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Quadratically presented grade three Gorenstein ideals [☆]

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ABSTRACT

Let J be a quadratically presented grade three Gorenstein ideal in the standard graded polynomial ring $R = \mathbf{k}[x, y, z]$, where \mathbf{k} is a field. Assume that R/J satisfies the weak Lefschetz property. We give the presentation matrix for J in terms of the coefficients of a Macaulay inverse system for J . (This presentation matrix is an alternating matrix and J is generated by the maximal order Pfaffians of the presentation matrix.) Our formulas are computer friendly; they involve only matrix multiplication; they do not involve multilinear algebra or complicated summations. As an application, we give the presentation matrix for $J_1 = (x^{n+1}, y^{n+1}, z^{n+1}) : (x + y + z)^{n+1}$, when n is even and the characteristic of \mathbf{k} is zero. Generators for J_1 had been identified previously; but the presentation matrix for J_1 had not previously been known. The first step in our proof is to give improved formulas for the presentation matrix of a linearly presented grade three Gorenstein ideal I in terms of the coefficients of the Macaulay inverse system for I .

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[☆] We used Macaulay2 [9] to test our formulas.

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1. Introduction

Let $R = \mathbf{k}[x, y, z]$ be a standard graded polynomial ring over the field \mathbf{k} , K be an ideal of R generated by homogeneous forms of degree n with R/K Artinian and Gorenstein, and s be the socle degree of R/K . If the ideal K is linearly presented, then $s = 2n - 2$ and the minimal resolution of R/K by free R -modules has the form

$$0 \rightarrow R(-2n - 1) \xrightarrow{b_3} R(-n - 1)^{2n+1} \xrightarrow{b_2} R(-n)^{2n+1} \xrightarrow{b_1} R. \tag{1.0.1}$$

If the ideal K is quadratically presented, then n is even, $s = 2n - 1$, and the minimal resolution of R/K by free R -modules has the form

$$0 \rightarrow R(-2n - 2) \xrightarrow{b_3} R(-n - 2)^{n+1} \xrightarrow{b_2} R(-n)^{n+1} \xrightarrow{b_1} R. \tag{1.0.2}$$

The theorem of Buchsbaum and Eisenbud [4] guarantees that the matrix b_2 is an alternating matrix of linear (respectively, quadratic) forms, b_1 is the row vector of signed maximal order Pfaffians of b_2 , and b_3 is the transpose of b_1 .

The socle of the graded Artinian Gorenstein \mathbf{k} -algebra R/K is the one dimensional vector space $(R/K)_s$. Fix an isomorphism $\sigma : (R/K)_s \rightarrow \mathbf{k}$. Let $\Phi_s : R_s \rightarrow \mathbf{k}$ represent the composition

$$R_s \xrightarrow{\text{natural quotient map}} (R/K)_s \xrightarrow{\sigma} \mathbf{k}. \tag{1.0.3}$$

(The homomorphism $\Phi_s \in \text{Hom}_{\mathbf{k}}(R_s, \mathbf{k})$ is called a Macaulay inverse system for K .) We give explicit formulas for the alternating matrix b_2 in terms of the constants $\{\Phi_s(\mu_s)\}$ as μ_s roams over the monomials of R of degree s . Our formulas involve only matrix multiplication; they do not involve multilinear algebra or complicated summations; and they are computer friendly. In the $s = 2n - 2$ case, no further hypothesis is needed and our procedure determines if K is linearly presented; and, if so, then gives the precise formula for b_2 . In the $s = 2n - 1$ case, our procedure requires that we know a weak Lefschetz element for R/K ; once this element is identified, then our procedure determines if K is quadratically presented; and, if so, then gives the precise formula for b_2 .

Our answer in the linearly presented case is an extension and reformulation of the result in [7]. The new version is significantly easier to apply than the original version. In particular, we apply the new version to the $s = 2n - 1$ case.

Our main results concern the $s = 2n - 1$ case. We became interested in this problem in 2010 when we first suspected that if \mathbf{k} is a field of characteristic zero and n is even, then the grade three Gorenstein ideal

$$J = (x^{n+1}, y^{n+1}, z^{n+1}) : (x + y + z)^{n+1} \tag{1.0.4}$$

is quadratically presented. In 2010, we produced $n + 1$ elements in the ideal J of degree n and we conjectured that these elements generate J . This conjecture is established in [12, Prop. 5.7]. It is easy to show that x is a weak Lefschetz element on R/J and a Macaulay inverse system for J is well known. In Section 5 we obtain the presentation matrix for the ideal J of (1.0.4) as an application of our main result, Theorem 4.2.

In Section 6 we give a Macaulay2 script which implements the ideas of Theorems 3.1 and 4.2.

An Artinian standard graded algebra A over a field \mathbf{k} satisfies the weak Lefschetz property (WLP) property if there is a linear form ℓ in A so that, for each index i , the map $A_i \rightarrow A_{i+1}$, which is given by multiplication by ℓ , is either injective or surjective. (In this case, ℓ is a weak Lefschetz element for A .) It is difficult to determine which graded Artinian \mathbf{k} -algebras satisfy the WLP. The problem has been attacked from many points of view, including representation theory, topology, vector bundle theory, plane partitions, splines, and differential geometry, among others; see for example, [15,14,10]. It is not known if all standard graded codimension three Artinian Gorenstein algebras over a field of characteristic zero satisfy the WLP. Indeed, it is not known if all quotient rings R/J resolved by (1.0.2) necessarily satisfy the WLP.

We describe the approach taken in this note.

1.1. Let $J \subseteq R$ be a homogeneous grade three ideal so that the minimal homogeneous resolution of R/J by free R -modules has the Betti numbers of (1.0.2). Let U be the vector space of homogeneous elements of R of degree 1. (So R is equal to the symmetric algebra $\text{Sym}_\bullet U$.) The divided power algebra $D_\bullet U^* = \bigoplus_{i=0}^\infty D_i U^*$, with $D_i U^* = \text{Hom}_{\mathbf{k}}(\text{Sym}_i U, \mathbf{k})$, is the graded dual of $\text{Sym}_\bullet U$. The algebras $\text{Sym}_\bullet U$ and $D_\bullet U^*$ are modules over one another and the ideal J is equal to $\text{ann}_{\text{Sym}_\bullet U} \Phi_{2n-1}$, for some $\Phi_{2n-1} \in D_{2n-1} U^*$. The element Φ_{2n-1} in $D_{2n-1} U^*$ is called a Macaulay inverse system of J . This element is unique up to multiplication by a non-zero constant of \mathbf{k} . In the case of (1.0.4),

$$\Phi_{2n-1} = (x + y + z)^{n+1} ((x^n y^n z^n)^*). \tag{1.1.1}$$

The symbols and module action in (1.1.1) are explained in 2.5.

The hypothesis that R/J satisfies the WLP ensures that there is a non-zero element ℓ of U for which the ideal $I = \text{ann}_{\text{Sym}_{\bullet} U} \ell(\Phi_{2n-1})$ of R is a linearly presented grade three Gorenstein ideal. We choose a basis x, y, z for U with $x = \ell$; and we let U_0 be the subspace of U which is spanned by y and z . The minimal homogeneous resolution of R/I by free R -modules is given in [7]. This resolution has the form

$$\mathbb{B} : 0 \rightarrow B_3 \xrightarrow{b_3} B_2 \xrightarrow{b_2} B_1 \xrightarrow{b_1} B_0,$$

with $B_3 = B_0 = R$,

$$B_2 = \left\{ \begin{array}{c} R \otimes_{\mathbf{k}} \text{Sym}_{n-1} U_0 \\ \oplus \\ R \otimes_{\mathbf{k}} D_n U_0^* \end{array} \right. \quad \text{and} \quad B_1 = \left\{ \begin{array}{c} R \otimes_{\mathbf{k}} D_{n-1} U_0^* \\ \oplus \\ R \otimes_{\mathbf{k}} \text{Sym}_n U_0. \end{array} \right.$$

The matrix for b_2 looks like

$$\left[\begin{array}{c|c} A & B \\ \hline -B^T & D \end{array} \right], \tag{1.1.2}$$

with A and D alternating matrices, $A = xA'$ where A' is an $n \times n$ alternating matrix of constants. The graded Betti numbers of \mathbb{B} are given in (1.0.1). Explicit formulas for the differentials of \mathbb{B} are given in Theorem 3.1.

The hypothesis that the ideal J is quadratically presented forces the matrix A' to be invertible. It follows quickly (see Theorem 4.2) that the resolution of R/J is

$$0 \rightarrow R \xrightarrow{b_1'^*} R \otimes_{\mathbf{k}} D_n U_0^* \xrightarrow{B^T(A')^{-1}B + xD} R \otimes_{\mathbf{k}} \text{Sym}_n U_0 \xrightarrow{b_1'} R,$$

where b_1' is the restriction of b_1 to the summand $R \otimes_{\mathbf{k}} \text{Sym}_n U_0$ of B_1 . The parity of n plays a critical role because, if n had been odd, then the $n \times n$ alternating matrix of constants A' would necessarily be singular.

2. Notation, conventions, and elementary results

In this paper \mathbf{k} is always a field and the symbol “ $D_i G^*$ ” always means $D_i(G^*)$. See [5, Thms. 21.5 and 21.6] for a quick treatment of Artinian Gorenstein rings and Macaulay duality.

2.1. When it is clear that “ A ” is the ambient ring, we use $(-)^*$, Sym , D , \otimes , $:$, and ann to mean $\text{Hom}_A(-, A)$, Sym^A , D^A , \otimes_A , $:_A$, and ann_A , respectively.

2.2. The *grade* of an ideal I in a commutative Noetherian ring R is the length of a maximal regular sequence on R which is contained in I .

2.3. If M is a graded module, then M_i denotes the component of M of homogeneous elements of degree i and $M = \bigoplus_i M_i$.

2.4. For any set of variables $\{x_1, \dots, x_r\}$ and any degree s , we write $\binom{x_1, \dots, x_r}{s}$ for the set of monomials of degree s in the variables x_1, \dots, x_r .

2.5. We explain the symbols and module action of

$$(x + y + z)^{n+1} ((x^n y^n z^n)^*)$$

from (1.1.1).

If A is a commutative ring and G is a finitely generated free A -module, then $D_{\bullet}^A G^*$ is the graded dual of the A -module $\text{Sym}_{\bullet}^A G$. In other words,

$$D_{\bullet}^A G^* = \bigoplus_{i=0}^{\infty} D_i^A G^*$$

and $D_i^A G^* = \text{Hom}_A(\text{Sym}_i^A G, A)$. The A -module $D_{\bullet}^A G^*$ is a $\text{Sym}_{\bullet}^A G$ -module under the action $u_i \in \text{Sym}_i^A G$ sends $w_j \in D_j^A G^*$ to the element $u_i(w_j)$ in $D_{j-i}^A G^*$, where $u_i(w_j)$ sends u_{j-i} to $(u_{j-i} u_i)(w_j)$, for u_{j-i} in $\text{Sym}_{j-i}^A G$. If x_1, \dots, x_g is basis for G , then

$$\binom{x_1, \dots, x_g}{i} = \text{the set of monomials of degree } i \text{ in } x_1, \dots, x_g$$

is a basis for $\text{Sym}_i^A G$, and $\{m^* \mid m \in \binom{x_1, \dots, x_g}{i}\}$ is a basis for $D_i G^*$, where for each $m \in \binom{x_1, \dots, x_g}{i}$, $m^* : \text{Sym}_i^A G \rightarrow A$ is defined by

$$m^*(m') = \begin{cases} 1, & \text{if } m' = m, \\ 0, & \text{if } m' \neq m, \end{cases}$$

for $m' \in \binom{x_1, \dots, x_g}{i}$. Observe that

$$x_{\ell}(m^*) = \begin{cases} 0, & \text{if } x_{\ell} \nmid m, \\ (\frac{m}{x_{\ell}})^*, & \text{if } x_{\ell} \mid m, \end{cases}$$

for $m \in \binom{x_1, \dots, x_g}{i}$.

2.6. The graded ring $R = \bigoplus_{0 \leq i} R_i$ is a *standard graded R_0 -algebra* if R is generated as an R_0 -algebra by R_1 and R_1 is finitely generated as an R_0 -module.

