TCU*
50%	65	394	1,454	101	302	1,310	123	253	1,243
40%	61	368	1,413	94	282	1,277	114	236	1,213
30%	56	342	1,378	87	262	1,249	106	219	1,187
20%	52	315	1,351	81	241	1,228	98	203	1,167
10%	48	289	1,332	74	221	1,213	90	186	1,154
0%	43	263	1,325	67	201	1,208	82	169	1,149
−10%	39	2367	1,333	60	181	1,214	74	152	1,154
−20%	35	210	1,361	54	161	1,234	65	135	1,172
−30%	30	184	1,413	47	141	1,272	57	118	1,206
−40%	26	158	1,496	40	121	1,332	49	101	1,259
−50%	22	131	1,622	34	101	1,421	41	84	1,336

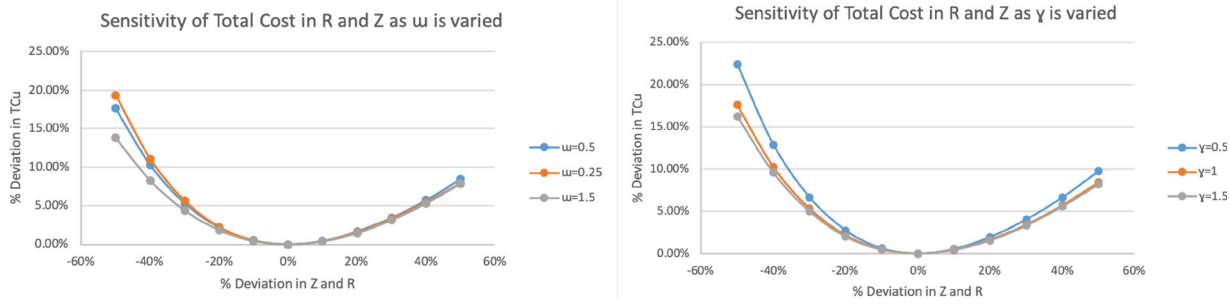

Figure 8. Sensitivity of the total cost with respect to deviations in Z and R, as the rates of preventive maintenance and random breakdown vary.

4.5. Asymmetric Laplace distribution

This subsection assumes that the duration of random breakdown, T_R, and duration of preventive maintenance, T_M, are random variables that follow the Asymmetric Laplace (AL) distribution, and investigates the effect of the AL distribution of T_R and T_M on the optimal model decisions and cost function. The literature shows that most of the supply chain models with preventive maintenance and/or random breakdown use the exponential probability distribution (Groenevelt, Pintelon, and Seidmann 1992; Zequeira, Valdes, and Berenguer 2008; Cheung and Hausman 1997), the Gamma distribution (Chelbi and Ait-Kadi 2004; Dhouib, Gharbi, and Aziza 2012), or the lognormal probability distribution (Gharbi, Kenne, and Beit 2007) to model the random breakdown or duration of preventive maintenance. According to Zequeira, Valdes, and Berenguer (2008), the exponential distribution is more tractable analytically than the Lognormal distribution, and is a good model for the time to repair in a number of situations, since it gives a greater probability to short durations of the maintenance action. Other recent studies compare different probability distributions used to model durations in supply chains (Tanai and Guiffrida 2015) and discuss concerns related to using the Gamma distribution, as it is easy to evaluate the cost functions when using the AL rather than gamma distribution. Tanai and Guiffrida (2015) propose the AL distribution for delivery times in supply chain models and note that they are more tractable than gamma distributions and the two-parameter exponential family in evaluating the supply chain cost models. Depending on the selected parameter values, the AL probability density function can have various skewness and kurtosis. Also, the AL distribution can be viewed as a special case of the generalised hyperbolic distribution introduced by Barndorff-Nielsen and Halgreen (1977) as the superclass of, among others, the hyperbolic, lognormal, and variance-gamma distributions, and is useful due to its recognised versatility. Therefore, the AL distribution is proposed as an attractive probability density function for modelling corrective and preventive durations.

The probability density and cumulative distribution functions of the AL distribution are as follows:

$$f_{m,\lambda,\kappa}(x) = \left(\frac{\lambda}{\kappa + 1/\kappa}\right) \begin{cases} e^{-\lambda\kappa(x-m)} & \text{if } x \geq m \\ e^{(\lambda/\kappa)(x-m)} & \text{if } x < \kappa \end{cases} \tag{19}$$

$$F_{m,\lambda,\kappa}(x) = \begin{cases} 1 - \dfrac{1}{1+\kappa^2} e^{-\lambda\kappa(x-m)} & \text{if } x > m \\ \dfrac{k^2}{1+k^2} e^{(\lambda/\kappa)(x-m)} & \text{if } x \leq m \end{cases}, \tag{20}$$

where κ is the asymmetry parameter, m is the location parameter, and λ is the scale parameter. The mean and the variance of the AL are as follow.

