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RESEARCH

The 1-jet determination of stationary discs attached to generic CR submanifolds

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Abstract
The existence of a non-defective stationary disc attached to a non-degenerate model quadric in $\mathbb{C}^N$ is a necessary condition to ensure the unique 1-jet determination of the lifts of a key family of stationary discs [6]. In this paper, we give an elementary proof of the equivalence when the model quadric is strongly pseudoconvex, recovering a result of Tumanov [14]. Our proof is based on the explicit expression of stationary discs, and opens up a conjecture for the unique 1-jet determination to hold when the model is not necessarily strongly pseudoconvex.

Introduction

In the paper [6], we gave an explicit construction of stationary discs attached to a strongly Levi non-degenerate model quadric, and obtained a necessary condition, namely the existence of a non-defective stationary disc, to ensure the unique 1-jet determination of their lifts. As emphasized in [5], this is a crucial step to deduce the 2-jet determination of CR automorphisms of strongly Levi non-degenerate CR submanifolds. Tumanov proved in [14] that the existence of a non-defective stationary disc is also a sufficient condition if the model is strongly pseudoconvex. His proof is based on the fact that, in the strongly pseudoconvex case, lifts of stationary discs can be made "supporting" in a suitable (non-linear) system of coordinates (Proposition 3.9 in [14], see also Proposition 2.2' in [13]).

One purpose of the present paper is to address the question of recovering Tumanov's result using the explicit construction of stationary discs, without performing a non-linear change of coordinates that would not preserve the model quadric (Theorem 4.3). The elementary proof we give reveals a possible generalization to merely strongly Levi non-degenerate model quadric of this unique 1-jet determination (Sect. 4.1 and its open questions).

Another purpose of this paper is to discuss the link between various definitions related to the model quadric (see Lemma 1.2), and also between different technics used to obtain finite jet determination of biholomorphisms preserving CR submanifolds. We thus generalize, for a class of Levi generating submanifolds, the 2-jet determination of biholomorphisms preserving strongly Levi non-degenerate CR submanifolds obtained in [4] to merely Levi non-degenerate CR submanifolds (see Theorem 1.6). We recall that the 2-jet determination of biholomorphisms preserving (Levi generating) Levi non-degenerate CR real-analytic submanifolds does not hold in general, due to the counterexamples obtained in [9]. We refer to the surveys of Zaitsev [16] and, Lamel and Mir [11], and references therein, for an overview of finite jet determination problems.

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1 Hermitian matrices associated to the Levi map of generic submanifolds

In this section, we recall various definitions related to the Levi map (see p. 41 [1]) of a generic submanifold via its associated Hermitian matrices. We also recall recent results about 2-jet determination of CR mappings in higher codimension, and obtain a new result in this direction.

Let $M \subset \mathbb{C}^{n+d}$ be a $\mathcal{C}^4$ generic real submanifold of real codimension $d \geq 1$ through $p = 0$ given locally by the following system of equations

$$\begin{cases} \Re e w_1 = {}^t\bar{z}A_1 z + O(3) \\ \vdots \\ \Re e w_d = {}^t\bar{z}A_d z + O(3) \end{cases} \tag{1.1}$$

where $A_1, \ldots, A_d$ are Hermitian matrices of size n. In the remainder O(3), the variables z and $\Im m w$ are respectively of weight one and two. We associate to M the model quadric M_H given by

$$\begin{cases} \Re e w_1 = {}^t\bar{z}A_1 z \\ \vdots \\ \Re e w_d = {}^t\bar{z}A_d z. \end{cases} \tag{1.2}$$

Recall that M is of finite type at 0 with 2 the only Hörmander number if and only M is Levi generating at 0 if and only if the matrices $A_1, \ldots, A_d$ are linearly-independent (see [7] for instance). We say that M is *Levi non-degenerate at* 0 in case $\cap_{j=1}^d \mathrm{Ker} A_j = \{0\}$. The submanifold M is *strongly Levi non-degenerate at* 0 (resp. *strongly pseudoconvex at* 0) if there exists $b \in \mathbb{R}^d$ such that the matrix $\sum_{j=1}^d b_j A_j$ is invertible (resp. positive definite). Note that in case M is strongly pseudoconvex, the matrix $\sum_{j=1}^d b_j A_j$ may be chosen to be the identity matrix after a linear holomorphic change of variables. We also recall the following definition from [5].