2.7. If M is a matrix, then M^T is the transpose of M . The matrix M is an *alternating matrix* if $M + M^T = 0$ and the entries of M on the main diagonal are all zero. Let M be

an $m \times m$ alternating matrix. The Pfaffian of M is a square root of a determinant of M . (For a more detailed formulation see, for example [11, 3.7] or [4, Appendix and Sect. 2].) If m is odd, then the Pfaffian of M is zero and the row vector of signed maximal order Pfaffians of M is

$$[M_1 \ \dots \ M_m],$$

where M_j is equal to $(-1)^{j+1}$ times the Pfaffian of M with row and column j deleted. The product $[M_1 \ \dots \ M_m]M$ is equal to zero.

2.8. Let U be the vector space, over the field \mathbf{k} , with basis x, y, z ; and let U_0 be the subspace of U spanned by y and z . We make matrices for \mathbf{k} -module homomorphisms between $\text{Sym}_i X$ and $D_i X^*$ for various i and for X equal to either U or U_0 . Here are the bases we use.

If i is a positive integer, then our favorite ordered basis for $\text{Sym}_i U$ is

$$x^i, x^{i-1}y, x^{i-1}z, x^{i-2}y^2, x^{i-2}yz, x^{i-2}z^2, \dots, y^i, y^{i-1}z, \dots, yz^{i-1}, z^i.$$

In other words, if $a + b + c$ and $\alpha + \beta + \gamma$ are both equal to i , then $x^a y^b z^c$ comes before $x^\alpha y^\beta z^\gamma$ if

$$\alpha < a; \quad \text{or else, } \alpha = a \text{ and } \beta < b.$$

In particular, if $m_1, \dots, m_{\binom{i+1}{2}}$ is our favorite basis for $\text{Sym}_{i-1} U$, then our favorite basis for $\text{Sym}_i U$ is

$$xm_1, \dots, xm_{\binom{i+1}{2}}, y^i, y^{i-1}z, \dots, yz^{i-1}, z^i; \tag{2.8.1}$$

our favorite basis for $D_{i-1}U^*$ is $(m_1)^*, \dots, (m_{\binom{i+1}{2}})^*$; our favorite ordered basis for $\text{Sym}_i U_0$ is

$$y^i, y^{i-1}z, \dots, yz^{i-1}, z^i;$$

and our favorite ordered basis for $D_i U_0^*$ is

$$(y^i)^*, (y^{i-1}z)^*, \dots, (yz^{i-1})^*, (z^i)^*.$$

2.9. Let I be an ideal in a ring A , N be an A -module, and L and M be submodules of N . Then

$$L :_I M = \{x \in I \mid xM \subseteq L\} \quad \text{and} \quad L :_M I = \{m \in M \mid Im \subseteq L\}.$$

If L is the zero module, then we also use “annihilator notation” to describe these “colon modules”; that is,

$$\text{ann}_A M = 0 :_A M \quad \text{and} \quad \text{ann}_N I = 0 :_N I.$$

2.10. If A is a local ring with maximal ideal \mathfrak{m} and M is an A -module, then the *socle* of M is the vector space $\text{socle}(M) = 0 :_M \mathfrak{m}$. If A is a graded Artinian local Gorenstein \mathbf{k} -algebra, then

$$s = \max\{i | A_i \neq 0\}$$

is called the *socle degree* of A .

2.11. Let R be a Noetherian graded \mathbf{k} -algebra. A finitely generated R -module M is *linearly presented* if the minimal homogeneous resolution of M by free R -modules has the form

$$\dots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

and all of the entries of the matrix d_2 are linear forms in R .

The following result gives many conditions which are equivalent to the statement “the grade three Gorenstein ideal I is linearly presented”. The initial hypotheses of the Proposition ensure that Φ_s is a Macaulay inverse system for I and that the socle degree of R/I is s .

Proposition 2.12. [6, Prop. 1.8] *Fix positive integers n and s . Let \mathbf{k} be a field, U be a three dimensional vector space over \mathbf{k} , Φ_s be an element of $D_s^{\mathbf{k}}U^*$, and I be the ideal $\text{ann}_{\text{Sym}U} \Phi_s$ in the standard graded polynomial ring $R = \text{Sym}_{\bullet}^{\mathbf{k}}U$. Then the following statements are equivalent:*

- (a) I is generated by homogeneous forms of degree n and I is linearly presented,
- (b) the minimal homogeneous resolution of R/I by free R -modules has the form

$$0 \rightarrow R(-2n - 1) \rightarrow R(-n - 1)^{2n+1} \rightarrow R(-n)^{2n+1} \rightarrow R,$$

- (c) all of the minimal generators of I have degree n and $s = 2n - 2$,
- (d) $I_{n-1} = 0$ and $(R/I)_{2n-1} = 0$,
- (e) $s = 2n - 2$ and the homomorphism $p : \text{Sym}_{n-1}^{\mathbf{k}}U \rightarrow D_{n-1}^{\mathbf{k}}U^*$, which is given by

$$p(\mu_{n-1}) = \mu_{n-1}(\Phi_s), \quad \text{for } \mu_{n-1} \in \text{Sym}_{n-1}U,$$

is an isomorphism, and

- (f) $s = 2n - 2$ and the $\binom{n+1}{2} \times \binom{n+1}{2}$ matrix

$$(\Phi_s(m_i m_j))_{1 \leq i, j \leq \binom{n+1}{2}},$$

is invertible, where $m_1, \dots, m_{\binom{n+1}{2}}$ is the ordered basis of $\text{Sym}_{n-1} U$ which is given in 2.8.

The matrix of item (f) has the element $\Phi_s(m_i m_j)$ of \mathbf{k} in row i and column j .

3. A computer friendly form of [7]

In a sequence of three papers, we gave explicit formulas for the differentials in the minimal homogeneous resolution of R/I by free R -modules, where $R = \mathbf{k}[x_1, \dots, x_g]$ is a standard-graded polynomial ring over the field \mathbf{k} and I is a grade g homogeneous Gorenstein ideal, provided the resolution is Gorenstein-linear in the sense that the resolution has the form

$$0 \rightarrow F_g \xrightarrow{f_g} F_{g-1} \xrightarrow{f_{g-1}} F_{g-2} \xrightarrow{f_{g-2}} \dots \xrightarrow{f_3} F_2 \dots \xrightarrow{f_2} F_1 \dots \xrightarrow{f_1} R$$

and all the entries of all of the matrices f_i , with $2 \leq i \leq g - 1$, are linear. The formulas are given in terms of the coefficients of a Macaulay inverse system for I . The paper [6] proves that the project can be done; [7] carries out the project when $g = 3$, and [8] carries out the project for all g .

Theorem 3.1, which is adapted from [7, Thm. 4.1], is the starting point for the present paper.

Theorem 3.1. *Let \mathbf{k} be a field, U be a vector space over \mathbf{k} of dimension 3 with basis x, y, z , and U_0 be the subspace of U spanned by y and z . Let n be a positive integer, Φ_{2n-1} be an element of $D_{2n-1} U^*$, R be the polynomial ring $\text{Sym}_{\bullet}^{\mathbf{k}} U$, and I be the ideal*

$$I = \text{ann}_{\text{Sym} U}(x\Phi_{2n-1})$$

of R . Let

$$m_1, \dots, m_{\binom{n+1}{2}} \quad \text{and} \quad m_{0,1}, \dots, m_{0,n+1}$$

be the ordered bases of $\text{Sym}_{n-1} U$ and $\text{Sym}_n U_0^*$, respectively, which are given in 2.8. Let \mathbf{p} and \mathbf{r} be the matrices

$$\mathbf{p} = (x m_i m_j (\Phi_{2n-1}))_{\substack{1 \leq i, j \leq \binom{n+1}{2}}} \quad \text{and} \quad \mathbf{r} = (m_i m_{0,j} (\Phi_{2n-1}))_{\substack{1 \leq i \leq \binom{n+1}{2} \\ 1 \leq j \leq n+1}}$$

Then the following statements hold.

- (a) The ideal I is linearly presented if and only if the matrix \mathbf{p} is invertible.
- (b) If \mathbf{p} is invertible, then the minimal homogeneous resolution of R/I by free R -modules is

$$0 \rightarrow R(-2n - 1) \xrightarrow{b_3} R^{2n+1}(-n - 1) \xrightarrow{b_2} R^{2n+1}(-n) \xrightarrow{b_1} R,$$

where

$$b_2 = \left[\begin{array}{c|c} A & B \\ \hline -B^T & D \end{array} \right],$$

b_1 is the row vector of signed maximal order Pfaffians of b_2 , $b_3 = b_1^T$,

$$\begin{aligned} A &= xA', \quad A' = A_0 - A_0^T, \quad B = xB_1 + B_2, \quad D = x(D_0 - D_0^T), \\ A_0 &= ([\mathbf{r}^T \mathbf{p}^{-1} \mathbf{r}]) \text{ with row } n + 1 \text{ and column } 1 \text{ deleted}, \\ B_0 &= \text{the right most } n \text{ columns of } \mathbf{r}^T \mathbf{p}^{-1}, \\ B_1 &= \left[\begin{array}{c|c} B_0 \text{ with} & \\ \hline 0_{n \times 1} & \text{row } n + 1 \\ & \text{deleted} \end{array} \right] - \left[\begin{array}{c|c} B_0 \text{ with} & \\ \hline \text{row } 1 & \\ & \text{deleted} \end{array} \middle| 0_{n \times 1} \right], \\ B_2 &= [zI_n \mid 0_{n \times 1}] - [0_{n \times 1} \mid yI_n], \\ D_0 &= \left[\begin{array}{c|c} 0_{n \times 1} & \text{the bottom right } n \times n \text{ submatrix of } \mathbf{p}^{-1} \\ \hline 0_{1 \times 1} & 0_{1 \times n} \end{array} \right]. \end{aligned}$$

Remarks 3.2.

- (a) There are concrete examples of the matrices of Theorem 3.1 (and also Theorem 4.2) in 5.4, 5.5, and 5.6. All details are recorded in those examples.
- (b) The Macaulay2 script PM, which is given in Example 6.1, makes the matrices of Theorems 3.1 and 4.2.
- (c) We write I_a for the $a \times a$ identity matrix and $0_{a \times b}$ for the zero matrix with a rows and b columns.
- (d) The entry of \mathbf{p} in row i and column j is the element

$$xm_i m_j(\Phi_{2n-1}) = \Phi_{2n-1}(xm_i m_j)$$

of \mathbf{k} ; similarly, the entry of \mathbf{r} in row i and column j is the element

$$m_i m_{0,j}(\Phi_{2n-1}) = \Phi_{2n-1}(m_i m_{0,j})$$

of \mathbf{k} .

- (e) The element $x\Phi_{2n-1} \in D_{2n-2}U^* = \text{Hom}_{\mathbf{k}}(\text{Sym}_{2n-2}U, \mathbf{k})$ is a Macaulay inverse system for I . If one starts with a Macaulay inverse system Φ_{2n-2} for I , then one may take Φ_{2n-1} to be any element of $D_{2n-1}U^*$ with $x\Phi_{2n-1} = \Phi_{2n-2}$. In particular, if

Table 1
The shapes of the matrices of Theorem 3.1.

matrix	number of rows	number of columns
\mathbf{r}	$\binom{n+1}{2}$	$n + 1$
\mathbf{p}	$\binom{n+1}{2}$	$\binom{n+1}{2}$
A	n	n
B	n	$n + 1$
D	$n + 1$	$n + 1$
B_0	$n + 1$	n
A_0	n	n
B_1	n	$n + 1$
B_2	n	$n + 1$
D_0	$n + 1$	$n + 1$
b_1	1	$2n + 1$
b_2	$2n + 1$	$2n + 1$
b_3	$2n + 1$	1

$$\Phi_{2n-2} = \sum_{m \in \binom{x,y,z}{2n-2}} \alpha_m m^*,$$

with $\alpha_m \in \mathbf{k}$, then one may take Φ_{2n-1} to be

$$\sum_{m \in \binom{x,y,z}{2n-2}} \alpha_m (xm)^*; \tag{3.2.1}$$

however one is not required to make this choice (Table 1).