$$E[x] = m + \frac{1-\kappa^2}{\lambda\kappa} \tag{21}$$

$$\text{var}[x] = \frac{1+\kappa^4}{\lambda^2\kappa^2}. \tag{22}$$

In our numerical analysis, T_M follows the AL distribution with parameters $m = 10$ days, $\lambda = 730$, and $\kappa = 0.05$. Similarly, T_R follows AL distribution with parameters $m = 8$ days, $\lambda = 913$, and $\kappa = 0.05$. Note that time will be converted to years. The parameters of the AL distributions are chosen so they yield the same expected value and variance as the exponential distribution adopted in the earlier sections. The convexity plot of the total cost rate with respect to Z and R is depicted in Figure 9.

For the set of parameters in Table 1 and AL distribution parameters, the optimal values of the just-in-time buffer, reserve level R, and total cost are as follows: $Z^* = 55$ units, $R^* = 318$ units, and $TCU = 1{,}437$\$, as reported in the first row of Table 7. Next, a comparative analysis of our model with those existing in the literature is performed. The results of this comparison are reported in Table 7, and indicate that, if only just-in-time buffer

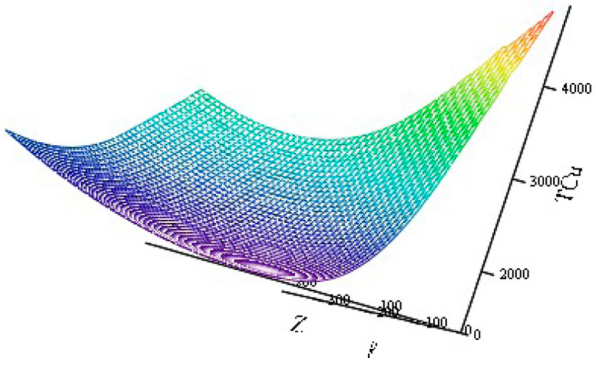

Figure 9. Joint convexity plot of the total cost rate with respect to Z and R under AL distribution.

Table 7. Optimal cost for different models of reserve and JIT buffer inventory under AL distribution.

	R	Z	TCU
Optimal policy	55	318	**$1,437**
Optimal $Z, R = 0$	0	244	$2,812
Optimal $R, Z = 0$	345	0	$1,462

Table 8. Optimal decision variables and cost rate when varying the expected duration of preventive maintenance E(T_M) and random breakdown E(T_R).

Varying E(T_M), $m_2 = 10$ days				Varying E(T_R), $m_1 = 10$ days			
m_1	Z	R	TCU	m_2	Z	R	TCU
6	49	311	$1,421	2	91	266	$1,355
8	55	318	$1,437	4	83	280	$1,377
10	61	327	$1,455	6	76	295	$1,401
12	67	336	$1,475	8	68	311	$1,427
14	73	346	$1,496	10	61	327	$1,455

Table 9. Optimal decision variables and cost rate when varying the average time between breakdowns and PM, ω and γ (inverse of failure rate) respectively.

ω	Z	R	TCU	γ	Z	R	TCU
0.50	55	318	1,437	0.75	45	344	1,482
1	68	290	1,139	1.00	55	318	1,437
2	74	276	985	2.00	81	263	1,342
3	77	271	932	3.00	96	234	1,295
4	78	269	906	4.00	106	215	1,266
5	79	267	890	5.00	113	202	1,246
10	81	264	858	10.00	134	167	1,198

inventory is maintained but not reserve stock ($R = 0$), the optimal total cost increases to $2,812. On the other hand, if only reserve stock is maintained, but not just-in-time buffer, ($Z = 0$), the optimal cost also increases to $1,462, indicating that the optimal cost is the lowest when both JIT buffer and reserve stock are jointly considered as proposed in our model.

Table 8 depicts the impact of varying the average maintenance rate λ_M and average rate of random breakdown λ_R on the optimal decisions and cost rate. To do so, the location parameters m_1 and m_2 are varied respectively. First, the values of m_1 are varied while keeping $m_2 = 10$. As m_1 changes from $m_1 = 6$ days to $m_1 = 14$ days, in increments of 2, the average rate of preventive maintenance λ_M increases (computed using the expected value of the AL distribution in (21)). In this case, the buffer inventory Z increases and the reserve level R increases slightly. On the other hand, keeping $m_1 = 10$ days constant, and varying the average rate of breakdown, λ_R (by varying m_2), decreases the buffer inventory Z while increasing R considerably. In both cases, the total cost increases. These results are consistent with those in Tables 3 and 4 for the exponential distribution.

