



Notes

Bump functions with monotone Fourier transforms satisfying decay bounds

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Received 20 September 2021; received in revised form 3 January 2022; accepted 3 March 2022

Available online 12 March 2022

Communicated by M. Buhmann

Abstract

The existence of a smooth, nonnegative, compactly supported function with monotone (on the half-line) Fourier transform satisfying two-sided decay bounds is demonstrated.

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Keywords: Monotone; Fourier; Transform; Asymptotic; Bump

The uncertainty principle in harmonic analysis roughly states that a function and its Fourier transform cannot both be made to decay too fast. In particular, it is easy to see that it is impossible for a compactly supported function to have an exponentially decaying Fourier transform, as this would force the function to be analytic. The Beurling–Malliavin multiplier theorem (see e.g. [3]) guarantees the existence of smooth, compactly supported functions whose Fourier transforms have sub-exponential decay (i.e. an exponential of a power less than one). However, one may need further information, e.g. a lower bound of the transform or whether it is monotonic.

The goal of this brief note is to prove the following

Theorem. *For any $\delta \in (0, 1)$ and any $C > 0$ there is a function $F(x)$ which is C^∞ , real, even, nonnegative, supported in $[-1, 1]$ and whose Fourier transform $\widehat{F}(\xi)$ is monotone decreasing for $\xi \geq 0$ and satisfies the following double inequality*

$$e^{-(1+\epsilon)C\xi^\delta} \lesssim \widehat{F}(\xi) \lesssim e^{-(1-\epsilon)C\xi^\delta},$$

for any $\epsilon > 0$.

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<https://doi.org/10.1016/j.jat.2022.105742>

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Above we have used the notation $f_1(\xi) \lesssim f_2(\xi)$ to mean $f_1(\xi) \leq c f_2(\xi)$ for some constant c . Our convention for the Fourier transform is $\widehat{f}(\xi) = \int e^{-2\pi i \xi x} f(x) dx$. To reduce clutter in some formulas below, we will set $\tilde{\xi} = 2\pi \xi$.

First, we prove a lemma of interest¹ in its own right.

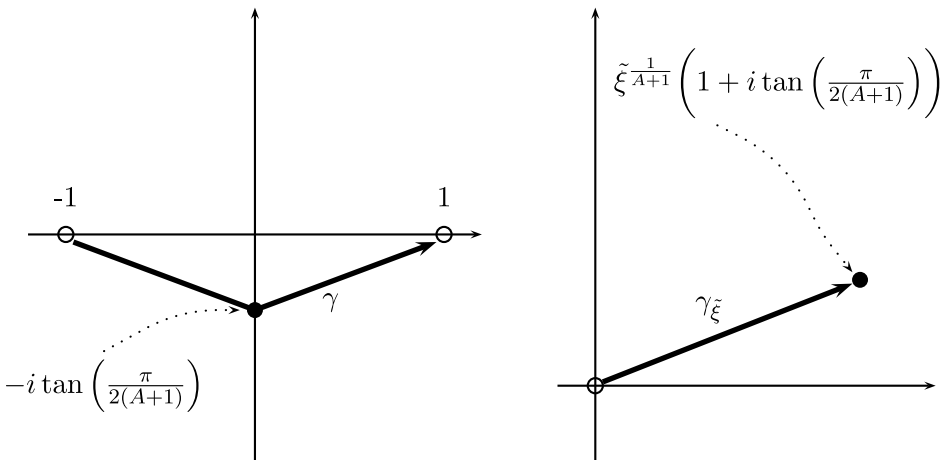
Lemma. For $A, B \in (0, \infty)$, let the smooth function $\phi_{A,B} : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\phi_{A,B}(x) = \begin{cases} e^{-\frac{B}{(1-x)^A}} e^{-\frac{B}{(1+x)^A}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

then, as $\xi \rightarrow \infty$, its Fourier transform $\widehat{\phi_{A,B}}(\xi)$ is asymptotic to a constant multiple of

$$\frac{\cos(\tilde{\xi} - \alpha \tilde{\xi}^{\frac{A}{A+1}}) e^{-\beta \tilde{\xi}^{\frac{A}{A+1}}}}{\tilde{\xi}^{\frac{A+2}{2A+2}}}$$

for constants α, β . Moreover, for a fixed A , B can be chosen to give any prescribed positive value of β .