Theorem 3.1 is obtained from [7] in three steps. Theorem 3.5 is the multilinear algebra version of the result from [7] (from a different point of view); Theorem 3.6 is a crucial extension of Theorem 3.5; and Theorem 3.1 is a matrix version of Theorem 3.6. The conversion from the language of Theorem 3.6 to the language of Theorem 3.1 is given in 3.7. Theorem 3.1 is easier to apply than anything in [7].

There are two differences between the resolution as given in [7] and the resolution of Theorem 3.5. In [7] the coefficients of a Macaulay inverse system are treated as variables and are specialized to field elements only at the last step. In the present paper, we work in $\mathbf{k}[x, y, z]$ rather than

$$\mathfrak{R} = \mathbb{Z}[x, y, z, \{t_m \mid m \text{ is a monomial in } x, y, z \text{ of degree } 2n - 2\}].$$

In [7], the map p of Data 3.3 is a map of free \mathfrak{R} -modules. Each entry of the matrix for p is a variable t_m of the polynomial ring \mathfrak{R} . In [7], the map p does not have an inverse until δ (which is essentially the determinant of p) is inverted. In the present paper, p is an isomorphism of vector spaces; hence, p has an inverse. We write p^{-1} and avoid q (which is essentially the classical adjoint of p) and δ . Theorem 4.1 of [7], with the above two changes of point of view, is recorded as Theorem 3.5.

In fact, we make one more change to [7, 4.1] before we can use it in the present situation. The paper [7] makes use of the “ $\tilde{\Phi}_{2n-1}$ ” of Data 3.3; however, [7] makes no mention of the Φ_{2n-1} of Data 3.3. This last fact is unfortunate from our point of view, because Φ_{2n-1} is a Macaulay inverse system for the quadratically presented ideal J , which is the topic of study in the present paper. In Theorem 3.6, we prove that if one uses Φ_{2n-1} in place of $\tilde{\Phi}_{2n-1}$ in the maps of Theorem 3.5, then one obtains a resolution of R/I which is isomorphic to the resolution of Theorem 3.5. We emphasize that Theorem 3.6 is an extension of Theorem 3.5; it is more than a change of notation. The element of (3.2.1) is the “ $\tilde{\Phi}_{2n-1}$ ” of Data 3.3. The technique of [7] becomes more valuable when one allows Φ_{2n-1} to be any element of $D_{2n-1}U^*$ with $x(\Phi_{2n-1}) = \Phi_{2n-2}$ rather than insisting that Φ_{2n-1} be the element “ $\tilde{\Phi}_{2n-1}$ ” of (3.2.1) and Data 3.3.

Data 3.3. Let \mathbf{k} be a field, U be a vector space over \mathbf{k} of dimension 3, R be the polynomial ring $R = \text{Sym}_{\bullet} U$, and n be a positive integer. Decompose U as $\mathbf{k}x \oplus U_0$ where x is a non-zero element of U and U_0 is a two-dimensional subspace of U . Let Φ_{2n-1} be an element of $D_{2n-1}U^*$. Assume that the homomorphism

$$p : \text{Sym}_{n-1} U \rightarrow D_{n-1}U^*,$$

which is given by

$$p(\mu_{n-1}) = \mu_{n-1}(x(\Phi_{2n-1})), \quad \text{for } \mu_{n-1} \in \text{Sym}_{n-1} U,$$

is an isomorphism. Let $\rho_{0,2n-1}$ be the element of $D_{2n-1}U_0^*$ with

$$\mu_{0,2n-1}(\Phi_{2n-1} - \rho_{0,2n-1}) = 0, \quad \text{for all } \mu_{0,2n-1} \in \text{Sym}_{2n-1} U_0.$$

Let

$$\tilde{\Phi}_{2n-1} = \Phi_{2n-1} - \rho_{0,2n-1} \in D_{2n-1}U^*.$$

Let I be the following ideal of R :

$$I = \{ \mu \in R \mid \mu(x(\Phi_{2n-1})) = 0 \}.$$

Remarks 3.4.

- (a) Here is an alternate description of the element $\rho_{0,2n-1}$. For each index j , let $(x^j)^*$ be the element of $D_j U^* = \text{Hom}_{\mathbf{k}}(\text{Sym}_j U, \mathbf{k})$ with $(x^j)^*(x^j) = 1$ and $(x^j)^*(\text{Sym}_j U_0) = 0$. Write Φ_{2n-1} in the form $\Phi_{2n-1} = \sum_{i=0}^{2n-1} \rho_{0,i}(x^{2n-1-i})^*$ with $\rho_{0,i} \in D_i U_0^*$. Then $\rho_{0,2n-1}$ is the element described in Data 3.3.
- (b) The hypothesis that the homomorphism p of Data 3.3 is an isomorphism is equivalent to the hypothesis that I is linearly presented; see, Proposition 2.12.

(c) The $\widetilde{\mathbb{B}}$ of Theorem 3.5 may be found as [7, Def. 2.7]. Other versions of $\widetilde{\mathbb{B}}$ may also be found as [7, Observation 4.4], or [7, Proposition 5.5], or [8, Definition 3.1].

Theorem 3.5. [7, Thm. 4.1] *Adopt Data 3.3. One minimal homogeneous resolution of R/I by free R -modules is*

$$\widetilde{\mathbb{B}} : 0 \rightarrow B_3 \xrightarrow{\widetilde{b}_3} B_2 \xrightarrow{\widetilde{b}_2} B_1 \xrightarrow{\widetilde{b}_1} B_0$$

with $B_3 = B_0 = R$,

$$B_2 = \begin{cases} R \otimes \text{Sym}_{n-1} U_0 \\ \oplus \\ R \otimes D_n U_0^* \end{cases} \quad \text{and} \quad B_1 = \begin{cases} R \otimes D_{n-1} U_0^* \\ \oplus \\ R \otimes \text{Sym}_n U_0, \end{cases}$$

$$\widetilde{b}_1(\nu_{0,n-1}) = xp^{-1}(\nu_{0,n-1}),$$

$$\widetilde{b}_1(\mu_{0,n}) = \mu_{0,n} - xp^{-1}(\mu_{0,n}(\widetilde{\Phi}_{2n-1})),$$

$$\widetilde{b}_2 \begin{bmatrix} \mu_{0,n-1} \\ 0 \end{bmatrix} = \begin{cases} +x \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(z\mu_{0,n-1})(\widetilde{\Phi}_{2n-1}) \right] \right) \left[(ym_1)(\widetilde{\Phi}_{2n-1}) \right] \otimes m_1^* \\ -x \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(y\mu_{0,n-1})(\widetilde{\Phi}_{2n-1}) \right] \right) \left[(zm_1)(\widetilde{\Phi}_{2n-1}) \right] \otimes m_1^* \\ \hline +x \sum_{m_2 \in \binom{y,z}{n}} \left(p^{-1} \left[(z\mu_{0,n-1})(\widetilde{\Phi}_{2n-1}) \right] \right) (y(m_2^*)) \otimes m_2 \\ -x \sum_{m_2 \in \binom{y,z}{n}} \left(p^{-1} \left[(y\mu_{0,n-1})(\widetilde{\Phi}_{2n-1}) \right] \right) (z(m_2^*)) \otimes m_2 \\ +y \otimes z\mu_{0,n-1} - z \otimes y\mu_{0,n-1}, \end{cases}$$

$$\widetilde{b}_2 \begin{bmatrix} 0 \\ \nu_{0,n} \end{bmatrix} = \begin{cases} -x \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(zm_1)(\widetilde{\Phi}_{2n-1}) \right] \right) (y(\nu_{0,n})) \otimes m_1^* \\ +x \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(ym_1)(\widetilde{\Phi}_{2n-1}) \right] \right) (z(\nu_{0,n})) \otimes m_1^* \\ -y \otimes z(\nu_{0,n}) + z \otimes y(\nu_{0,n}) \\ \hline +x \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(z(\nu_{0,n}))][y(m_2^*)] \otimes m_2 \\ -x \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(y(\nu_{0,n}))][z(m_2^*)] \otimes m_2, \end{cases}$$

and

$$\widetilde{b}_3(1) = \begin{bmatrix} \sum_{m_1 \in \binom{y,z}{n-1}} \widetilde{b}_1(m_1^*) \otimes m_1 \\ \sum_{m_2 \in \binom{y,z}{n}} \widetilde{b}_1(m_2) \otimes m_2^* \end{bmatrix},$$

for $\mu_{0,i} \in \text{Sym}_i U_0$ and $\nu_{0,i} \in D_i U_0^*$.

The graded Betti numbers of $\widetilde{\mathbb{B}}$ from Theorem 3.5, and also the graded Betti numbers of \mathbb{B} from Theorem 3.6, are given in (1.0.1).

Theorem 3.6. *Adopt Data 3.3. A second minimal homogeneous resolution of R/I by free R -modules is*

$$\mathbb{B} : 0 \rightarrow B_3 \xrightarrow{b_3} B_2 \xrightarrow{b_2} B_1 \xrightarrow{b_1} B_0, \tag{3.6.1}$$

with B_i as given in Theorem 3.5, and each \widetilde{b}_i replaced by b_i , where the formula for b_i is exactly the same as the formula for \widetilde{b}_i , except $\widetilde{\Phi}_{2n-1}$ is replaced with Φ_{2n-1} .

Proof. It is not difficult to see that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_3 & \xrightarrow{b_3} & B_2 & \xrightarrow{b_2} & B_1 & \xrightarrow{b_1} & B_0 \\ & & \downarrow = & & \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow = \\ 0 & \longrightarrow & B_3 & \xrightarrow{\widetilde{b}_3} & B_2 & \xrightarrow{\widetilde{b}_2} & B_1 & \xrightarrow{\widetilde{b}_1} & B_0, \end{array}$$

with

$$\begin{aligned} \theta_1 \left(\begin{bmatrix} \nu_{0,n-1} \\ \mu_{0,n} \end{bmatrix} \right) &= \begin{bmatrix} \nu_{0,n-1} - \mu_{0,n}(\rho_{0,2n-1}) \\ \mu_{0,n} \end{bmatrix} \quad \text{and} \\ \theta_2 \left(\begin{bmatrix} \mu_{0,n-1} \\ \nu_{0,n} \end{bmatrix} \right) &= \begin{bmatrix} \mu_{0,n-1} \\ \mu_{0,n-1}(\rho_{0,2n-1}) + \nu_{0,n} \end{bmatrix}, \end{aligned}$$

for $\mu_{0,i} \in \text{Sym}_i U_0$ and $\nu_{0,i} \in D_i U_0^*$, is an isomorphism of complexes. \square

3.7. *The proof of Theorem 3.1.* The map b_2 of Theorem 3.6 is

$$b_2 \begin{bmatrix} \mu_{0,n-1} \\ \nu_{0,n} \end{bmatrix} = \begin{bmatrix} x \otimes f_1(\mu_{0,n-1}) + x \otimes f_2(\nu_{0,n}) + f_3(\nu_{0,n}) \\ x \otimes f_4(\mu_{0,n-1}) + f_5(\mu_{0,n-1}) + x \otimes f_6(\nu_{0,n}) \end{bmatrix},$$

where

$$\begin{aligned} f_1 : \text{Sym}_{n-1} U_0 &\rightarrow D_{n-1} U_0^*, & f_2 : D_n U_0^* &\rightarrow D_{n-1} U_0^*, \\ f_3 : D_n U_0^* &\rightarrow R \otimes_{\mathbf{k}} D_{n-1} U_0^* & f_4 : \text{Sym}_{n-1} U_0 &\rightarrow \text{Sym}_n U_0 \\ f_5 : \text{Sym}_{n-1} U_0 &\rightarrow R \otimes_{\mathbf{k}} \text{Sym}_n U_0, \text{ and} & f_6 : D_n U_0^* &\rightarrow \text{Sym}_n U_0, \end{aligned}$$

are given by

$$f_1(\mu_{0,n-1}) = \begin{cases} \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(z\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) \left[(ym_1)(\Phi_{2n-1}) \right] \cdot m_1^* \\ - \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(y\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) \left[(zm_1)(\Phi_{2n-1}) \right] \cdot m_1^*, \end{cases}$$

$$\begin{aligned}
 f_2(\nu_{0,n}) &= \begin{cases} \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(ym_1)(\Phi_{2n-1}) \right] \right) (z(\nu_{0,n})) \cdot m_1^* \\ - \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(zm_1)(\Phi_{2n-1}) \right] \right) (y(\nu_{0,n})) \cdot m_1^*, \end{cases} \\
 f_3(\nu_{0,n}) &= -y \otimes z(\nu_{0,n}) + z \otimes y(\nu_{0,n}), \\
 f_4(\mu_{0,n-1}) &= \begin{cases} \sum_{m_2 \in \binom{y,z}{n}} \left(p^{-1} \left[(z\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) (y(m_2^*)) \cdot m_2 \\ - \sum_{m_2 \in \binom{y,z}{n}} \left(p^{-1} \left[(y\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) (z(m_2^*)) \cdot m_2, \end{cases} \\
 f_5(\mu_{0,n-1}) &= y \otimes z\mu_{0,n-1} - z \otimes y\mu_{0,n-1}, \text{ and} \\
 f_6(\nu_{0,n}) &= \begin{cases} \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(z(\nu_{0,n}))][y(m_2^*)] \cdot m_2 \\ - \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(y(\nu_{0,n}))][z(m_2^*)] \cdot m_2. \end{cases}
 \end{aligned}$$