Definition 1.1 Let M be strongly Levi non-degenerate at 0 given by (1.1) and let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is invertible. We say that M is $\mathfrak{D}$*-non-degenerate* at 0 if there exists $V \in \mathbb{C}^n$ such that, if we set D_0 to be the $n \times d$ matrix whose j^{th} column is $A_j V$, then $\Re e({}^t\overline{D}_0(\sum_{j=1}^d b_j A_j)^{-1} D_0)$ is invertible.

The following lemma was pointed out to us by Bernhard Lamel; we thank him for that. This lemma shows the connection between the Segre sets introduced by Baouendi, Ebenfelt and Rothschild (see Chapter 10 in [1]) and the matrix D_0 introduced in the previous definition.

Lemma 1.2 *The model quadric M_H given by* (1.2) *is of finite type with Segre number* 2 *at* 0 *if and only if there exists a V such that the $n \times d$ matrix whose j^{th} column is $A_j V$ has rank d.*

Proof Let $S_2(0)$ be the Segre set of order 2 at 0. A direct computation yields to

$$S_2(0) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^d \mid \exists \zeta \in \mathbb{C}^n, \ \ w_j = {}^t\zeta A_j z\}.$$

Let $d_2(0)$ be the generic rank of the map

$$pr : (z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n \longrightarrow (z, {}^t\zeta A_1 z, \ldots, {}^t\zeta A_d z) \in \mathbb{C}^{n+d}. \tag{1.3}$$

According to Theorem 10.5.5 in [1], the quadric M_H is of finite type at 0 with Segre number 2 at 0 if and only if $d_2(0) = n+d$. According to the form of the map pr given by (1.3), we obtain that $d_2(0) = n+d$ if and only if the $n \times d$ matrix whose j^{th} column is $A_j V$ has rank d for some V. This achieves the proof. □

We now highlight two recent results on 2-jet determination of CR automorphisms. Tumanov obtained in [15] the following theorem for strongly pseudoconvex submanifolds.

Theorem 1.3 [15] *Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold given by* (1.1)*. Assume that M is strongly pseudoconvex and that the matrices $A_1, \ldots, A_d$ are linearly-independent. Then any germ at* 0 *of CR automorphism of M of class $\mathcal{C}^3$ is uniquely determined by its* 2*-jet at* 0.

In [5], we proved that 2-jet determination holds under the $\mathfrak{D}$-non-degeneracy of the submanifold (see also [4]). More precisely,

Theorem 1.4 [5] *Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold given by* (1.1)*. Assume that M is $\mathfrak{D}$-non-degenerate at* 0*. Then any germ at* 0 *of CR automorphism of M of class $\mathcal{C}^3$ is uniquely determined by its* 2*-jet at* 0.

In light of Lemma 1.2, we may rewrite the following theorem due to Baouendi, Ebenfelt and Rothschild (Theorem 12.3.11 and Remark 12.3.13 in [1]) for the model quadric M_H.

Theorem 1.5 [1] *Let $M_H \subset \mathbb{C}^N$ be a model quadric given by* (1.2)*. Assume that M_H is Levi non-degenerate at* 0 *and that there exists $V \in \mathbb{C}^n$ such that the $n \times d$ matrix D_0 whose j^{th} column is $A_j V$ is of rank d. Then any biholomorphism sending M_H into itself is uniquely determined by its* 2*-jet at* 0.

Proof This is a direct application of Lemma 1.2 and Proposition 11.1.12 in [1]. □

Using Theorem 1.5, we then obtain the following new result which is a direct application of Theorem 3.10 in [8] for smooth generic submanifolds:

Theorem 1.6 *Let $M \subset \mathbb{C}^N$ be a smooth submanifold given by* (1.1)*. Assume that M is Levi non-degenerate at* 0 *and that the matrix D_0 whose j^{th} column is $A_j V$ is of rank d. Then any biholomorphism sending M into itself is uniquely determined by its* 2*-jet at* 0.