The impact of varying the average time between breakdowns per year, γ, (the inverse of the constant failure rate), and the average time between starting with two PM cycles per year, ω, is also tested. Table 9 summarises the results. It is observed that as the time between scheduled PM (ω) increases, i.e. a lower rate of scheduled PM, the buffer stock Z increases and reserve R level decreases, and the optimal cost decreases since less buffer stock is being held. Similar observations hold when the average time between breakdowns increases, equivalent to a decreasing failure rate. When γ increases (i.e. decreasing failure rate), the buffer stock Z increases and reserve R decreases, with a decrease of the optimal cost. This is also illustrated in Figures 10–11.

4.6. Managerial insights

Our model provides significant insights and important implications for operations and inventory managers to properly incorporate disruption and risk management in their inventory planning. By building inventory strategies in anticipation of delayed deliveries and interruption risk, a company can strengthen its ability to build resilience into its operations and stabilises the company's performance, enabling continuous delivery to customers.

The joint implementation of just-in-time and reserve stock proposed in this paper is a strategy that requires managers to invest in disruption mitigation strategies, before any disruption occurs. Due to the very low likelihood of disruptions, managers may not be inclined to value such investments (Chopra and Sodhi 2014). In fact, keeping excess inventory as one of the mitigation strategies challenges the lean management practices that support operating under low inventory levels. However, the risk management view promotes availability of excess capacity in order to respond to disruptions. Alicke, Azcue, and Barriball (2020) explains that for most companies around the world, the most important consideration from the first ten weeks of the 2019-nCoV pandemic

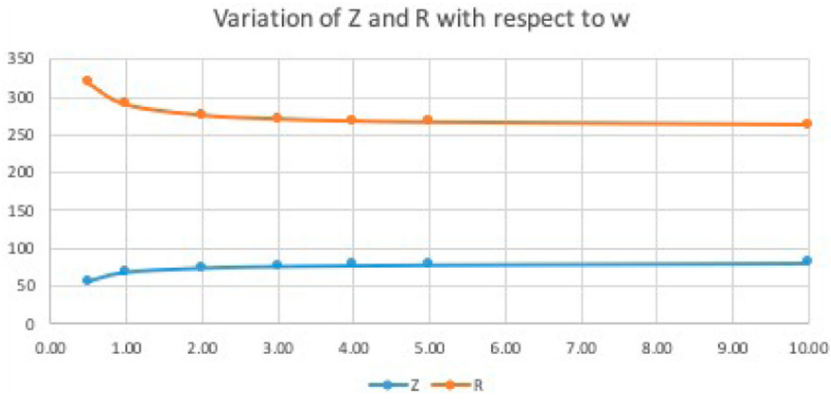

Figure 10. Sensitivity of R and Z with respect to ω under AL.

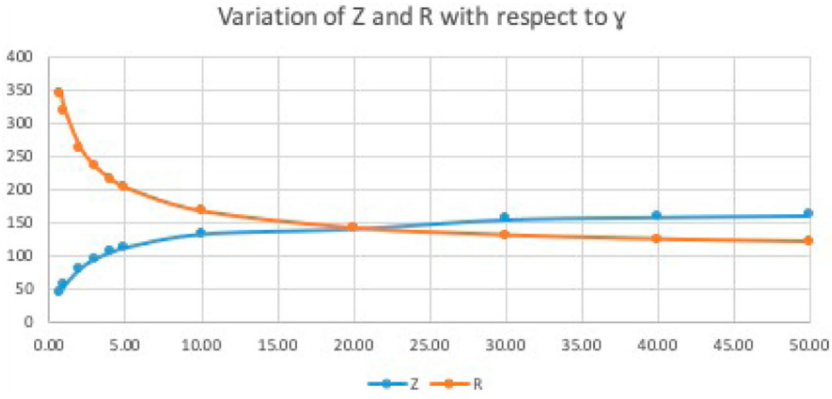

Figure 11. Sensitivity of R and Z with respect to γ under AL.

has been the effect on supply chains. Many disruptions were felt across the supply chains. As a result, inventories were facing delays, which has resulted in expected shortages for companies across various industries in the European Union and the United States. In the face of such extreme disruption, inventory disruption mitigation strategies are needed. Explicitly considering the trade-offs between investing in lean management and risk mitigation strategies can help managers build resilience into their operations.

To build more resilient operations, inventory managers may need to work simultaneously with just-in-time buffer and reserve stock to mitigate the effect of random disruptions. Failing to address the impact of disruptions in the design of operational strategies and the development of disruption mitigation strategies, could have significant negative consequences on a firm's performance and cost. The importance of incorporating risk into inventory planning should be stressed more by managers, since the firm has less control over disruption risk. Implementing jointly a buffer inventory and reserve stock is a hedging strategy against supply disruptions that is cost efficient. Manufacturing organisations tend to keep a reserve supply to cope with unexpected disruptions. These systems that only account for reserve stock but no just-in-time buffer inventory risk the possibility of shortages when the duration of PM is high. Hence, manufacturing organisations can recover significant cost savings by incorporating joint decisions into its inventory planning.