The coefficients in $f_1, f_2, f_4,$ and f_6 all are elements of \mathbf{k} ; so we have written

$$\text{coefficient} \cdot \text{basis vector.}$$

On the other hand, the coefficients of f_3 and f_5 are homogeneous of degree one in R ; so we continue to write

$$\text{coefficient} \otimes \text{basis vector.}$$

We complete the proof of Theorem 3.1 by showing that the matrices of $f_1, \dots, f_6,$ with respect to the ordered bases of 2.8 are

map	matrix	map	matrix
f_1	A'	f_2	B_1
f_3	B_2	f_4	$-B_1^T$
f_5	$-B_2^T$	f_6	$D_0 - D_0^T$

The matrices \mathbf{p} and \mathbf{r} are defined in the statement of Theorem 3.1. Notice that \mathbf{p} is the matrix for the map $p : \text{Sym}_{n-1} U \rightarrow D_{n-1} U^*$ of Data 3.3 in the bases of 2.8. Let

$$r : \text{Sym}_n U_0 \rightarrow D_{n-1} U^*$$

be the map defined by

$$r(\mu_{0,n}) = \mu_{0,n}(\Phi_{2n-1}),$$

for $\mu_{0,n} \in \text{Sym}_n U_0$. Observe that \mathbf{r} is the matrix for r in the bases of 2.8.

The module $\text{Sym}_n U$ is equal to the direct sum

$$x \text{Sym}_{n-1} U \oplus \text{Sym}_n U_0.$$

The choice of basis for $\text{Sym}_n U$, as described in (2.8.1), leads to the observation that the matrix for $\text{Sym}_n U \rightarrow D_{n-1}U^*$, given by

$$\mu_n \mapsto \mu_n(\Phi_{2n-1}), \tag{3.7.1}$$

for $\mu_n \in \text{Sym}_n U$, is

$$\left[\mathbf{p} \mid \mathbf{r} \right]. \tag{3.7.2}$$

Furthermore, the dual of (3.7.1) is the map

$$\text{Sym}_{n-1} U \rightarrow D_n U^*, \tag{3.7.3}$$

which is given by

$$\mu_{n-1} \mapsto \mu_{n-1}(\Phi_{2n-1}).$$

The matrix for (3.7.3) is

$$\left[\mathbf{p} \mid \mathbf{r} \right]^T = \begin{bmatrix} \mathbf{p}^T \\ \mathbf{r}^T \end{bmatrix}.$$

Let f_{11} and f_{12} be the homomorphisms

$$\text{Sym}_{n-1} U_0 \rightarrow D_{n-1}U_0^*$$

given by

$$\begin{aligned} f_{11}(\mu_{0,n-1}) &= \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(z\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) \left[(ym_1)(\Phi_{2n-1}) \right] \cdot m_1^* \text{ and} \\ f_{12}(\mu_{0,n-1}) &= \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(y\mu_{0,n-1})(\Phi_{2n-1}) \right] \right) \left[(zm_1)(\Phi_{2n-1}) \right] \cdot m_1^*. \end{aligned}$$

We first prove that A_0 is the matrix for f_{11} . Observe that

$$f_{11}(\mu_{0,n-1}) = y \left(\text{proj}_{D_n U_0^*} \left[(p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])(\Phi_{2n-1}) \right] \right). \tag{3.7.4}$$

(See Lemma 3.8 for details.) Thus, f_{11} is the composition of the following six maps:

$$\text{Sym}_{n-1} U_0 \xrightarrow{\text{map}_1} \text{Sym}_n U_0 \xrightarrow{\text{map}_2} D_{n-1} U^* \xrightarrow{\text{map}_3} \text{Sym}_{n-1} U \xrightarrow{\text{map}_4} D_n U^* \xrightarrow{\text{map}_5} D_n U_0^* \xrightarrow{\text{map}_6} D_{n-1} U_0^*,$$

where

$$\begin{aligned} \text{map}_1(\mu_{0,n-1}) &= z\mu_{0,n-1}, & \text{map}_2 &= r, & \text{map}_3 &= p^{-1}, \\ \text{map}_4(\mu_{n-1}) &= \mu_{n-1}(\Phi_{2n-1}), & \text{map}_5((x\mu_{n-1})^*) &= 0, & & (3.7.5) \\ \text{map}_5((\mu_{0,n})^*) &= (\mu_{0,n})^*, \text{ and} & \text{map}_6(\nu_{0,n}) &= y(\nu_{0,n}), & & \end{aligned}$$

for

$$\mu_{0,i} \in \text{Sym}_i U_0, \quad \mu_i \in \text{Sym}_i U, \quad \text{and} \quad \nu_{0,i} \in D_i U_0^*.$$

The matrix for map_1 in the bases of 2.8 is

	y^{n-1}	$y^{n-2}z$	\dots	yz^{n-2}	z^{n-1}
y^n	0	0	\dots	0	0
$y^{n-1}z$	1	0	\dots	0	0
$y^{n-2}z^2$	0	1	\dots	0	0
\vdots	\vdots	\ddots	\ddots	\ddots	\vdots
yz^{n-1}	0	0	\dots	1	0
z^n	0	0	\dots	0	1

the matrix for map_2 is \mathbf{r} ; and the matrix for map_3 is \mathbf{p}^{-1} . Observe that map_4 is (3.7.3); hence, the matrix for map_4 is

$$\begin{bmatrix} \mathbf{p}^T \\ \mathbf{r}^T \end{bmatrix}.$$

Use the convention of 2.8 to see that the matrix for map_5 is

$$\left[0_{(n+1) \times \binom{n+1}{2}} \mid I_{n+1} \right];$$

and the matrix for map_6 is

	$(y^n)^*$	$(y^{n-1}z)^*$	\dots	$(y^2z^{n-2})^*$	$(yz^{n-1})^*$	$(z^n)^*$
$(y^{n-1})^*$	1	0	\dots	0	0	0
$(y^{n-2}z)^*$	0	1	\ddots	0	0	0
\vdots	\vdots	\ddots	\ddots	\ddots	\vdots	\vdots
$(yz^{n-2})^*$	0	0	\ddots	1	0	0
$(z^{n-1})^*$	0	0	\dots	0	1	0

The matrix for $\text{map}_5 \circ \text{map}_4 \circ \text{map}_3 \circ \text{map}_2$ is easily seen to be $\mathbf{r}^T \mathbf{p}^{-1} \mathbf{r}$. The matrix for map_1 deletes column 1 and the matrix for map_6 deletes row $n + 1$. It follows that A_0 is the matrix for f_{11} .

In a similar manner the matrix for f_{12} is

$$A'_0 = [\mathbf{r}^T \mathbf{p}^{-1} \mathbf{r}] \text{ with row 1 and column } n + 1 \text{ deleted.}$$

The matrix \mathbf{p} is symmetric; hence, the matrix $[\mathbf{r}^T \mathbf{p}^{-1} \mathbf{r}]$ is symmetric and A'_0 is equal to the transpose of A_0 . We conclude that the matrix for f_1 , which is equal to $f_{11} - f_{12}$, is $A_0 - A_0^T = A'$.

Let f_{21} and f_{22} be the homomorphisms

$$D_n U_0^* \rightarrow D_{n-1} U_0^*$$

given by

$$f_{21}(\nu_{0,n}) = \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(y m_1)(\Phi_{2n-1}) \right] \right) (z(\nu_{0,n})) \cdot m_1^* \text{ and}$$

$$f_{22}(\nu_{0,n}) = \sum_{m_1 \in \binom{y,z}{n-1}} \left(p^{-1} \left[(z m_1)(\Phi_{2n-1}) \right] \right) (y(\nu_{0,n})) \cdot m_1^*.$$

Use the techniques of Lemma 3.8 to see that

$$f_{21}(\nu_{0,n}) = y \left[\text{proj}_{D_n U_0^*} \left([p^{-1}(z(\nu_{0,n}))](\Phi_{2n-1}) \right) \right].$$

Thus, the map f_{21} is the composition of the following six maps:

$$D_n U_0^* \xrightarrow{\text{map}_7} D_{n-1} U_0^* \xrightarrow{\text{map}_8} D_{n-1} U^* \xrightarrow{\text{map}_3} \text{Sym}_{n-1} U \xrightarrow{\text{map}_4} D_n U^* \xrightarrow{\text{map}_5} D_n U_0^* \xrightarrow{\text{map}_6} D_{n-1} U_0^*,$$

where $\text{map}_7(\nu_{n,0}) = z(\nu_{n,0})$, map_8 is inclusion, and $\text{map}_3, \text{map}_4, \text{map}_5$, and map_6 are given defined in (3.7.5). The matrix for map_7 in the bases of 2.8 is

	$(y^n)^*$	$(y^{n-1}z)^*$	$(y^{n-2}z^2)^*$	\dots	$(yz^{n-1})^*$	$(z^n)^*$
$(y^{n-1})^*$	0	1	0	0	\dots	0
$(y^{n-2}z)^*$	0	0	1	0	\dots	0
\vdots	\vdots	\vdots	0	\ddots	\ddots	\vdots
$(yz^{n-2})^*$	0	\vdots	\vdots	\ddots	1	0
$(z^{n-1})^*$	0	0	0	\dots	0	1

$$= \left[0_{n \times 1} \mid I_n \right]; \tag{3.7.6}$$

the matrix for map_8 is

$$\left[\begin{array}{c} 0_{\binom{n}{2} \times n} \\ I_n \end{array} \right]; \tag{3.7.7}$$

the matrix for $\text{map}_6 \circ \text{map}_5 \circ \text{map}_4 \circ \text{map}_3$ continues to be

$$\left[I_n \mid 0_{n \times 1} \right] \mathbf{r}^T \mathbf{p}^{-1}.$$

Observe that

$$\mathbf{r}^T \mathbf{p}^{-1} \left[\begin{array}{c} 0_{\binom{n}{2} \times n} \\ I_n \end{array} \right] = [\text{the right most } n \text{ columns of } \mathbf{r}^T \mathbf{p}^{-1}] = B_0.$$

It follows that the matrix for f_{21} is

$$\left[I_n \mid 0_{n \times 1} \right] B_0 \left[0_{n \times 1} \mid I_n \right] = \left[0_{n \times 1} \mid B_0 \text{ with row } n + 1 \text{ deleted} \right].$$

The map f_{22} is obtained from the map f_{21} by exchanging the role of y and z ; hence the matrix for f_{22} is

$$\left[0_{n \times 1} \mid I_n \right] B_0 \left[I_n \mid 0_{n \times 1} \right] = \left[B_0 \text{ with row } 1 \text{ deleted} \mid 0_{n \times 1} \right].$$

It follows that the matrix for $f_2 = f_{21} - f_{22}$ is

$$\left[0_{n \times 1} \mid B_0 \text{ with row } n + 1 \text{ deleted} \right] - \left[B_0 \text{ with row } 1 \text{ deleted} \mid 0_{n \times 1} \right] = B_1.$$

We have already seen that the matrix for $\text{map}_7(\nu_{0,n}) = z(\nu_{0,n})$ is $\left[0_{n \times 1} \mid I_n \right]$ and the matrix for $\text{map}_6(\nu_{0,n}) = y(\nu_{0,n})$ is $\left[I_n \mid 0_{n \times 1} \right]$. It follows that the matrix for f_3 is

$$\left[zI_n \mid 0_{n \times 1} \right] - \left[0_{n \times 1} \mid yI_n \right] = B_2.$$

The maps f_2 and f_4 satisfy

$$[f_2(\nu_{0,n})](\mu_{0,n-1}) + \nu_{0,n}[f_4(\mu_{0,n-1})] = 0;$$

hence the matrix for f_4 is minus the transpose of the matrix for f_2 . Similarly,

$$[f_3(\nu_{0,n})](\mu_{0,n-1}) + \nu_{0,n}[f_5(\mu_{0,n-1})] = 0;$$

hence the matrix for f_5 is minus the transpose of the matrix for f_3 .