Proof Let $hol(M_H, 0)$ be the set of germs of real-analytic infinitesimal CR automorphisms of M_H at 0. By Theorem 1.5, the elements of $hol(M_H, 0)$ are determined by their 2-jets. Hence, a direct application of Theorem 3.10 in [8] yields to the result. □

Remark 1.7 The second author recently wrote a paper with Kolář [10] in which they observe that it is enough to assume that M is of class $\mathcal{C}^3$ in the previous theorem.

2 Stationary minimal submanifolds

Let M be a strongly Levi non-degenerate real submanifold given by (1.1). We fix $b \in \mathbb{R}^d$ such that $\sum_{j=1}^d b_j A_j$ is invertible, and let $a \in \mathbb{C}^d$ be sufficiently small. We consider the following quadratic matrix equation

$$PX^2 + AX + {}^t\overline{P} = 0 \tag{2.1}$$

where $P := \sum_{j=1}^d a_j A_j$ and $A := \sum_{j=1}^d (b_j - a_j - \overline{a_j})A_j$. Consider X to be the unique $n \times n$ matrix solution of (2.1) such that $\|X\| < 1$. Note that X depends on both a and b. Moreover, the existence and unicity of X is provided by the Implicit Function Theorem (see Theorem 6.3.8 [1]). The following definition, inspired by the work of Tumanov in the strongly pseudoconvex setting [14], was introduced in [6].

Definition 2.1 Let M be a strongly Levi non-degenerate (at 0) real submanifold given by (1.1). Let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is invertible and let $V \in \mathbb{C}^n$. Consider $a \in \mathbb{C}^d$ sufficiently small and the solution X of (2.1) with $\|X\| < 1$. We say that M is *stationary minimal at* 0 *for* $(a, b - a - \overline{a}, V)$ if the matrices $A_1, \ldots, A_d$, as linear transformations, restricted to the orbit space

$$\mathcal{O}_{X,V} := \mathrm{span}_{\mathbb{R}}\{V, XV, X^2V, \ldots, X^kV, \ldots\}$$

are $\mathbb{R}$-linearly-independent.

Note that the above definition is independent of the choice of holomorphic coordinates and that M is stationary minimal for $(0, b, V)$ in case it is $\mathfrak{D}$-non-degenerate. Tumanov proved that if $A_1, \ldots, A_d$ are d linearly-independent Hermitian matrices, then there exist $a \in \mathbb{R}^d$ and $V \in \mathbb{C}^n$ such that the matrices $A_1, \ldots, A_d$ restricted to the space

$$\mathrm{span}_{\mathbb{R}}\{V, PV, P^2V, \ldots, P^kV, \ldots\}$$

are $\mathbb{R}$-linearly-independent (Theorem 5.3 in [15]). In particular, this implies that strongly pseudoconvex Levi generating submanifolds are stationary minimal. For the sake of clarity, we provide a proof.

Theorem 2.2 [15] *Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold given by* (1.1). *Assume that M is strongly pseudoconvex and that the matrices $A_1, \ldots, A_d$ are linearly-independent. Then there exists $a, b \in \mathbb{R}^d$ and $V \in \mathbb{C}^n$ such that M is stationary minimal at 0 for $(a, b - 2a, V)$.*

Proof Let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is positive definite; without loss of generality, we assume that $\sum_{j=1}^d b_j A_j$ is the identity matrix. According to Theorem 5.3 in [15], there exists $a_0 \in \mathbb{R}^d$ and $V \in \mathbb{C}^n$ such that the matrices $A_1, \ldots, A_d$ are $\mathbb{R}$-linearly-independent on the space

$$\mathcal{O}_{(a_0,V)} := \mathrm{span}_{\mathbb{R}}\{V, PV, P^2V, \ldots, P^sV, \ldots\},$$

with $P = \sum a_{0j}A_j$. Note that since a_0 can be chosen as small as necessary, the matrix $A := \sum_{j=1}^d (b_j - 2a_{0j})A_j$ is positive definite. Taking s large enough, we assume that

$$\mathcal{O}_{(a_0,V)} = \mathrm{span}_{\mathbb{R}}\{V, PV, P^2V, \ldots, P^sV\}.$$