5. Conclusion

This paper provides inventory and operations decision makers with an inventory strategy to hedge against random disruptions when regular preventive maintenance actions are performed at regular time instants. Production could stop following random disruption such as external shocks, machine breakdowns, labour strikes or supply interruption, or to undergo regular PM actions. We assume that the times between PM actions and breakdowns are constant, while the PM and corrective actions have random durations that follow a general probability distribution.

The following inventory policy is proposed: Shortly before the beginning of a preventive maintenance cycle, a buffer inventory is built up at the maximum production rate to cover at least the demand during PM

actions. Moreover, to cope with random disruptions caused by uncertain events, a reserve stock is also maintained. The reserve stock is also used when the preventive maintenance exceeds the buffer supply time. When the reserve stock level is depleted, instantaneous production/replenishment brings the reserve level back to its optimal level again.

Implementing jointly a JIT buffer inventory and reserve stock is a powerful and effective strategy for manufacturing systems to extend their lives, reduce operating costs, and avoid economic losses. Additionally, it ensures continuous availability of supply and ability to meet demand. Therefore, a manufacturing organisation is better off maintaining a temporary buffer and a reserve stock to cope with interruptions, and ensure constant supply. The optimal levels of the buffer inventory and reserve level are unique and can be determined easily by our model.

This research work contributes in several ways to the related research work in the literature. First, an in depth study is presented of the joint implementation of a just-in-time buffer inventory accumulated before preventive maintenance actions and reserve level to hedge against random breakdowns and supply interruptions. Second, an extensive numerical analysis is performed showing the significant advantage that the proposed inventory strategy presents to operations managers. Specifically, implementing jointly a buffer stock and reserve inventory reduces the total inventory cost as opposed to considering one or the other as assumed in the literature. Third, the joint convexity of the total cost function is established with respect to the buffer stock and reserve level, under general distribution functions of preventive and corrective actions. Fourth, analysis of the proposed model is performed first under exponential distributions of the duration of breakdown and preventive maintenance, then under Asymmetric Laplace (AL) distributions. AL distribution is suggested as an attractive distribution to model the durations of preventive and corrective actions. Fifth, the proposed policy outperforms traditional models that study preventive maintenance and random breakdowns separately. Sixth, managerial insights are derived to help decision makers manage the trade-off between buffer stock and reserve level to minimise inventory costs while hedge against random interruptions.

The results indicate that as the shortage cost increases, the reserve level increases as well as the total cost, indicating a more dependence on the reserve stock in this case. There is also a trade-off between the JIT buffer and reserve stock holding costs. As the holding cost of JIT buffer increases, the JIT level is reduced, while the reserve level is increased. This is the opposite when the holding cost of reserve stock increases, in which case the reserve level decreases and JIT buffer increases. Varying the duration of preventive maintenance and breakdowns shows that higher level of reserve stock allows us to hedge against higher uncertainties caused by the PM and breakdown random durations. Also, as the system is more susceptible to uncertainty and disruptions, i.e. as the rates on PM and breakdown increase, it is optimal to raise the reserve stock levels while reducing the JIT buffer. Numerical and sensitivity analysis were conducted for the exponential and Asymmetric Laplace distributions, which are proposed as attractive probability distribution in the context of random preventive and corrective durations.

This work could be extended to allow the time to failure to be stochastic and the failure rate to be an increasing function of time, as assumed in some literature works (Groenevelt, Pintelon, and Seidmann 1992; Moinzadeh and Aggarwal 1997, nov). Demand can be assumed to be stochastic, in which case a long term expected cost function would need to be developed and analysed.

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Notes on contributors

Lama Moussawi-Haidar, PhD is a tenured Associate Professor of Management Science in the Suliman Olayan School of Business at the American University of Beirut where she is currently serving as the Associate Dean for Research and Faculty Development. Dr. Moussawi earned a bachelor degree in Computer Science from the American University of Beirut (AUB), and an MBA, an M.S. degree in Supply Chain Management, and a Ph.D. in Management Science from the University of Texas at Dallas. Her current research centers on applying models of Operations Research and Management Science to various business areas. Dr. Moussawi's research also focuses on deploying data analytics to promote gender diversity and equity in the Middle East and North Africa region (MENA). On a global level, she is the founding Chair of the Women in Data Science (WiDS) conference at AUB, where she is the founding chair of the AUB Committee on the Lives and Careers of Women at AUB. She is a co-founder of the Center for Inclusive Business and Leadership for Women at AUB, the first gender center in the MENA region that focuses on advancing inclusive workplaces and dignified work for women. Dr. Moussawi also serves as a co-PI on several US funded leading research projects focusing on women's economic empowerment.