Let f_{61} and f_{62} be the homomorphisms

$$D_n U_0^* \rightarrow \text{Sym}_n U_0,$$

which are given by

$$f_{61}(\nu_{0,n}) = \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(z(\nu_{0,n}))][y(m_2^*)] \cdot m_2 \text{ and}$$

$$f_{62}(\nu_{0,n}) = \sum_{m_2 \in \binom{y,z}{n}} [p^{-1}(y(\nu_{0,n}))][z(m_2^*)] \cdot m_2.$$

Use the techniques of Lemma 3.8 to see that

$$f_{61}(\nu_{0,n}) = y \cdot \text{proj}_{\text{Sym}_{n-1} U_0} [p^{-1}(z(\nu_{0,n}))].$$

The map f_{61} is the composition

$$D_n(U_0^*) \xrightarrow{\text{map}_7} D_{n-1}U_0^* \xrightarrow{\text{map}_3 \circ \text{map}_8} \text{Sym}_{n-1} U \xrightarrow{\text{map}_9} \text{Sym}_{n-1} U_0 \xrightarrow{\text{map}_{10}} \text{Sym}_n U_0,$$

where map_9 is projection, and map_{10} is multiplication in $\text{Sym}_\bullet U_0$ by y . The matrix for map_7 is given in (3.7.6); the matrix for $\text{map}_3 \circ \text{map}_8$ is the product of \mathbf{p}^{-1} and (3.7.7); the matrix for map_9 is

$$\left[\begin{array}{c|c} 0_{n \times \binom{n}{2}} & I_n \end{array} \right];$$

and the matrix for map_{10} is

$$\left[\begin{array}{c} I_n \\ \hline 0_{1 \times n} \end{array} \right].$$

It follows that the matrix for f_{61} is the product

$$\begin{aligned} & \left[\begin{array}{c} I_n \\ \hline 0_{1 \times n} \end{array} \right] \left[\begin{array}{c|c} 0_{n \times \binom{n}{2}} & I_n \end{array} \right] \left[\begin{array}{c} \text{The right most } n \text{ columns} \\ \hline \text{of } \mathbf{p}^{-1}. \end{array} \right] \left[\begin{array}{c|c} 0_{n \times 1} & I_n \end{array} \right] \\ &= \left[\begin{array}{c} I_n \\ \hline 0_{1 \times n} \end{array} \right] \left[\text{the bottom right } n \times n \text{ submatrix of } \mathbf{p}^{-1} \right] \left[\begin{array}{c|c} 0_{n \times 1} & I_n \end{array} \right] \\ &= \left[\begin{array}{c|c} 0_{n \times 1} & \text{the bottom right } n \times n \text{ submatrix of } \mathbf{p}^{-1} \\ \hline 0_{1 \times 1} & 0_{1 \times n} \end{array} \right] = D_0. \end{aligned}$$

The matrix for f_{62} is obtained from the matrix for f_{61} by exchanging the role of y and z . In other words, the matrix for f_{62} is

$$\begin{aligned} & \left[\begin{array}{c} 0_{1 \times n} \\ \hline I_n \end{array} \right] \left[\text{the bottom right } n \times n \text{ submatrix of } \mathbf{p}^{-1} \right] \left[\begin{array}{c|c} I_n & 0_{n \times 1} \end{array} \right] \\ &= \left[\begin{array}{c|c} 0_{1 \times n} & 0_{1 \times 1} \\ \hline \text{the bottom right } n \times n \text{ submatrix of } \mathbf{p}^{-1} & 0_{n \times 1} \end{array} \right]. \end{aligned} \tag{3.7.8}$$

The matrix \mathbf{p} is symmetric; so (3.7.8) is the transpose of the matrix for f_{61} and the matrix for $f_6 = f_{61} - f_{62}$ is $D_0 - D_0^T$. \square

At (3.7.4) we promised to exhibit the following calculation.

Lemma 3.8. *The elements*

$$\sum_{m_1 \in \binom{y,z}{n-1}} (p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])[(ym_1)(\Phi_{2n-1})] \cdot m_1^*$$

and

$$y(\text{proj}_{D_n U_0^*} [(p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])](\Phi_{2n-1}))$$

of $D_{n-1}U_0^*$ are equal.

Proof. Recall that $p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})]$ and y and m_1 are all elements of the commutative ring $\text{Sym}_\bullet U$ and Φ_{2n-1} is an element of the $\text{Sym}_\bullet U$ -module $D_\bullet U^*$. It follows that

$$\begin{aligned} & (p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])[(ym_1)(\Phi_{2n-1})] \\ &= m_1 [y((p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])](\Phi_{2n-1}), \end{aligned}$$

where

$$\begin{aligned} & (p^{-1}[(z\mu_{0,n-1})(\Phi_{2n-1})])(\text{an element of } D_\bullet U^*), \\ & y(\text{an element of } D_\bullet U^*), \text{ and} \\ & m_1(\text{an element of } D_\bullet U^*) \end{aligned}$$

all represent the module action of $\text{Sym}_\bullet U$ on $D_\bullet U^*$. If X is an element of $D_{n-1}U_0^*$, then

$$\sum_{m_1 \in \binom{y,z}{n-1}} m_1(X) \cdot m_1^* = \text{proj}_{D_{n-1}U_0^*} X$$

because

$$\sum_{m_1 \in \binom{y,z}{n-1}} m_1(X_0) \cdot m_1^* = X_0,$$

if $X_0 \in D_{n-1}U_0^*$; and

$$\sum_{m_1 \in \binom{y,z}{n-1}} m_1(X_1) \cdot m_1^* = 0,$$

if X_1 is in the submodule of $D_{n-1}U$ which is generated by

$$\left\{ (xm)^* \mid m \in \binom{x, y, z}{n-2} \right\}.$$

Thus,

$$\begin{aligned} & \sum_{m_1 \in \binom{y, z}{n-1}} (p^{-1}[(z\mu_{0, n-1})(\Phi_{2n-1})])[(ym_1)(\Phi_{2n-1})] \cdot m_1^* \\ &= \text{proj}_{D_{n-1}U_0^*} (y[(p^{-1}[(z\mu_{0, n-1})(\Phi_{2n-1})])](\Phi_{2n-1})) \\ &= y(\text{proj}_{D_n U_0^*} [(p^{-1}[(z\mu_{0, n-1})(\Phi_{2n-1})])](\Phi_{2n-1})). \end{aligned}$$

The final equality holds because the diagram

$$\begin{array}{ccc} D_n U^* & \xrightarrow{\text{proj}} & D_n U_0^* \\ \downarrow y & & \downarrow y \\ D_{n-1} U^* & \xrightarrow{\text{proj}} & D_{n-1} U_0^* \end{array}$$

commutes. \square

Remark 3.9. We proved Theorem 3.1 by proving that the b_2 of Theorem 3.1 is the matrix for the coordinate-free map b_2 of Theorem 3.6. In each of the Theorems, 3.1 and 3.6, the image of b_3 is the kernel of b_2 and $b_1 = \text{Hom}_R(b_3, R)$. It follows that b_1 from Theorem 3.6 is equal to a unit of \mathbf{k} times b_1 from Theorem 3.1. There are times that the explicit formulas of Theorem 3.6 are more useful than the instruction “take the maximal order Pfaffians of this alternating matrix”.

4. The main theorem

Theorem 4.2 is the main result in the paper.

Observation 4.1. Let $R = \mathbf{k}[x, y, z]$ be a standard graded polynomial ring over a field. If J is a quadratically presented grade three Gorenstein ideal in R , then the Betti numbers of the minimal free resolution of R/J by free R -modules have the form of (1.0.2) for some positive even number n . In particular, the socle degree of R/J is $2n - 1$ and the minimal homogeneous generators of J all have degree n .

Proof. Apply the Theorem of Buchsbaum and Eisenbud [4] to see that J is minimally generated by the maximal order Pfaffians of some alternating matrix X of odd size, whose entries, by our hypothesis, are quadratic forms in R . Let X have $n+1$ rows and columns, where n is even; and let \underline{x} be the row vector of signed maximal order Pfaffians of X .

(See 2.7, if necessary.) The minimal homogeneous resolution of R/J by free R -modules has the form

$$0 \rightarrow R(-(2n - 2)) \xrightarrow{\underline{x}^T} R(-(n + 2))^{n+1} \xrightarrow{X} R(-n)^{n+1} \xrightarrow{\underline{x}} R. \quad \square$$

In Theorem 4.2, J is a homogeneous grade three Gorenstein ideal $R = \mathbf{k}[x, y, z]$. We assume that the socle degree of R/J is $2n - 1$, for some positive even integer n , and that all homogeneous generators of J have degree at least n . We also assume that x is a weak Lefschetz element on R/J . We determine if J is quadratically presented; and if so, then we give the presentation matrix of J in terms of the Macaulay inverse system for J .

Theorem 4.2. *Let $R = \mathbf{k}[x, y, z]$ be a standard graded polynomial ring over the field \mathbf{k} and n be a positive even integer. Let U be the vector space R_1 , Φ_{2n-1} be an element of $D_{2n-1}U^*$ and J be the ideal*

$$J = \text{ann}_{\text{Sym}_{\bullet} U} \Phi_{2n-1}$$

of $R = \text{Sym}_{\bullet} U$. Assume that $J_{n-1} = 0$ and that x is a weak Lefschetz element on R/J . Let

$$I = \text{ann}_{\text{Sym}_{\bullet} U} x(\Phi_{2n-1}).$$

Then the following statements hold.

- (a) *The ideal I is a linearly presented grade three Gorenstein ideal in R and the minimal homogeneous resolution of R/I by free R -modules is*

$$0 \rightarrow R(-2n - 1) \xrightarrow{b_3} R^{2n+1}(-n - 1) \xrightarrow{b_2} R^{2n+1}(-n) \xrightarrow{b_1} R,$$

where

$$b_2 = \left[\begin{array}{c|c} xA' & B \\ \hline -B^T & D \end{array} \right],$$

as described in Theorem 3.1.

- (b) *Let g_i be $(-1)^{i+1}$ times the Pfaffian of the submatrix of b_2 obtained by deleting row and column i . Then g_{n+1}, \dots, g_{2n+1} is part of a minimal generating set for J .*
- (c) *The ideal J is quadratically presented if and only if the matrix A' is invertible.*
- (d) *If J is quadratically presented, then $J = (g_{n+1}, \dots, g_{2n+1})$ and the minimal homogeneous resolution of R/J by free R -modules is*

$$0 \rightarrow R(-2n - 2) \xrightarrow{c_3} R(-n - 2)^{n+1} \xrightarrow{c_2} R(-n)^{n+1} \xrightarrow{c_1} R, \tag{4.2.1}$$

where

$$c_2 = B^T(A')^{-1}B + xD$$

is an alternating matrix, c_1 is equal to the row vector of signed (see (2.7)) maximal order Pfaffians of c_2 , and $c_3 = c_1^T$.