We set

$$\tilde{V} = (V, PV, P^2V, \ldots, P^sV) \in \mathbb{C}^{(s+1)n},$$

and we define $\tilde{A}_j$, $j = 1 \ldots, d$, and $\tilde{P}$ to be the following $(s+1)n \times (s+1)n$ matrices

$$\tilde{A}_j := \begin{pmatrix} A_j & & & (0) \\ & A_j & & \\ & & \ddots & \\ (0) & & & A_j \end{pmatrix}, \quad \tilde{P} := \begin{pmatrix} P & & & (0) \\ & P & & \\ & & \ddots & \\ (0) & & & P \end{pmatrix}.$$

It follows that the vectors $\tilde{A}_1\tilde{V}, \ldots, \tilde{A}_d\tilde{V}$ are $\mathbb{R}$-linearly-independent. We may then find an invertible $d \times d$ matrix C_{hom} whose j^{th} column is given by well chosen (independent of j) components of $\tilde{A}_j\tilde{V}$. Let X be the matrix solution of (2.1) with $\|X\| < 1$. We replace $\tilde{V}$ by $\tilde{V}_X$, where

$$\tilde{V}_X = (V, XV, X^2V, \ldots, X^sV) \in \mathbb{C}^{(s+1)n},$$

with X evaluated at a_0, and we consider the $d \times d$ matrix C obtained from $\tilde{A}_1\tilde{V}_X, \ldots, \tilde{A}_d\tilde{V}_X$ by keeping the same rows as the ones of C_{hom}. Since X is of the form

$$\begin{aligned} X &= -\left(I - 2\sum_{j=1}^d a_{0j}A_j\right)^{-1} {}^t\overline{P} + O(|a_0|^3) \\ &= -{}^t\overline{P} + O(|a_0|^2), \end{aligned}$$

it follows that C has a determinant with a first non-zero homogeneous term of degree d depending on a_0, actually the determinant of C_{hom}. Therefore, expanding the determinant of C into homogeneous terms, and replacing a_0 by λa_0, we obtain a function $h(\lambda)$ with respect to λ whose first term coefficient of order d is non-zero. Therefore, we may choose λ_0 such that $h(\lambda_0) \neq 0$. This shows that M is stationary minimal for $a = \lambda_0 a_0$. □

3 Stationary discs

Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold of codimension d given by (1.1). Following Lempert [12] and Tumanov [14], a holomorphic disc $f : \Delta \to \mathbb{C}^N$ continuous up to $\partial\Delta$ and such that $f(\partial\Delta) \subset M$ is *stationary for* M if there exist d real valued functions $c_1, \dots, c_d : \partial\Delta \to \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta)\partial r_j(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map denoted by $\tilde{f}$

$$\zeta \mapsto \zeta \sum_{j=1}^{d} c_j(\zeta)\partial r_j\left(f(\zeta), \overline{f(\zeta)}\right)$$

defined on $\partial\Delta$ extends holomorphically on Δ. In that case, the disc $\boldsymbol{f} = (f, \tilde{f})$ is a holomorphic lift of f to the cotangent bundle $T^*\mathbb{C}^N$ and the set of all such lifts $\boldsymbol{f} = (f, \tilde{f})$, with f non-constant, is denoted by $\mathcal{S}(M)$.

We now consider a strongly Levi non-degenerate quadric M_H given by (1.2). We fix $b \in \mathbb{R}^d$ such that $\sum_{j=1}^d b_j A_j$ is invertible and we define $\mathcal{S}_0(M_H) \subset \mathcal{S}(M_H)$ to be the subset of lifts whose value at $\zeta = 1$ is $(0, 0, 0, b/2)$. Consider an initial disc $\boldsymbol{f_0} \in \mathcal{S}_0(M_H)$ given by

$$\vec{f_0} = \Big((1-\zeta)V_0, 2(1-\zeta)^t\overline{V_0}A_1V_0, \dots, 2(1-\zeta)^t\overline{V_0}A_dV_0, (1-\zeta)^t\overline{V_0}(\sum b_jA_j), \frac{\zeta}{2}b\Big),$$

with $V_0 \in \mathbb{C}^n$. In [6], we obtained the explicit expression of lifts of stationary discs near $\boldsymbol{f_0}$. Actually, that result is due to Tumanov [14] when the quadric M_H is strongly pseudoconvex. More precisely lifts of stationary discs $\boldsymbol{f} = (h, g, \tilde{h}, \tilde{g}) \in \mathcal{S}_0(M_H)$ near $\boldsymbol{f_0}$ are of the form