Hoda Daou, PhD is the Head of Applied Analytics at MBC Group Media Service. Prior to her current role, she was an Assistant Professor in Business Information and Decision Systems at the Olayan School of Business, AUB. She holds a PhD in Computer and Electrical Engineering, from McGill University. During her doctoral studies, her research work focused on the area of healthcare, where she worked on analyzing biomedical signals and developing compression techniques targeting EEG signals of patients suffering from epileptic seizures. The research work she has done, in both academic and corporate paths, focused on multiple disciplines such as information theory, data mining and distributed computing.

Khalil Y. Khalil is an experienced Cost Consultant with a demonstrated history of working in the construction industry. Skilled in Budgeting, Value Engineering, Microsoft Office, and risk analysis. Graduated as a Civil Engineer from Notre Dame University – Zouk Mosbeh specialized in lightweight concrete construction. Achieved a Master's Degree in Engineering Management focused in Financial Engineering from the American University of Beirut. Submitted his thesis titled 'Reserve Stock and the Effects of Diversification'. Currently working as a Project engineer for several construction projects. In addition, Khalil is preparing for several research papers focusing on supply chain management and reserve stock analysis.

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Appendix

A.1 Proof of Lemma 3.1

$$\begin{aligned}\frac{\mathrm{d}TCU(Z,R)}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}(nk + h_M n A_{JIT}) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}Z}\left(h_R\left[R - nE(A_M) - \frac{E(A_R)}{\gamma}\right]\right) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}Z}\left(\rho\left[nE(\tilde{t}_M) + \frac{E(\tilde{t}_R)}{\gamma}\right]\right) \\ &= \frac{h_M n\alpha}{\beta(\alpha-\beta)}Z + \frac{h_R n}{\beta}\left[\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}(\beta t - Z)f(t)\,\mathrm{d}t\right] \\ &\quad \times \frac{n}{\beta}[h_R R - \rho]\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t\end{aligned}$$

Taking the first order optimality condition with respect to Z, we get

$$\begin{aligned}\frac{\mathrm{d}TCU(Z,R)}{\mathrm{d}Z} &= \frac{h_M n\alpha}{\beta(\alpha-\beta)}Z + \frac{h_R n}{\beta}\left[\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}(\beta t - Z)f(t)\,\mathrm{d}t\right] \\ &\quad \times \frac{n}{\beta}[h_R R - \rho]\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t \\ &= 0\end{aligned}$$

where we have

$$\begin{aligned}\frac{\mathrm{d}E(A_M)}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\left(\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}\frac{(\beta t - Z)^2}{2\beta}f(t)\,\mathrm{d}t\right) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}Z}\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left[-\frac{R^2}{(2\beta)} - \frac{ZR}{\beta} + Rt\right]f(t)\,\mathrm{d}t\right) \\ &= \int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}\frac{(Z-\beta t)}{\beta}f(t)\,\mathrm{d}t - \frac{R}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t \\ \frac{\mathrm{d}E(A_R)}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\int_0^{\frac{R}{\beta}}\frac{\beta t^2}{2}g(t)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}Z}\int_{\frac{R}{\beta}}^{\infty}\left[Rt - \frac{R^2}{2\beta}\right]g(t)\,\mathrm{d}t \\ &= 0 \\ \frac{\mathrm{d}E(\tilde{t}_M)}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\int_{\frac{Z+R}{\beta}}^{\infty}\left(t - \frac{Z+R}{\beta}\right)f(t)\,\mathrm{d}t \\ &= -\frac{1}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t \\ \frac{\mathrm{d}E(\tilde{t}_R)}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\left[\int_{\frac{R}{\beta}}^{\infty}\left(t - \frac{R}{\beta}\right)g(t)\,\mathrm{d}t\right] \\ &= 0\end{aligned}$$

The second order optimality condition with respect to Z is

$$\begin{aligned}\frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z^2} &= \frac{\mathrm{d}^2}{\mathrm{d}Z^2}(nk + h_M n A_{JIT}) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left(h_R\left[R - nE(A_M) - \frac{E(A_R)}{\gamma}\right]\right) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left(\rho\left[nE(\tilde{t}_M) + \frac{E(\tilde{t}_R)}{\gamma}\right]\right) \\ &= \frac{h_M n\alpha}{\beta(\alpha-\beta)} - \frac{h_R n}{\beta}\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t \\ &\quad + \frac{\rho n}{\beta^2}f\left(\frac{Z+R}{\beta}\right)\end{aligned}$$