Proof. We need to estimate

$$\int_{-1}^1 e^{-i \tilde{\xi} x} \phi_{A,B}(x) dx.$$

Choosing the principal branch of the logarithm (to define the exponents in $\phi_{A,B}$), we see that the integrand is an analytic function in the region $|\Re(z)| < 1, \Im(z) \leq 0$. We can thus replace integration along the real axis with the integral along the contour γ shown in the figure above. The reason for this choice will be apparent momentarily. Note that no trouble arises at the endpoints of the contour as the integrand remains bounded (in fact it tends to zero) as we approach them.

¹ See [1] for a related heuristic discussion.

It is easy to see that the integral along the left segment is the conjugate of the one on the right. Doing now the change of variables $z \rightarrow \tilde{\xi}^{\frac{1}{A+1}}(1 - z)$ we see that the integral above is equal to

$$2\Re\left(-\frac{e^{-i\tilde{\xi}}}{\tilde{\xi}^{\frac{1}{A+1}}} \int_{\gamma_{\tilde{\xi}}} e^{\tilde{\xi}^{\frac{A}{A+1}}g(z)} h_{\tilde{\xi}}(z) dz\right)$$

where

$$g(z) = iz - \frac{B}{z^A}$$

$$h_{\tilde{\xi}}(z) = e^{-B\left(2 - \tilde{\xi}^{\frac{-1}{A+1}}z\right)^{-A}}$$

and $\gamma_{\tilde{\xi}}$ is the contour shown in the figure.

It is straightforward to verify that, provided $\tilde{\xi}$ is large enough, then g has a single critical point on the contour located at $z_0 = (AB)^{\frac{1}{A+1}} e^{\frac{i\pi}{2(A+1)}}$. Moreover, $\Re(g)$ has a maximum there. We can now deform the contour in a neighbourhood around z_0 so that it follows the path of steepest descent there and apply the steepest descent method² to obtain that, up to a constant³

$$\int_{\gamma_{\tilde{\xi}}} e^{\tilde{\xi}^{\frac{A}{A+1}}g(z)} h_{\tilde{\xi}}(z) dz \sim \frac{\exp\left[\tilde{\xi}^{\frac{A}{A+1}}\left(ie^{\frac{i\pi}{2(A+1)}}(AB)^{\frac{1}{A+1}}\left(1 + \frac{1}{A}\right)\right)\right]}{\tilde{\xi}^{\frac{A}{2(A+1)}}$$

Putting everything together, we have what we want with

$$\alpha = (AB)^{\frac{1}{A+1}}\left(1 + \frac{1}{A}\right) \cos\left(\frac{\pi}{2A + 2}\right)$$

$$\beta = (AB)^{\frac{1}{A+1}}\left(1 + \frac{1}{A}\right) \sin\left(\frac{\pi}{2A + 2}\right).$$

Clearly, for a fixed A , β takes all values in $(0, \infty)$ by varying B . \square

We are now ready to prove the theorem.

Proof of theorem. Let ϕ be a real, even, nonnegative, C_0^∞ function whose support is contained in $[-\frac{1}{2}, \frac{1}{2}]$. Thus, $\widehat{\phi}$ is real and even. It follows that $\phi * \phi$ is a real, even, nonnegative, C_0^∞ function whose support is contained in $[-1, 1]$, and moreover, $\widehat{\phi * \phi}(\xi) = (\widehat{\phi}(\xi))^2 \geq 0$. Let $\psi(x) = i(\phi * \phi)'(x)$. We then have that $\widehat{\psi}(\xi) = -(2\pi)\xi(\widehat{\phi}(\xi))^2$. Therefore, $\widehat{\psi}(\xi)$ is a real, odd function, which is nonpositive for $\xi \geq 0$. Moreover, by Paley–Wiener’s theorem (see e.g. [4]) we have that $\widehat{\psi}$ has an extension to an entire function such that, for every natural N , there is a constant C_N such that

$$|\widehat{\psi}(\xi)| \leq C_N \frac{e^{2\pi|\Im(\xi)|}}{(1 + |\xi|)^N}.$$

Now, set

$$I = - \int_0^\infty \widehat{\psi}(t) dt,$$

² See e.g. [2]. Note that one needs to extend the result there as $h_{\tilde{\xi}}$ and $\gamma_{\tilde{\xi}}$ both depend on $\tilde{\xi}$, but such an extension is trivial in this case.

³ The constant, whose explicit value will not play a role in the following is of course $\frac{2\pi}{\sqrt{-g''(z_0)}}$.

and define the function⁴ \widehat{f} by

$$\widehat{f}(\xi) = I + \int_{\gamma} \widehat{\psi}(z) dz,$$

where γ is some curve in the complex plane going from 0 to ξ . We see at once that \widehat{f} is an entire function whose derivative is $\widehat{\psi}$ and that, when restricted to real values, then \widehat{f} is a real, even and nonnegative Schwartz function.