$$\begin{cases} h(\zeta) = V - \zeta(I - \zeta X)^{-1}(I - X)V \\ g_j(\zeta) = {}^t\overline{V}A_jV - 2{}^t\overline{V}A_j\zeta(I - \zeta X)^{-1}(I - X)V \\ \qquad + {}^t\overline{V}({}^t\overline{X}K_j - K_jX))V + \\ \qquad {}^t\overline{V}(I - {}^t\overline{X})K_j(I + 2\zeta X(I - \zeta X)^{-1})(I - X)V, \\ \tilde{h}(\zeta) = -\zeta\, {}^t\overline{h(\zeta)}\left(\sum_{j=1}^d (a_j\overline{\zeta} + (b_j - a - \overline{a}) + \overline{a_j}\zeta)A_j\right) \\ \tilde{g}(\zeta) = \dfrac{a + (b - a - \overline{a})\zeta + \overline{a}\zeta^2}{2} \end{cases} \tag{3.1}$$

where $V \in \mathbb{C}^n$ (close to V_0), $a \in \mathbb{C}^d$ is sufficiently small, X is the unique $n \times n$ matrix solution of (2.1) (with $b - a - \overline{a}$) with $\|X\| < 1$, and for $j = 1, \dots, d$

$$K_j = \sum_{r=0}^{\infty} {}^t\overline{X}^r A_j X^r. \tag{3.2}$$

As a consequence of that explicit expression, it is possible to characterize non-defective discs in $\mathcal{S}_0(M_H)$. Recall from [2] that a stationary disc f is *defective* if it admits a lift $\boldsymbol{f} = (f, \tilde{f}) : \Delta \to T^*\mathbb{C}^N$ such that $(f, \tilde{f}/\zeta)$ is holomorphic on Δ. As proved in [6], a disc f of the form (3.1) is non-defective if and only if the quadric M_H is stationary minimal at 0 for $(a, b - a - \overline{a}, h(0))$. Together with Theorem 2.2, we then recover

Theorem 3.1 [15] *Let M_H be a strongly pseudoconvex quadric given by* (1.2)*. Assume that matrices $A_1, \dots, A_d$ are linearly-independent. Then there exists a non-defective stationary disc f of the form* (3.1)*.*

4 The 1-jet map for stationary discs of the model quadric

Let M_H be a model quadric of the form (1.2). Consider the 1-jet map at $\zeta = 1$ defined on $\mathcal{S}_0(M_H)$

$$\mathrm{j}_1 : \boldsymbol{f} \mapsto (\boldsymbol{f}(1), \boldsymbol{f}'(1)).$$

Since $\boldsymbol{f}(1) = (0, 0, 0, b/2)$ where $b \in \mathbb{R}^d$ is fixed, the 1-jet map is identified with the derivative map $\boldsymbol{f} \mapsto \boldsymbol{f}'(1)$ at $\zeta = 1$. Using the explicit expression of lifts (3.1), we note in [6] that, after changes of variables in both the source and the target spaces, the 1-jet map $\mathrm{j}_1 : \mathbb{C}^d \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{C}^d$ at $\zeta = 1$ may be written as

$$\mathrm{j}_1 : (a, V) \mapsto (V, {}^t\overline{V}(I - {}^t\overline{X})K_j(I - X)V, \Im m a). \tag{4.1}$$

We prove in [6] that the fact that M_H is stationary minimal at 0 is a necessary condition for the 1-jet map j_1 to be a local diffeomorphism. In this section, we give an elementary proof of the equivalence when the model quadric M_H is strongly pseudoconvex.

Denote by $\mathcal{M}_n(\mathbb{C})$ the space of square matrices of size n with complex coefficients. Let P and X be given by (2.1). We consider the map

$$\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}),$$

defined by

$$\varphi(N) = N + P(NX + XN). \tag{4.2}$$

For $a \in \mathbb{C}^d$ small enough, the map φ is invertible with inverse of the form

$$\varphi^{-1}(N) = N + \sum_{r=0}^{\infty} Q_r(P, X)NX^r,$$

where Q_r is a convergent power series in P and X; e.g. $Q_0 = \sum_{k\geq 1}(-1)^k(PX)^k$. Moreover $Q_r(P, X) = \sum_{k\geq 1} R_{k,r}(P, X)$ where $R_{k,r}(P, X)$ is a homogeneous polynomial of degree k in P and $k-r$ in X. Note that if $a = 0$ then φ is the identity map. In what follows, we denote by $X_{\Re ea_s}$ the derivative $\dfrac{\partial X}{\partial \Re ea_s}$, $s = 1, \ldots, d$. We have the following lemma.