where we have

$$\begin{aligned}\frac{\mathrm{d}^2E(A_M)}{\mathrm{d}Z^2} &= \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left(\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}\frac{(\beta t - Z)^2}{2\beta}f(t)\,\mathrm{d}t\right) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left[-\frac{R^2}{(2\beta)} - \frac{ZR}{\beta} + Rt\right]f(t)\,\mathrm{d}t\right) \\ &= \frac{1}{\beta}\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t \\ \frac{\mathrm{d}^2E(A_R)}{\mathrm{d}Z^2} &= \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\int_0^{\frac{R}{\beta}}\frac{\beta t^2}{2}g(t)\,\mathrm{d}t \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\int_{\frac{R}{\beta}}^{\infty}\left[Rt - \frac{R^2}{2\beta}\right]g(t)\,\mathrm{d}t \\ &= 0 \\ \frac{\mathrm{d}^2E(\tilde{t}_M)}{\mathrm{d}Z^2} &= \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\int_{\frac{Z+R}{\beta}}^{\infty}\left(t - \frac{Z+R}{\beta}\right)f(t)\,\mathrm{d}t \\ &= \frac{1}{\beta^2}f\left(\frac{Z+R}{\beta}\right) \\ \frac{\mathrm{d}^2E(\tilde{t}_R)}{\mathrm{d}Z^2} &= \frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left[\int_{\frac{R}{\beta}}^{\infty}\left(t - \frac{R}{\beta}\right)g(t)\,\mathrm{d}t\right] \\ &= 0\end{aligned}$$

For the total cost function to be convex in Z, the following condition should hold

$$\begin{aligned}&h_M - h_R\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t \\ &\quad \geq 0, \quad \text{which implies that } \int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t \leq \frac{h_M}{h_R}.\end{aligned}$$

Since the maximum value of $\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t$ is 1, TCU(Z, R) is convex if $h_M \geq h_R$. In general, the holding of the reserve stock, h_R, is less than the working inventory h_M, and the result follows.

A.2 Proof of Lemma 3.2

$$\begin{aligned}\frac{\mathrm{d}TCU(Z,R)}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}(nk + h_M n A_{JIT}) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}R}\left(h_R\left[R - nE(A_M) - \frac{E(A_R)}{\gamma}\right]\right) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}R}\left(\rho\left[nE(\tilde{t}_M) + \frac{E(\tilde{t}_R)}{\gamma}\right]\right) \\ &= h_R\left[1 - n\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left(\frac{-R}{\beta} - \frac{Z}{\beta} + t\right)f(t)\,\mathrm{d}t\right)\right.\end{aligned}$$

$$- \frac{1}{\gamma}\left(\int_{\frac{R}{\beta}}^{\infty}(t-\frac{R}{\beta})g(t)\,\mathrm{d}t\right)\Bigg] - \frac{\rho}{\beta}\left[n\int_{\frac{Z+R}{\beta}}^{\infty}f(t)\,\mathrm{d}t+\frac{1}{\gamma}\int_{\frac{R}{\beta}}^{\infty}g(t)\,\mathrm{d}t\right]$$

The first order optimality condition with respect to R is

$$\begin{aligned}\frac{\mathrm{d}TCU(Z,R)}{\mathrm{d}R} &= h_R\left[1-n\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left(\frac{-R}{\beta}-\frac{Z}{\beta}+t\right)f(t)\,\mathrm{d}t\right)\right.\\ &\quad\left.-\frac{1}{\gamma}\left(\int_{\frac{R}{\beta}}^{\infty}\left(t-\frac{R}{\beta}\right)g(t)\,\mathrm{d}t\right)\right]\\ &\quad-\frac{\rho}{\beta}\left[n\int_{\frac{Z+R}{\beta}}^{\infty}f(t)\,\mathrm{d}t+\frac{1}{\gamma}\int_{\frac{R}{\beta}}^{\infty}g(t)\,\mathrm{d}t\right]\\ &= 0 \qquad \text{(A1)}\end{aligned}$$

where we have

$$\begin{aligned}\frac{\mathrm{d}E(A_M)}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}\left(\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}\frac{(\beta t-Z)^2}{2\beta}f(t)\,\mathrm{d}t\right)\\ &\quad+\frac{\mathrm{d}}{\mathrm{d}R}\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left[-\frac{R^2}{(2\beta)}-\frac{ZR}{\beta}+Rt\right]f(t)\,\mathrm{d}t\right)\\ &= \int_{\frac{Z+R}{\beta}}^{\infty}\left(\frac{-R}{\beta}-\frac{Z}{\beta}+t\right)f(t)\,\mathrm{d}t\\ \frac{\mathrm{d}E(A_R)}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}\int_0^{\frac{R}{\beta}}\frac{\beta t^2}{2}g(t)\,\mathrm{d}t+\frac{\mathrm{d}}{\mathrm{d}R}\int_{\frac{R}{\beta}}^{\infty}\left(Rt-\frac{R^2}{2\beta}\right)g(t)\,\mathrm{d}t\\ &= \int_{\frac{R}{\beta}}^{\infty}\left(t-\frac{R}{\beta}\right)g(t)\,\mathrm{d}t\\ \frac{\mathrm{d}E(\tilde{t}_M)}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}\left[\int_{\frac{Z+R}{\beta}}^{\infty}\left(t-\frac{Z+R}{\beta}\right)f(t)\,\mathrm{d}t\right]\\ &= -\frac{1}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty}f(t)\,\mathrm{d}t\\ \frac{\mathrm{d}E(\tilde{t}_R)}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}\left[\int_{\frac{R}{\beta}}^{\infty}\left(t-\frac{R}{\beta}\right)g(t)\,\mathrm{d}t\right]\\ &= -\frac{1}{\beta}\int_{\frac{R}{\beta}}^{\infty}g(t)\,\mathrm{d}t\end{aligned}$$