Proof. (a) Let $p : \text{Sym}_{n-1}U \rightarrow D_{n-1}U^*$ be the \mathbf{k} -module homomorphism with

$$p(\mu_{n-1}) = \mu_{n-1}(x\Phi_{2n-1}), \quad \text{for } \mu_{n-1} \in \text{Sym}_{n-1}U.$$

According to Proposition 2.12 it suffices to show that p is injective. Suppose μ_{n-1} , in $\text{Sym}_{n-1}U$, is in the kernel of p . Then $(x\mu_{n-1})(\Phi_{2n-1}) = 0$; $x\mu_{n-1} \in J$; and μ_{n-1} represents an element in the kernel of the \mathbf{k} -module map

$$(R/J)_{n-1} \xrightarrow{x} (R/J)_n, \tag{4.2.2}$$

which is given by multiplication by x . On the other hand, the hypothesis that x is a weak Lefschetz element on R/J guarantees that the rank of (4.2.2) is equal to

$$\min\{\dim_{\mathbf{k}}(R/J)_{n-1}, \dim_{\mathbf{k}}(R/J)_n\}.$$

The Hilbert function of the standard graded Artinian Gorenstein algebra R/J , with socle degree $2n - 1$, is symmetric in the sense that

$$\dim_{\mathbf{k}}(R/J)_i = \dim_{\mathbf{k}}(R/J)_{2n-1-i},$$

for all i . It follows that $(R/J)_{n-1}$ and $(R/J)_n$ have the same dimension; (4.2.2) is injective; and $\mu_{n-1} \in J_{n-1} = 0$.

(b) Recall that Theorem 3.1 is Theorem 3.6 written in the language of matrices rather than the language of multilinear algebra. The assertion of (b) is obvious in the language of multilinear algebra. Indeed, the row vector

$$[g_{n+1}, \dots, g_{2n+1}] \tag{4.2.3}$$

is equal to some unit of \mathbf{k} times the row vector

$$[b_1(m_1), \dots, b_1(m_{n+1})],$$

where m_1, \dots, m_{n+1} is the basis $y^n, y^{n-1}z, \dots, yz^{n-1}, z^n$ of $\text{Sym}_n U_0$, as given in 2.8, and

$$b_1(\mu_{0,n}) = \mu_{0,n} - xp^{-1}(\mu_{0,n}(\Phi_{2n-1})), \quad \text{for } \mu_{0,n} \in \text{Sym}_n U_0,$$

as defined in the statement of Theorem 3.6. Fix $\mu_{0,n} \in \text{Sym}_n U_0$. We show that $b_1(\mu_{0,n})$ is in J by showing that $[b_1(\mu_{0,n})](\Phi_{2n-1}) = 0$. Observe that

$$\begin{aligned} [xp^{-1}(\mu_{0,n}(\Phi_{2n-1}))](\Phi_{2n-1}) &= [p^{-1}(\mu_{0,n}(\Phi_{2n-1}))](x(\Phi_{2n-1})) \\ &= p[p^{-1}(\mu_{0,n}(\Phi_{2n-1}))] = \mu_{0,n}(\Phi_{2n-1}). \end{aligned}$$

Hence, g_{n+1}, \dots, g_{2n+1} are elements of J of degree n which form part of a minimal generating set of I . By hypothesis, there are no elements of J of degree less than n . It follows that g_{n+1}, \dots, g_{2n+1} is part of a minimal generating set of J . Let J' be the ideal $(g_{n+1}, \dots, g_{2n+1})$.

(c) and (d) We first show that if A' is not invertible, then there is a non-zero linear relation on the minimal generators of J .

Recall that A' is an $n \times n$ matrix of elements of \mathbf{k} . If A' is not invertible, then there exists a non-zero vector $v \in \mathbf{k}^n$ with $Av \neq 0$. The matrix b_2 appears in a minimal homogeneous resolution; so, the columns of b_2 are linearly independent and

$$b_2 \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -B^T v \end{bmatrix}$$

is a non-zero linear relation on b_1 . It follows that $-B^T v$ is a non-zero linear relation on (4.2.3). However, according to (b), g_{n+1}, \dots, g_{2n+1} is part of a minimal generating set for J . We conclude that if A' is not invertible, then there is a non-zero linear relation on a minimal generating set of J and J is not quadratically presented.

Now we assume that A' is invertible. We prove that J is quadratically presented and that the complex of (d) is the minimal homogeneous resolution of R/J by free R -modules.

Claim 4.2.4. *If A' is invertible, then the row vector (4.2.3) is a unit of \mathbf{k} times the row vector of signed maximal order Pfaffians of $c_2 = B^T(A')^{-1}B + xD$.*

Fix the index i with $1 \leq i \leq n + 1$; let B' be B with column i deleted and D' be D with row i and column i deleted. Let Pf mean “the Pfaffian of”. The element x of the polynomial ring R is a regular element. No damage is done if we compute in the ring $R[\frac{1}{x}]$. Observe that the Pfaffian of b_2 with row and column $n + i$ deleted is equal to

$$\begin{aligned} & \text{Pf} \left[\begin{array}{c|c} xA' & B' \\ \hline -(B')^T & D' \end{array} \right] \\ &= \text{Pf} \left(\left[\begin{array}{c|c} I_n & 0 \\ \hline \frac{1}{x}(B')^T(A')^{-1} & I_n \end{array} \right] \left[\begin{array}{c|c} xA' & B' \\ \hline -(B')^T & D' \end{array} \right] \left[\begin{array}{c|c} I_n & -\frac{1}{x}(A')^{-1}B' \\ \hline 0 & I_n \end{array} \right] \right) \\ &= \text{Pf} \left[\begin{array}{c|c} xA' & 0 \\ \hline 0 & \frac{1}{x}[(B')^T(A')^{-1}B'] + D' \end{array} \right] \end{aligned}$$

$$\begin{aligned}
 &= \text{Pf}(xA') \cdot \text{Pf}\left(\frac{1}{x} [(B')^T(A')^{-1}B' + xD']\right) \\
 &= x^{n/2} \text{Pf}(A') \frac{1}{x^{n/2}} \text{Pf} [(B')^T(A')^{-1}B' + xD'],
 \end{aligned}$$

which is equal to the Pfaffian of A' times the Pfaffian of c_2 with row and column i deleted. This completes the proof of Claim 4.2.4.

Claim 4.2.5. *The ideal J' is a quadratically presented grade three Gorenstein ideal and the minimal homogeneous resolution of R/J' is given in (4.2.1).*

The row vector (4.2.3) is the row vector of signed maximal order Pfaffians of the $(n + 1) \times (n + 1)$ alternating matrix c_2 , and $n + 1$ is odd. We show that

$$0 \rightarrow R(-2n - 2) \xrightarrow{(4.2.3)^T} R(-n - 2)^{n+1} \xrightarrow{c_2} R(-n)^{n+1} \xrightarrow{(4.2.3)} R$$

is the minimal homogeneous resolution of R/J' by free R -modules by showing that the grade of J' is at least three; see [4] or [3]. In particular, we show that (x, y, z) is contained in the radical of J' . The socle degree of R/I is $2n - 2$; so the ideal $(x, y, z)^{2n-1}$ is contained in I .

The fact that the columns of

$$\begin{bmatrix} xA' \\ -B^T \end{bmatrix}$$

are relations on $[g_1, \dots, g_{2n+1}]$, with A' invertible, $(g_{n+1}, \dots, g_{2n+1}) = J'$, and I equal to $(g_1, \dots, g_n) + J'$ ensures that $xI \subseteq J'$. Thus, $x^{2n} \in J'$.

The matrix B_2 is the $n \times (n + 1)$ matrix

$$B_2 = \begin{bmatrix} z & -y & 0 & \cdots & 0 & 0 \\ 0 & z & -y & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & -y & 0 \\ 0 & \cdots & \cdots & 0 & z & -y \end{bmatrix}.$$

The determinant of B_2 with column $n + 1$ deleted is z^n and the determinant of B_2 with column 1 deleted is y^n . It is well known that if A' is an alternating $n \times n$ matrix and M is an $n \times n$ matrix, then the Pfaffian of $M^T A' M$ is $\det M \text{Pf} A'$; see for example [1, Thm. 3.28]. It follows that z^n and y^n are in the ideal generated by the maximal order Pfaffians of c_2 , together with x . We conclude that (x, y, z) is contained in the radical of J' . This completes the proof of Claim 4.2.5.

We complete the proof by showing that $J' = J$. For this we use the Macaulay duality. The ideals under consideration are related by

$$J' \subseteq J. \tag{4.2.6}$$

The Macaulay inverse system

$$J = \text{ann}_{\text{Sym}_{\bullet} U} \Phi_{2n-1}$$

is known. The socle degree of R/J' is $2n - 1$. Let $\Psi_{2n-1} \in D_{2n-1}U^*$ be the Macaulay inverse system of J' ; hence $J' = \text{ann}_{\text{Sym}_{\bullet} U} \Psi_{2n-1}$. The inclusion of (4.2.6) gives rise to the inclusion

$$\text{ann}_{D_{\bullet}U^*} J \subseteq \text{ann}_{D_{\bullet}U^*} J'.$$

If Φ is in $D_{\bullet}U^*$, then

$$\text{ann}_{D_{\bullet}U^*}(\text{ann}_{\text{Sym}_{\bullet} U}(\Phi)) = (\Phi),$$

where (Φ) is the $\text{Sym}_{\bullet} U$ -submodule of $D_{\bullet}U^*$ generated by Φ . It follows that

$$(\Phi_{2n-1}) \subseteq (\Psi_{2n-1}).$$

Thus, $\Phi_{2n-1} = \alpha\Psi_{2n-1}$ for some unit α in \mathbf{k} and $J = J'$. \square

5. An example

Proposition 5.1. *If \mathbf{k} is a field of characteristic zero, $R = \mathbf{k}[x, y, z]$, n is an even integer, and*

$$J = (x^{n+1}, y^{n+1}, z^{n+1}) : (x + y + z)^{n+1},$$

then the graded Betti numbers of R/J are given in (1.0.2) and x is a weak Lefschetz element for R/J .

Proof. The field \mathbf{k} has characteristic zero. Apply [12, Obs. 3.13.(d)] (which is based on [16, Thm. 5]) to see that R/J is a compressed ring with socle degree $s = 2n - 1$. Now apply [12, Lem. 5.4] (which is based on [2, Prop. 3.2]) to see that the minimal homogeneous resolution of R/J by free R -modules has the form

$$0 \rightarrow R(-2n - 2) \rightarrow \begin{matrix} R(-n - 1)^v \\ \oplus \\ R(-n - 2)^{n+1} \end{matrix} \rightarrow \begin{matrix} R(-n)^{n+1} \\ \oplus \\ R(-n - 1)^v \end{matrix} \rightarrow R, \tag{5.1.1}$$

for some integer v . A minimal generating set for J which consists of $n + 1$ homogeneous forms of degree n is given in [12, Prop. 5.7] (with $\epsilon = 0$ and $d = n + 1$). In particular, J has $v = 0$ minimal generators of degree $n + 1$ and (5.1.1) is equal to (1.0.2).

We complete the argument by showing that x is a Lefschetz element for R/J . We have seen that the minimal generators of J all have degree n and that the socle of R/J has degree $s = 2n - 1$. In particular, $J_{(s-1)/2} = J_{n-1} = 0$ and $(R/J)_{(s-1)/2} = R_{n-1}$. Let $N = \binom{n+1}{2}$, let μ_1, \dots, μ_N be the monomials of R of degree $n - 1$, and Φ in $D_{2n-1}R_1^*$ be a Macaulay inverse system for J , where

$$(-)^* \text{ means } \text{Hom}_{\mathbf{k}}(-, \mathbf{k}).$$

Define $\psi : (R/J)_s \rightarrow \mathbf{k}$ by setting

$$\psi(\bar{\mu}) = \mu(\Phi)$$

where $\mu \in R_s$ and $\bar{\mu}$ is the image of μ in $(R/J)_s$. Observe that $\psi : (R/J)_s \rightarrow \mathbf{k}$ is an isomorphism. Apply Lemma 5.2. In order to show that x is a Lefschetz element for R/J , it suffices to show that $\det(\psi(\mu_i x \mu_j))$ is non-zero.