Lemma 4.1 *Let M_H be a strongly pseudoconvex quadric given by (1.2). Let $a \in \mathbb{R}^d$ be small enough and let X be the unique $n \times n$ matrix solution of (2.1) such that $\|X\| < 1$. Then, after a linear choice of coordinates, we have*

$$X_{\Re ea_s} = \varphi^{-1}(-A_s(I - X)^2) = -\varphi^{-1}(A_s)(I - X)^2,$$

where φ is given by (4.2).

Proof Since M_H is strongly pseudoconvex, we choose coordinates for which $\sum_{j=1}^d (b_j - a_j - \overline{a_j})A_j = I$. Differentiating Eq. (2.1) in $\Re ea_s$ and evaluating at $(a, b - a - \overline{a})$ implies that

$$A_s X^2 + P(XX_{\Re ea_s} + X_{\Re ea_s}X) - 2A_s X + X_{\Re ea_s} + A_s = 0$$

and so

$$X_{\Re ea_s} + P(XX_{\Re ea_s} + X_{\Re ea_s}X) = -A_s(I - X)^2$$

We then obtain

$$X_{\Re ea_s} = \varphi^{-1}(-A_s(I - X)^2) = -\varphi^{-1}(A_s)(I - X)^2.$$

□

We recall the following lemma.

Lemma 4.2 [6] *Let M_H be a strongly Levi non-degenerate quadric given by (1.2) and let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is positive definite. Let $a \in \mathbb{C}^d$ be small enough and let X be the unique $n \times n$ matrix solution of (2.1) such that $\|X\| < 1$.*

i. The 1-jet map $\mathfrak{j}_1$ is a local diffeomorphism at $(a, V) \in \mathbb{C}^d \times \mathbb{C}^n$ if and only if the $d \times d$ matrix

$$\left(\frac{\partial}{\partial \Re ea_s} {}^t\overline{V}(I - {}^t\overline{X})K_j(I - X)V\right)_{j,s}$$

is invertible.

ii. For any $s = 1, \ldots, d$, we have

$$\begin{aligned}\frac{\partial}{\partial \Re ea_s} {}^t\overline{V}\left(I - {}^t\overline{X}\right)K_j(I - X)V \\ = -2\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\left(I - {}^t\overline{X}\right)^2 {}^t\overline{X}^r K_j X_{\Re ea_s} X^r V\right).\end{aligned}$$

The main theorem is

Theorem 4.3 *Let M_H be a strongly pseudoconvex quadric given by (1.2) and let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is positive definite. Assume that M_H is stationary minimal at 0 for $(a, b - a - \overline{a}, V)$ with $a \in \mathbb{C}^d$ sufficiently small. Then the 1-jet map $\mathfrak{j}_1$ given by (4.1) is a local diffeomorphism at (a, V).*

Proof Assume that M_H is stationary minimal at 0 for $(a, b - a - \overline{a}, V)$ with $a \in \mathbb{C}^d$ sufficiently small. Due to the strong pseudoconvexity of M_H, we assume that the matrix $A := \sum_{j=1}^d (b_j - a_j - \overline{a_j})A_j$ is the identity. In light of Lemma 4.2, we need to show that

$$Y := \Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\left(I - {}^t\overline{X}\right)^2 {}^t\overline{X}^r K_j X_{\Re ea_s} X^r V\right)_{j,s}$$

is invertible. In fact, we will prove it is negative definite. Using Lemma 4.1, we have:

$$\begin{aligned}Y &= -\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\left(I - {}^t\overline{X}\right)^2 {}^t\overline{X}^r K_j\varphi^{-1}(A_s)X^r(I - X^2)V\right)_{j,s} \\ &= -\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V'}\,{}^t\overline{X}^r K_j\varphi^{-1}(A_s)X^r V'\right)_{j,s}\end{aligned}$$

where $V' = (I - X)^2V$. Since the quadric M_H is stationary minimal at 0 for $(a, b - a - \overline{a}, V)$, it is also stationary minimal at 0 for $(a, b - a - \overline{a}, (I - X)^2V)$ (see the proof of Lemma 6.7 [14], see also [6].) We then abusively write V in place of V', and we need to show that the matrix $\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\,{}^t\overline{X}^r K_j\varphi^{-1}(A_s)X^r V\right)_{j,s}$ is positive definite.