Now choose γ to be a straight line segment γ_1 along the real axis from 0 to $\Re(\xi)$, followed by a vertical one γ_2 to ξ . We then have

$$\left| \int_{\gamma_1} \widehat{\psi}(z) dz \right| \leq \int_0^{|\Re(\xi)|} |\widehat{\psi}(t)| dt \leq C_2 \int_{-\infty}^{\infty} \frac{1}{(1 + |t|)^2} dt < \pi C_2.$$

Also,

$$\begin{aligned} \left| \int_{\gamma_2} \widehat{\psi}(z) dz \right| &\leq \int_0^{|\Im(\xi)|} |\widehat{\psi}(\Re(\xi) + it)| dt \\ &\leq C_2 \int_0^{|\Im(\xi)|} \frac{e^{2\pi t}}{(1 + |\Re(\xi) + it|)^2} dt \leq C_2 e^{2\pi |\Im(\xi)|} |\xi|. \end{aligned}$$

Putting it all together, we have that \widehat{f} is an entire function which satisfies

$$|\widehat{f}(\xi)| \leq \pi C_2 (1 + |\xi|) e^{2\pi |\Im(\xi)|}.$$

It then follows, again by Paley–Wiener, that there is a Schwartz distribution f , supported in $[-1, 1]$ whose Fourier transform has \widehat{f} as its entire extension (hence the notation). However, we showed above that \widehat{f} is a Schwartz function when restricted to the real line, which implies that f is in fact a Schwartz function. Thus, \widehat{f} is a real, even, C_0^∞ function supported in $[-1, 1]$ whose Fourier transform’s derivative is $\widehat{\psi}$, and is thus negative for positive ξ ’s. Let us now prove the decay estimates.

Suppose $C > 0$ and $\delta \in (0, 1)$ are given. We shall take $\phi(x) = \phi_{A,B}(2x)$, choosing A and B such that up to an overall constant

$$\widehat{\phi}_{A,B} \left(\frac{\xi}{2} \right) \sim \frac{\cos\left(\frac{\xi}{2} - \frac{\alpha}{2\delta} \xi^\delta\right) e^{-\frac{C\xi^\delta}{2}}}{\xi^{\frac{A+2}{2A+2}}}.$$

Now, note that for $\xi \geq 0$ we have

$$\int_{\xi}^{\infty} \frac{e^{-Ct^\delta}}{t^{1-\delta}} dt = \int_{\xi^\delta}^{\infty} e^{-C\tau} d\tau = \frac{e^{-C\xi^\delta}}{C}.$$

It follows at once that for $\xi \geq 0$

$$\widehat{f}(\xi) \lesssim e^{-C\xi^\delta} \leq e^{-(1-\epsilon)C\xi^\delta}$$

for any $\epsilon > 0$, and we have the upper bound we need.

To obtain the lower bound, let Z be the set of zeros of the function $\cos^2\left(\frac{\xi}{2} - \frac{\alpha}{2\delta} \xi^\delta\right)$, and \tilde{Z} the preimage by this function of $[0, \frac{1}{4}]$. Given ξ , let $\{\xi_n\}_{n=1}^\infty$ be the sequence of elements of $Z \cap (\xi, \infty)$ arranged in increasing order, and let ξ_0 be the largest element in Z less or equal to ξ . Since $\delta < 1$, it is clear, by considering the derivative of the argument of the cosine, that

⁴ At the moment the hat is purely notational. It will be justified below.

$\xi_{n+1} - \xi_n \rightarrow 2\pi$ as $n \rightarrow \infty$. It also follows that \tilde{Z} consists of a collection of disjoint intervals, each containing a single ξ_n , with their lengths tending to $\frac{2\pi}{3}$.