Observe that

$$(\mu_i x \mu_j)(\Phi) = (\mu_i \mu_j)(x(\Phi))$$

and that $x(\Phi)$ is a Macaulay inverse system for

$$\mathcal{J} = (x^n, y^{n+1}, z^{n+1}) : (x + y + z)^{n+1}.$$

Apply [16, Thm. 5] to see that $\mathcal{J}_{n-1} = 0$. Observe that the socle degree of R/\mathcal{J} is the socle degree of $R/(x^n, y^{n+1}, z^{n+1})$ minus the degree of $(x + y + z)^{n+1}$; hence $2n - 2$. It follows (see, for example, Proposition 2.12) that \mathcal{J} is linearly presented; hence, in particular, the matrix $(\mu_i \mu_j(x\Phi))$ is invertible; see Proposition 2.12. The proof is complete. \square

Lemma 5.2. *Let R be a standard graded polynomial ring over a field \mathbf{k} , and J be a homogeneous ideal of R with R/J Gorenstein and Artinian with odd socle degree s . Fix an isomorphism $\psi : (R/J)_s \rightarrow \mathbf{k}$. Let ℓ be a linear form in R and μ_1, \dots, μ_N be a basis for $(R/J)_{(s-1)/2}$. Consider the $N \times N$ matrix M with entry $\psi(\mu_i \ell \mu_j)$ in row i and column j . Then ℓ is a Lefschetz element for R/J if and only if $\det M \neq 0$.*

Proof. (\Rightarrow) Assume that ℓ is a Lefschetz element for R/J ; thus, multiplication by ℓ from $(R/J)_i$ to $(R/J)_{i+1}$ is an injection whenever $\dim(R/J)_i \leq \dim(R/J)_{i+1}$. The vector spaces $(R/J)_{(s-1)/2}$ and $(R/J)_{(s+1)/2}$ have the same dimension because R/J is Gorenstein. It follows that the multiplication

$$\ell : (R/J)_{(s-1)/2} \rightarrow (R/J)_{(s+1)/2}$$

is an isomorphism. On the other hand, the map

$$(R/J)_{(s-1)/2} \otimes (R/J)_{(s+1)/2} \rightarrow \mathbf{k},$$

which sends $\theta \otimes \theta'$ to $\psi(\theta\theta')$ is a perfect pairing, again because R/J is Gorenstein. Thus, there exists a basis μ'_1, \dots, μ'_N for $(R/J)_{(s+1)/2}$ for which the matrix

$$\psi(\mu_i \mu'_j)$$

looks like the identity matrix. The elements μ'_1, \dots, μ'_N and the elements $\ell\mu_1, \dots, \ell\mu_N$ each form a basis for $(R/J)_{(s+1)/2}$; hence, the matrix M is obtained from the identity matrix by a change of basis. We conclude that $\det M$ is also a unit.

(\Leftarrow) If $\det M$ is a unit, then the elements $\ell\mu_1, \dots, \ell\mu_N$ of $(R/J)_{(s+1)/2}$ are linearly independent. Thus, multiplication by ℓ from $(R/J)_{(s-1)/2}$ to $(R/J)_{(s+1)/2}$ is injective and this is enough to show that ℓ is a Lefschetz element; see, for example, [13, Prop. 2.1]. \square

Corollary 5.3. *If \mathbf{k} is a field of characteristic zero, $R = \mathbf{k}[x, y, z]$, n is an even integer, and*

$$J = (x^{n+1}, y^{n+1}, z^{n+1}) : (x + y + z)^{n+1}, \tag{5.3.1}$$

then the minimal homogeneous resolution of R/J is given by (4.2.1) as described in Theorem 4.2.

Proof. Apply Proposition 5.1 to see that the hypotheses of Theorem 4.2 are satisfied. The element

$$\Phi_{2n-1} = (x + y + z)^{n+1} ((x^n y^n z^n)^*)$$

of $D_{2n-1}R_1^*$ is a Macaulay inverse system for J . \square

Example 5.4. If $n = 2$, then the matrices of Corollary 5.3 are

$$\begin{aligned} \Phi_3 &= (x + y + z)^3 ((x^2 y^2 z^2)^*) \\ &= 3(yz^2)^* + 3(y^2z)^* + 3(xz^2)^* + 3(x^2z)^* + 3(xy^2)^* + 3(x^2y)^* + 6(xyz)^*, \end{aligned}$$

$$\mathbf{p} = \begin{bmatrix} x^3\Phi_3 & x^2y\Phi_3 & x^2z\Phi_3 \\ x^2y\Phi_3 & xy^2\Phi_3 & xyz\Phi_3 \\ x^2z\Phi_3 & xyz\Phi_3 & xz^2\Phi_3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 3 & 6 \\ 3 & 6 & 3 \end{bmatrix},$$

$$\mathbf{r} = \begin{bmatrix} xy^2\Phi_3 & xyz\Phi_3 & xz^2\Phi_3 \\ y^3\Phi_3 & y^2z\Phi_3 & yz^2\Phi_3 \\ y^2z\Phi_3 & yz^2\Phi_3 & z^3\Phi_3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 0 \end{bmatrix},$$

$$A_0 = (\mathbf{r}^T \text{ with row 3 deleted})\mathbf{p}^{-1}(\mathbf{r} \text{ with column 1 deleted})$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 6 & 3 & 3 \end{bmatrix} \left(\frac{1}{6} \begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 6 & 3 \\ 3 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -6 & -3 \end{bmatrix},$$

$$A' = A_0 - A_0^T = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix},$$

$$B_0 = \mathbf{r}^T(\mathbf{p}^{-1} \text{ with column 1 deleted}) = \frac{1}{6} \begin{bmatrix} 3 & 0 & 3 \\ 6 & 3 & 3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} -x+z & -y & 0 \\ 0 & z & x-y \end{bmatrix}, \quad D = \frac{1}{6} \begin{bmatrix} 0 & -x & x \\ x & 0 & -x \\ -x & x & 0 \end{bmatrix}, \quad \text{and}$$

$$c_2 = \frac{1}{6} \begin{bmatrix} 0 & -x^2+xz-z^2 & 2x^2-xy-xz+yz \\ x^2-xz+z^2 & 0 & -x^2+xy-y^2 \\ -2x^2+xy+xz-yz & x^2-xy+y^2 & 0 \end{bmatrix}.$$

Example 5.5. If $n = 4$, then the matrices of Corollary 5.3 are

$$\mathbf{p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 5 & 10 & 10 & 5 \\ 0 & 0 & 0 & 5 & 10 & 10 & 5 & 20 & 30 & 20 \\ 0 & 0 & 0 & 10 & 10 & 5 & 20 & 30 & 20 & 5 \\ 0 & 5 & 10 & 5 & 20 & 30 & 0 & 10 & 30 & 30 \\ 0 & 10 & 10 & 20 & 30 & 20 & 10 & 30 & 30 & 10 \\ 0 & 10 & 5 & 30 & 20 & 5 & 30 & 30 & 10 & 0 \\ 5 & 5 & 20 & 0 & 10 & 30 & 0 & 0 & 10 & 20 \\ 10 & 20 & 30 & 10 & 30 & 30 & 0 & 10 & 20 & 10 \\ 10 & 30 & 20 & 30 & 30 & 10 & 10 & 20 & 10 & 0 \\ 5 & 20 & 5 & 30 & 10 & 0 & 20 & 10 & 0 & 0 \end{bmatrix},$$

$$\mathbf{p}^{-1} = \frac{1}{70} \begin{bmatrix} -35 & 15 & 15 & -5 & -10 & -5 & 1 & 3 & 3 & 1 \\ 15 & -15 & 5 & 9 & 2 & -7 & -3 & -3 & 3 & 3 \\ 15 & 5 & -15 & -7 & 2 & 9 & 3 & 3 & -3 & -3 \\ -5 & 9 & -7 & -9 & 6 & 1 & 5 & -1 & -3 & 3 \\ -10 & 2 & 2 & 6 & -9 & 6 & -6 & 3 & 3 & -6 \\ -5 & -7 & 9 & 1 & 6 & -9 & 3 & -3 & -1 & 5 \\ 1 & -3 & 3 & 5 & -6 & 3 & -5 & 5 & -3 & 1 \\ 3 & -3 & 3 & -1 & 3 & -3 & 5 & -7 & 6 & -3 \\ 3 & 3 & -3 & -3 & 3 & -1 & -3 & 6 & -7 & 5 \\ 1 & 3 & -3 & 3 & -6 & 5 & 1 & -3 & 5 & -5 \end{bmatrix},$$

$$\mathbf{r} = \begin{bmatrix} 5 & 20 & 30 & 20 & 5 \\ 0 & 10 & 30 & 30 & 10 \\ 10 & 30 & 30 & 10 & 0 \\ 0 & 0 & 10 & 20 & 10 \\ 0 & 10 & 20 & 10 & 0 \\ 10 & 20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 & 0 \end{bmatrix}, \quad A' = \frac{1}{2} \begin{bmatrix} 0 & 50 & 75 & 45 \\ -50 & 0 & 85 & 75 \\ -75 & -85 & 0 & 50 \\ -45 & -75 & -50 & 0 \end{bmatrix},$$

Table 2

The matrix c_2 in the resolution of R/J for J given (5.3.1), when $n = 4$. The top matrix is the first three columns of c_2 . The bottom matrix is the last two columns of c_2 .

$$\frac{1}{70} \begin{bmatrix} 0 & -10x^2 + 15xz - 10z^2 & 5x^2 - 10xy - 15xz + 10yz + 15z^2 \\ 10x^2 - 15xz + 10z^2 & 0 & -4x^2 - 5xy - 10y^2 - 3xz - 15yz - 9z^2 \\ -5x^2 + 10xy + 15xz - 10yz - 15z^2 & 4x^2 + 5xy + 10y^2 + 3xz + 15yz + 9z^2 & 0 \\ 2x^2 - 15xy - 16xz + 15yz + 17z^2 & -4x^2 - 4xy - 15y^2 - 4xz - 26yz - 15z^2 & 4x^2 + 3xy + 9y^2 + 5xz + 15yz + 10z^2 \\ -18x^2 + 17xy + 17xz - 17yz & 2x^2 - 16xy + 17y^2 - 15xz + 15yz & -5x^2 + 15xy - 15y^2 + 10xz - 10yz \end{bmatrix}$$

$$\frac{1}{70} \begin{bmatrix} -2x^2 + 15xy + 16xz - 15yz - 17z^2 & 18x^2 - 17xy - 17xz + 17yz \\ 4x^2 + 4xy + 15y^2 + 4xz + 26yz + 15z^2 & -2x^2 + 16xy - 17y^2 + 15xz - 15yz \\ -4x^2 - 3xy - 9y^2 - 5xz - 15yz - 10z^2 & 5x^2 - 15xy + 15y^2 - 10xz + 10yz \\ 0 & -10x^2 + 15xy - 10y^2 \\ 10x^2 - 15xy + 10y^2 & 0 \end{bmatrix}$$

$$B = \frac{1}{2} \begin{bmatrix} -2x + 2z & -x - 2y & 0 & 0 & 0 \\ 0 & -x + 2z & -2y & 0 & 0 \\ 0 & 0 & 2z & x - 2y & 0 \\ 0 & 0 & 0 & x + 2z & 2x - 2y \end{bmatrix},$$

$$D = \frac{1}{70} \begin{bmatrix} 0 & -5x & 5x & -3x & x \\ 5x & 0 & -4x & 5x & -3x \\ -5x & 4x & 0 & -4x & 5x \\ 3x & -5x & 4x & 0 & -5x \\ -x & 3x & -5x & 5x & 0 \end{bmatrix},$$

and the matrix c_2 appears in Table 2.

Example 5.6. If $n = 6$, then the matrix c_2 of Corollary 5.3 is the matrix given in Table 3.

6. A Macaulay2 script

The script PM (for Presentation matrix), which is given in Example 6.1, implements the techniques of Theorems 3.1 and 4.2. The input is a homogeneous polynomial Φ of degree $2n - 1$ in a polynomial ring $R = \mathbf{k}[x, y, z]$, where \mathbf{k} is a field and n is a positive EVEN integer. We think of Φ as an element of $D_{2n-1}(R_1^*)$.