Let $W \in \mathbb{C}^d$ be a non-zero unit vector. Since M is stationary minimal, we have

$$\begin{aligned}{}^t\overline{W}\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\,{}^t\overline{X}^r A_j A_s X^r V\right)_{j,s} \\ W = \sum_{r=0}^{\infty}\|D_r W\|^2 + \sum_{r=0}^{\infty}\|\overline{D_r}W\|^2 > 0,\end{aligned}$$

where D_r is the $n \times d$ matrix whose s^{th} column is $A_s X^r V$. Since X is solution of Eq. (2.1), the term $D_r W$ is a sum of homogeneous polynomials in $a, \overline{a}$ of degree greater than or equal to r (and congruent to r modulo 2). Denote by $k(r)$ the minimal degree appearing in this decomposition and define $k_0 := \min_{r\geq 0} k(r)$. We denote by I the finite set of integers r which realize the minimum, that is such that $D_r W$ contains terms of degree k_0. Note that if $r \notin I$ then $D_r W = O(\|a\|^{k_0+1})$. Note also that the minimal degree appearing $\sum_{r=0}^{\infty} \|D_r W\|^2$ is exactly $2k_0$. Moreover, the function k_0 is a upper semi-continuous in W, which by compactness implies that it is bounded by above; this fact is crucial since it allows to choose a smaller if necessary. Now define

$$S_r := {}^t\overline{W}\left({}^t\overline{V}\,{}^t\overline{X}^r K_j \varphi^{-1}(A_s) X^r V\right)_{j,s} W.$$

In case $r \in I$ then

$$S_r = \|D_r W\|^2 + O(\|a\|^{2k_0+2}) > 0.$$

We claim that if $r \notin I$ then $S_r = O(\|a\|^{2k_0+2})$. To show this, note that, in that case, S_r only contains terms of the form

$$\begin{aligned}&{}^t\overline{W}\left({}^t\overline{V}\,{}^t\overline{X}^{r+\ell_1} A_j X^{\ell_1} R_{k,\ell_2}(P,X) A_s X^{r+\ell_2} V\right)_{j,s}\\ &W = {}^t\overline{D_{r+\ell_1} W} X^{\ell_1} R_{k,\ell_2}(P,X) D_{r+\ell_2} W,\end{aligned}$$

where $R_{k,\ell_2}(P,X)$ is a homogeneous polynomial of degree k in P and $k-\ell_2$ in X; this is due to the form of the inverse of φ (see Eq. (4.2)). Whether $r+\ell_1$ and $r+\ell_2$ are in I or not, we have

$${}^t\overline{D_{r+\ell_1} W} X^{\ell_1} R_{k,\ell_2}(P,X) D_{r+\ell_2} W = O(\|a\|^{2k_0+2}).$$

Indeed, we distinguish the four following cases.

- If $r+\ell_1 \notin I$ and $r+\ell_2 \notin I$, this is clear since $D_{r+\ell_j} W = O(\|a\|^{k_0+1})$, $j=1,2$.
- The case $r+\ell_1 \notin I$ and $r+\ell_2 \in I$ can only occur if $k \geq 1$. In that case, we have $D_{r+\ell_1} W = O(\|a\|^{k_0+1})$ and $D_{r+\ell_2} W = O(\|a\|^{k_0})$. Due to the contribution of the term $R_{k,\ell_2}(P,X)$, we obtain $O(\|a\|^{2k_0+2})$.
- The case $r+\ell_1 \in I$ and $r+\ell_2 \notin I$ can only occur if $\ell_1 \geq 1$. We have $D_{r+\ell_1} W = O(\|a\|^{k_0})$, $D_{r+\ell_2} W = O(\|a\|^{k_0+1})$ and with the contribution of the term X^{ℓ_1} we obtain $O(\|a\|^{2k_0+2})$.
- Finally, the case $r+\ell_1 \in I$ and $r+\ell_2 \in I$ can only occur if $\ell_1 \geq 1$ and $k \geq 1$. We have $D_{r+\ell_j} W = O(\|a\|^{k_0})$, $j=1,2$, and with the contribution of the terms X^{ℓ_1} and $R_{k,\ell_2}(P,X)$ we also obtain $O(\|a\|^{2k_0+2})$.