We now have that, for all sufficiently large ξ , and for any $\epsilon > 0$

$$\begin{aligned} \widehat{f}(\xi) &\gtrsim \int_{\xi}^{\infty} \frac{\cos^2(\frac{t}{2} - \frac{\alpha}{2\delta}t^\delta)e^{-Ct^\delta}}{t^{\frac{1}{A+1}}} dt \\ &\gtrsim \int_{\tilde{Z}^c \cap [\xi, \infty)} \frac{e^{-Ct^\delta}}{t^{\frac{1}{A+1}}} dt \\ &\gtrsim \int_{\tilde{Z}^c \cap [\xi, \infty)} e^{-(1+\frac{\epsilon}{2})Ct^\delta} dt. \end{aligned}$$

Define $\lambda \in [0, 1]$ via $\xi = (1 - \lambda)\xi_0 + \lambda\xi_1$ and let $I_n = [(1 - \lambda)\xi_{n-1} + \lambda\xi_n, (1 - \lambda)\xi_n + \lambda\xi_{n+1}]$. We then see that

$$\begin{aligned} \int_{\tilde{Z}^c \cap [\xi, \infty)} e^{-(1+\frac{\epsilon}{2})Ct^\delta} dt &= \sum_{n=1}^{\infty} \int_{\tilde{Z}^c \cap I_n} e^{-(1+\frac{\epsilon}{2})Ct^\delta} dt \\ &\geq \sum_{n=1}^{\infty} \int_{[\frac{\xi_{n-1} + \xi_n}{2}, (1-\lambda)\xi_n + \lambda\xi_{n+1}]} e^{-(1+\frac{\epsilon}{2})Ct^\delta} dt \\ &\geq \sum_{n=1}^{\infty} \int_{I_n} e^{-(1+\epsilon)Ct^\delta} dt \\ &= \int_{\xi}^{\infty} e^{-(1+\epsilon)Ct^\delta} dt \gtrsim e^{-(1+\epsilon)C\xi^\delta}, \end{aligned}$$

where to go to the penultimate line we have used the fact that

$$\frac{\int_x^{x+L(x)} e^{-(1+\epsilon)Ct^\delta} dt}{\int_{x+\frac{L(x)}{2}}^{x+L(x)} e^{-(1+\frac{\epsilon}{2})Ct^\delta} dt} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

as long as $L(x)$ is bounded away from 0 and ∞ , which is the case above since $L(x)$ will be contained in a narrow range around 2π .

We have thus obtained a function f satisfying all the requirements of the theorem with the possible exception of nonnegativity.

Let $F(x) = f^2(x)$, then F is C^∞ , real, even, nonnegative and is supported in $[-1, 1]$. What remains is showing the monotonicity of its Fourier transform and the decay bounds.

Let $\xi \geq 0$. We then have that

$$\frac{d\widehat{F}(\xi)}{d\xi} = \int_0^\infty (\widehat{f}(\omega - \xi) - \widehat{f}(\omega + \xi)) \frac{d\widehat{f}}{d\omega}(\omega) d\omega.$$

Since \widehat{f} is monotone and even, the term in brackets is positive ($|\omega - \xi| \leq \omega + \xi$) while $\frac{d\widehat{f}}{d\omega}(\omega)$ is negative. Therefore \widehat{F} is monotone decreasing on the positive axis.

Now, note that

$$\begin{aligned} \widehat{F}(\xi) &= \int \widehat{f}(\xi - \omega) \widehat{f}(\omega) d\omega \\ &\lesssim \int e^{-(1-\epsilon)C|\xi - \omega|^\delta} e^{-(1-\epsilon)C|\omega|^\delta} d\omega \\ &= \xi \int e^{-(1-\epsilon)C\xi^\delta (|1 - \chi|^\delta + |\chi|^\delta)} d\chi. \end{aligned}$$

It is easy to show, essentially by an application of Laplace’s method,⁵ that the last expression is asymptotic to

$$\frac{2e^{-(1-\epsilon)C\xi^\delta} \left(\int_{-\infty}^{\infty} e^{-|\chi|^\delta} d\chi \right)}{((1-\epsilon)C)^{\frac{1}{\delta}}} = \frac{4e^{-(1-\epsilon)C\xi^\delta} \Gamma(\frac{1}{\delta})}{\delta((1-\epsilon)C)^{\frac{1}{\delta}}},$$

which in turn gives that

$$\widehat{F}(\xi) \lesssim e^{-(1-\epsilon)C\xi^\delta},$$

and we have the decay estimate from the right. The one from the left is demonstrated in a similar way, which concludes the proof. \square

Acknowledgments

The author would like to thank J. Merhej and B. Shayya for reading a preliminary version of this paper and for numerous comments which greatly improved its readability. Additionally, the author would like to thank J. Steif for posing the question whether the function constructed can be taken to be positive.

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⁵ A little care is needed since the maxima of the exponent which dominate the asymptotics are located at 0 and 1 where the exponent is not regular. However, the extension of the method needed is completely straightforward in this case.