We assume that $\text{ann } x(\Phi)$ is linearly presented and $\text{ann } \Phi$ is quadratically presented. The script will complain if either of these hypotheses are not satisfied.

The output consists of the matrices \mathbf{p} , \mathbf{p}^{-1} , A' , B , D , c_2 as described in Theorems 3.1 and 4.2. In particular, $\mathbf{p} = (xm_i m_j(\Phi))_{1 \leq i, j \leq \binom{n+1}{2}}$, \mathbf{p}^{-1} is the inverse of \mathbf{p} ,

$$b_2 = \left[\begin{array}{c|c} xA' & B \\ \hline -B^T & D \end{array} \right]$$

is a $(2n + 1) \times (2n + 1)$ alternating matrix and the maximal order Pfaffians of b_2 form a minimal generating set for $\text{ann } x(\Phi)$. The matrix c_2 is an alternating $n + 1$ by

Table 3

The matrix c_2 in the resolution of R/J for J given (5.3.1), when $n = 6$. The top matrix is the first two columns of c_2 ; the second matrix is columns 3 and 4 of c_2 ; the third matrix is columns 5 and 6 of c_2 ; and the bottom matrix is the last column of c_2 .

$$\begin{array}{c}
 \frac{1}{1848} \left[\begin{array}{cc}
 0 & -252x^2 + 420xz - 252z^2 \\
 252x^2 - 420xz + 252z^2 & 0 \\
 -126x^2 + 252xy + 420xz - 252yz - 378z^2 & 84x^2 + 126xy + 252y^2 + 42xz + 378yz + 210z^2 \\
 56x^2 - 378xy - 434xz + 378yz + 434z^2 & -84x^2 - 112xy - 378y^2 - 56xz - 644yz - 350z^2 \\
 -21x^2 + 434xy + 448xz - 434yz - 455z^2 & 54x^2 + 63xy + 434y^2 + 45xz + 805yz + 425z^2 \\
 6x^2 - 455xy - 457xz + 455yz + 461z^2 & -24x^2 - 24xy - 455y^2 - 24xz - 886yz - 455z^2 \\
 -463x^2 + 461xy + 461xz - 461yz & 6x^2 - 457xy + 461y^2 - 455xz + 455yz
 \end{array} \right] \\
 \\
 \frac{1}{1848} \left[\begin{array}{cc}
 126x^2 - 252xy - 420xz + 252yz + 378z^2 & -56x^2 + 378xy + 434xz - 378yz - 434z^2 \\
 -84x^2 - 126xy - 252y^2 - 42xz - 378yz - 210z^2 & 84x^2 + 112xy + 378y^2 + 56xz + 644yz + 350z^2 \\
 0 & -60x^2 - 70xy - 210y^2 - 50xz - 350yz - 200z^2 \\
 60x^2 + 70xy + 210y^2 + 50xz + 350yz + 200z^2 & 0 \\
 -75x^2 - 75xy - 350y^2 - 75xz - 625yz - 350z^2 & 60x^2 + 50xy + 200y^2 + 70xz + 350yz + 210z^2 \\
 54x^2 + 45xy + 425y^2 + 63xz + 805yz + 434z^2 & -84x^2 - 56xy - 350y^2 - 112xz - 644yz - 378z^2 \\
 -21x^2 + 448xy - 455y^2 + 434xz - 434yz & 56x^2 - 434xy + 434y^2 - 378xz + 378yz
 \end{array} \right] \\
 \\
 \frac{1}{1848} \left[\begin{array}{cc}
 21x^2 - 434xy - 448xz + 434yz + 455z^2 & -6x^2 + 455xy + 457xz - 455yz - 461z^2 \\
 -54x^2 - 63xy - 434y^2 - 45xz - 805yz - 425z^2 & 24x^2 + 24xy + 455y^2 + 24xz + 886yz + 455z^2 \\
 75x^2 + 75xy + 350y^2 + 75xz + 625yz + 350z^2 & -54x^2 - 45xy - 425y^2 - 63xz - 805yz - 434z^2 \\
 -60x^2 - 50xy - 200y^2 - 70xz - 350yz - 210z^2 & 84x^2 + 56xy + 350y^2 + 112xz + 644yz + 378z^2 \\
 0 & -84x^2 - 42xy - 210y^2 - 126xz - 378yz - 252z^2 \\
 84x^2 + 42xy + 210y^2 + 126xz + 378yz + 252z^2 & 0 \\
 -126x^2 + 420xy - 378y^2 + 252xz - 252yz & 252x^2 - 420xy + 252y^2
 \end{array} \right] \\
 \\
 \frac{1}{1848} \left[\begin{array}{c}
 463x^2 - 461xy - 461xz + 461yz \\
 -6x^2 + 457xy - 461y^2 + 455xz - 455yz \\
 21x^2 - 448xy + 455y^2 - 434xz + 434yz \\
 -56x^2 + 434xy - 434y^2 + 378xz - 378yz \\
 126x^2 - 420xy + 378y^2 - 252xz + 252yz \\
 -252x^2 + 420xy - 252y^2 \\
 0
 \end{array} \right]
 \end{array}$$

$n + 1$ matrix and the maximal order Pfaffians of c_2 form a minimal generating set for $\text{ann } \Phi$.

Example 6.1. We use the script PM to calculate the matrices of Example 5.4.

```

Macaulay2, version 1.17
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases,
MinimalPrimes, PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone
ii : input "PM"

ii2 : PM=(Phi)->(
  R1:=ring(Phi);
  Phil:=Phi;
  x1:=R_0;
  y1:=R_1;
  z1:=R_2;

  if not isHomogeneous(Phil) then
    error "Phi must be homogeneous.";

  DegPhil=(degree (Phil))_0;
  n1:=(DegPhil+1)//2;

```

```

if not DegPhi+1 = 4*((DegPhi+1)//4) then
error "The degree of Phi must be congruent to 3 mod 4.";

r1:=contract (
(transpose( gens (ideal(x1,y1,z1))^(nl-1)))
*(gens (ideal(y1,z1))^(nl),Phi);

leftbasis:=transpose (gens(ideal(x1,y1,z1))^(nl-1));
topbasis:=gens ( ideal(x1,y1,z1))^(nl-1);
pl:=contract (x1*leftbasis*topbasis,Phi);

if det pl==0 then
error "The ideal ann x(Phi) is not linearly presented.";

pinversel:= inverse pl;
ql:=numColumns(pinversel);

B0l:=submatrix ((transpose r1)*pinversel
,,{(ql-nl)..(ql-1)});

A0l:=submatrix'(transpose r1*pinversel*r1,{nl},{0});
Z1nl:=map(Rl^1,Rl^nl,(i,j)->0);
Zn1l:=map(Rl^nl,Rl^1,(i,j)->0);

B1l:=((transpose Z1nl|submatrix'(B0l,{nl},))
-(submatrix'(B0l,{0},)|(transpose Z1nl)));

B2l:=((z1*id_(Rl^nl))| Zn1l) -( Zn1l|yl*id_(Rl^nl));

D0l:=((transpose Z1nl|
submatrix(pinversel,{(ql-nl)..(ql-1)},
{(ql-nl)..(ql-1)}))||
(matrix{{0}}|Z1nl);

A'1:=A0l-transpose A0l;

if det A'1==0 then
error "The ideal ann Phi is not quadratically presented.";

B1:=x1*B1l+B2l;
D1:=(x1*(D0l-transpose D0l));
L1:= (x1*A'1|B1)|((-transpose B1)|D1);
Q1:=(transpose B1)*(inverse A'1)* B1+x1*D1;
Ll:=map(Rl^{2*nl+1:-nl},,L1);
Ql:=map(Rl^{nl+1:-nl},,Q1);
return(pl,pinversel,r1,A'1,B1,D1,Q1)
)

oo2 = PM
oo2 : FunctionClosure
ii3 :
ii4 : input "runme"
ii5 : R=QQ[x,y,z]
oo5 = R
oo5 : PolynomialRing
ii6 : n=2
oo6 = 2
ii7 : Phi=contract((x+y+z)^(n+1),(x*y*z)^n)
oo7 = 
$$3x^2y + 3x^2y + 3x^2z + 6x^2yz + 3y^2z + 3x^2z + 3y^2z$$

oo7 : R
ii8 : (p,pinverse,r,A',B,D,c2)=PM(Phi);
ii9 : p
oo9 = {1} | 0 3 3 |
      {1} | 3 3 6 |
      {1} | 3 6 3 |
oo9 : Matrix R  $\leftarrow$  R
ii10 : r
oo10 = {1} | 3 6 3 |
      {1} | 0 3 3 |
      {1} | 3 3 0 |
oo10 : Matrix R  $\leftarrow$  R

```

```

iii1 : A'
oo11 = {-1} | 0 6 |
      {-1} | -6 0 |
      2      2
oo11 : Matrix R <--- R
iii2 : B
oo12 = | -x+z -y 0 |
      | 0 z x-y |
      2      3
oo12 : Matrix R <--- R
iii3 : D
oo13 = | 0 -1/6x 1/6x |
      | 1/6x 0 -1/6x |
      | -1/6x 1/6x 0 |
      3      3
oo13 : Matrix R <--- R
iii4 : c2
oo14 = {2} | 0 -1/6x2+1/6xz-1/6z2 1/3x2-1/6xy-1/6xz+1/6yz |
      {2} | 1/6x2-1/6xz+1/6z2 0 -1/6x2+1/6xy-1/6y2 |
      {2} | -1/3x2+1/6xy+1/6xz-1/6yz 1/6x2-1/6xy+1/6y2 0 |
      3      3
oo14 : Matrix R <--- R

```

Data availability

No data was used for the research described in the article.

References

- [1] E. Artin, *Geometric Algebra*, John Wiley & Sons, Inc., New York, 1988.
- [2] M. Boij, Betti numbers of compressed level algebras, *J. Pure Appl. Algebra* 134 (1999) 111–131.
- [3] D. Buchsbaum, D. Eisenbud, What makes a complex exact?, *J. Algebra* 25 (1973) 259–268.
- [4] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Am. J. Math.* 99 (1977) 447–485.
- [5] D. Eisenbud, *Commutative Algebra. With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
- [6] S. El Khoury, A. Kustin, Artinian Gorenstein algebras with linear resolutions, *J. Algebra* 420 (2014) 402–474.
- [7] S. El Khoury, A. Kustin, The explicit minimal resolution constructed from a Macaulay inverse system, *J. Algebra* 440 (2015) 145–186.
- [8] S. El Khoury, A. Kustin, The structure of Gorenstein-linear resolutions of Artinian algebras, *J. Algebra* 453 (2016) 492–560.
- [9] D. Grayson, M. Stillman, *Macaulay2*, a software system for research in algebraic geometry, Available at <https://math.uiuc.edu/Macaulay2/>.
- [10] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, J. Watanabe, *The Lefschetz Properties*, Lecture Notes in Mathematics, vol. 2080, Springer, Heidelberg, 2013.
- [11] A. Kustin, C. Polini, B. Ulrich, A matrix of linear forms which is annihilated by a vector of indeterminates, *J. Algebra* 469 (2017) 120–187.
- [12] A.R. Kustin, R. R.G., A. Vraciu, The resolution of (x^N, y^N, z^N, w^N) , *J. Algebra* 590 (2022) 338–393.
- [13] J. Migliore, R. Miró-Roig, U. Nagel, Monomial ideals, almost complete intersections and the weak Lefschetz property, *Trans. Am. Math. Soc.* 363 (2011) 229–257.
- [14] J. Migliore, U. Nagel, Survey article: a tour of the weak and strong Lefschetz properties, *J. Commut. Algebra* 5 (2013) 329–358.
- [15] R. Miró-Roig, Q.H. Tran, The weak Lefschetz property for Artinian Gorenstein algebras of codimension three, *J. Pure Appl. Algebra* 224 (2020) 106305.
- [16] L. Reid, L. Roberts, M. Roitman, On complete intersections and their Hilbert functions, *Can. Math. Bull.* 34 (1991) 525–535.