The second order optimality condition of the total cost function is

$$\begin{aligned}\frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R^2} &= \frac{\mathrm{d}^2}{\mathrm{d}R^2}(nk+h_MnA_{JIT})\\ &\quad+\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(h_R\left[R-nE(A_M)-\frac{E(A_R)}{\gamma}\right]\right)\\ &\quad+\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(\rho\left[nE(\tilde{t}_M)+\frac{E(\tilde{t}_R)}{\gamma}\right]\right)\\ &= \frac{h_R}{\beta}\left[n\int_{\frac{Z+R}{\beta}}^{\infty}f(t)\,\mathrm{d}t+\frac{1}{\gamma}\int_{\frac{R}{\beta}}^{\infty}g(t)\,\mathrm{d}t\right)\\ &\quad+\frac{\rho}{\beta^2}\left[nf\left(\frac{Z+R}{\beta}\right)+\frac{1}{\gamma}g\left(\frac{R}{\beta}\right)\right],\end{aligned}$$

where we have

$$\begin{aligned}\frac{\mathrm{d}^2E(A_M)}{\mathrm{d}R^2} &= \frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}}\frac{(\beta t-Z)^2}{2\beta}f(t)\,\mathrm{d}t\right)\\ &\quad+\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left[-\frac{R^2}{(2\beta)}-\frac{ZR}{\beta}+Rt\right]f(t)\,\mathrm{d}t\right.\\ &= -\frac{1}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty}f(t)\,\mathrm{d}t\\ \frac{\mathrm{d}^2E(A_R)}{\mathrm{d}R^2} &= \frac{\mathrm{d}^2}{\mathrm{d}R^2}\int_0^{\frac{R}{\beta}}\frac{\beta t^2}{2}g(t)\,\mathrm{d}t\\ &\quad+\frac{\mathrm{d}^2}{\mathrm{d}R^2}\int_{\frac{R}{\beta}}^{\infty}\left(Rt-\frac{R^2}{2\beta}\right)g(t)\,\mathrm{d}t\\ &= -\frac{1}{\beta}\int_{\frac{R}{\beta}}^{\infty}g(t)\,\mathrm{d}t\\ \frac{\mathrm{d}^2E(\tilde{t}_M)}{\mathrm{d}R^2} &= \frac{\mathrm{d}^2}{\mathrm{d}R^2}\int_{\frac{Z+R}{\beta}}^{\infty}\left(t-\frac{Z+R}{\beta}\right)f(t)\,\mathrm{d}t\\ &= \frac{1}{\beta^2}f\left(\frac{Z+R}{\beta}\right)\\ \frac{\mathrm{d}^2E(\tilde{t}_R)}{\mathrm{d}R^2} &= \frac{\mathrm{d}^2}{\mathrm{d}R^2}\left[\int_{\frac{R}{\beta}}^{\infty}\left(t-\frac{R}{\beta}\right)g(t)\,\mathrm{d}t\right]\\ &= \frac{1}{\beta^2}g\left(\frac{R}{\beta}\right)\end{aligned}$$

Since the $\frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R^2}$ is positive, total cost function is convex with respect to R.

A.3 Proof of Lemma 3.3

To show that the expected total cost function in (16) is jointly convex in Z and R, we need to show that the corresponding Hessian matrix is positive. The Hessian matrix of the expected total cost function in (16) is given as

$$H = \begin{pmatrix}\frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z^2} & \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z\,\mathrm{d}R}\\ \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z\,\mathrm{d}R} & \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R^2}\end{pmatrix}$$

Now, define

$$H_1 = \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z^2},$$

which is shown in the previous subsection to be positive. Also define

$$H_2 = \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z^2} * \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R^2} - \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z}.$$

To show that H is positive definite, we need to show that $H_2 > 0$.

In the expression of H_2 above, note that if we show that we have

$$\text{(i)}\quad \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}Z^2} \geq \frac{\mathrm{d}^2TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z}, \qquad \text{(A2)}$$

$$\text{(ii)} \quad \frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R^2} \geq \frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z}, \tag{A3}$$

then $H_2 > 0$ and the result follows. In the following steps, we show that $H_2 > 0$ by showing that conditions (i) and (ii) hold. Note that the second derivative of the expected total cost function with respect to R and Z is obtained as follows

$$\begin{aligned}\frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z} &= \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}(nk + h_M n A_{JIT}) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\left(h_R\left[R - nE(A_M) - \frac{E(A_R)}{\gamma}\right]\right) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\left(\rho\left[nE(\tilde{t}_M) + \frac{E(\tilde{t}_R)}{\gamma}\right]\right) \\ &= \frac{h_R n}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t + \frac{\rho n}{\beta^2} f\left(\frac{Z+R}{\beta}\right)\end{aligned} \tag{A4}$$

where we have

$$\begin{aligned}\frac{\mathrm{d}^2 E(A_M)}{\mathrm{d}R\,\mathrm{d}Z} &= \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\left(\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} \frac{(\beta t - Z)^2}{2\beta} f(t)\,\mathrm{d}t\right) \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\left(\int_{\frac{Z+R}{\beta}}^{\infty}\left[-\frac{R^2}{(2\beta)} - \frac{ZR}{\beta} + Rt\right] f(t)\,\mathrm{d}t\right) \\ &= \frac{-1}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t \\ \frac{\mathrm{d}^2 E(A_R)}{\mathrm{d}R\,\mathrm{d}Z} &= \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\int_0^{\frac{R}{\beta}} \frac{\beta t^2}{2} g(t)\,\mathrm{d}t \\ &\quad + \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\int_{\frac{R}{\beta}}^{\infty}\left(Rt - \frac{R^2}{2\beta}\right) g(t)\,\mathrm{d}t \\ &= 0 \\ \frac{\mathrm{d}^2 E(\tilde{t}_M)}{\mathrm{d}R\,\mathrm{d}Z} &= \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\int_{\frac{Z+R}{\beta}}^{\infty}\left(t - \frac{Z+R}{\beta}\right) f(t)\,\mathrm{d}t \\ &= \frac{1}{\beta^2} f\left(\frac{Z+R}{\beta}\right) \\ \frac{\mathrm{d}^2 E(\tilde{t}_R)}{\mathrm{d}R\,\mathrm{d}Z} &= \frac{\mathrm{d}^2}{\mathrm{d}R\,\mathrm{d}Z}\left[\int_{\frac{R}{\beta}}^{\infty}\left(t - \frac{R}{\beta}\right) g(t)\,\mathrm{d}t\right] \\ &= 0\end{aligned}$$

(i) First, we show that

$$\frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}Z^2} \geq \frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z}.$$

We have

$$\begin{aligned}&\frac{h_M n\alpha}{\beta(\alpha-\beta)} - \frac{h_R n}{\beta}\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t + \frac{\rho n}{\beta^2} f\left(\frac{Z+R}{\beta}\right) \\ &\quad > \frac{h_R n}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t + \frac{\rho n}{\beta^2} f\left(\frac{Z+R}{\beta}\right).\end{aligned}$$

Thus we get

$$\frac{h_M n\alpha}{\beta(\alpha-\beta)} > \frac{h_R n}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t + \frac{h_R n}{\beta}\int_{\frac{Z}{\beta}}^{\frac{Z+R}{\beta}} f(t)\,\mathrm{d}t$$

Taking the maximum values of both sides of the equation above, and since $h_M \geq h_R$, condition (i) is satisfied.

(ii) Second, we show that

$$\frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R^2} \geq \frac{\mathrm{d}^2 TCU(Z,R)}{\mathrm{d}R\,\mathrm{d}Z}.$$

$$\begin{aligned}&\frac{h_R}{\beta}\left[n\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t + \frac{1}{\gamma}\int_{\frac{R}{\beta}}^{\infty} g(t)\,\mathrm{d}t)\right] \\ &\quad + \frac{\rho}{\beta^2}\left[nf\left(\frac{Z+R}{\beta}\right) + \frac{1}{\gamma} g\left(\frac{R}{\beta}\right)\right] \\ &\quad > \frac{h_R n}{\beta}\int_{\frac{Z+R}{\beta}}^{\infty} f(t)\,\mathrm{d}t + \frac{\rho n}{\beta^2} f\left(\frac{Z+R}{\beta}\right)\end{aligned}$$

Thus we get

$$\frac{h_R}{\beta}\left[\frac{1}{\gamma}\int_{\frac{R}{\beta}}^{\infty} g(t)\,\mathrm{d}t)\right] > 0.$$

Condition (ii) is satisfied, thus $H_2 > 0$, and the Hessian matrix is positive definite. Thus, the total cost function is jointly convex with respect to Z and R.