At this stage, we have proved that

$$\sum_{r=0}^{\infty} S_r = \sum_{r\in I} \|D_r W\|^2 + O(\|a\|^{2k_0+2})$$

and so is positive since a can be taken smaller if necessary. This proves that Y is negative definite and concludes the proof of the theorem. □

The previous theorem and Theorem 2.2 imply the following result.

Corollary 4.4 *Let M_H be a strongly pseudoconvex quadric given by (1.2). Assume that the matrices $A_1,\ldots,A_d$ are linearly-independent. Then there exist $a \in \mathbb{R}^d$ and $V \in \mathbb{C}^n$ such that the 1-jet map $\mathfrak{j}_1$ is a local diffeomorphism at (a, V).*

As a direct consequence, we recover Theorem 1.3 by the usual stationary disc method [3–5].

4.1 Open questions

An interesting feature of Theorem 4.3 is the fact that its proof gives a roadmap for extending the result to quadrics and submanifolds which are not necessarily strongly pseudoconvex. In this vein, we define

Definition 4.5 Let M be a real submanifold given by (1.1). Assume M is strongly Levi non-degenerate at 0 and let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is invertible. Let $a \in \mathbb{C}^d$ be sufficiently small and set $A := \sum_{j=1}^d (b_j - a_j - \overline{a_j}) A_j$. We say that M is (resp. *strongly*) $\mathfrak{D}(a)$*-non-degenerate* at 0 if there exists $V \in \mathbb{C}^n$ such that the matrix

$$\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\,{}^t\overline{X}^r A_j A^{-1} A_s X^r V\right)_{j,s}$$

is non-degenerate (resp. positive definite).

Note that when $a = 0$, we recover the definition of $\mathfrak{D}$-non-degeneracy. The following lemma is immediate.

Lemma 4.6 *Definition 4.5 is independent of the choice of holomorphic coordinates.*

We also have

Lemma 4.7 *If M is $\mathfrak{D}(a)$-non-degenerate at 0 then M is stationary minimal at 0 for $(a, b-a-\overline{a}, V)$ for some $V \in \mathbb{C}^n$.*

Proof Assume that $\lambda_1,\ldots,\lambda_d \in \mathbb{R}$ are such that $\sum_{j=1}^d \lambda_j A_j X^r V = 0$ for all $r = 0,1,2,\ldots$. Set $W = {}^t(\lambda_1,\ldots,\lambda_d) \in \mathbb{R}^d$ and we consider the $n \times d$ matrix D_r whose s^{th} column is $A_s X^r V$. We then have $D_r W = 0$ for all $r = 0,1,2,\ldots$. It follows that

$$\begin{aligned}&\Re e\left(\sum_{r=0}^{\infty} {}^t\overline{V}\,{}^t\overline{X}^r A_j A^{-1} A_s X^r V\right)_{j,s}\\ &W = {}^t\overline{D_r} A^{-1} D_r W + {}^t D_r \overline{A}^{-1}\overline{D_r} W\\ &= {}^t D_r\,{}^t A^{-1}\overline{D_r W} = 0\end{aligned}$$

Since M is $\mathfrak{D}(a)$-non-degenerate, it follows that $W = 0$. □

We also note that in case M is strongly pseudoconvex then if M is stationary minimal at 0 for $(a, b - a - \overline{a}, V)$ then it is (strongly) $\mathfrak{D}(a)$-non-degenerate at 0; in fact, according to Theorem 2.2, if M is strongly pseudoconvex and Levi generating then it is (strongly) $\mathfrak{D}(a)$-non-degenerate for some $a \in \mathbb{C}^d$.

In [5], we conjectured that if M is a $\mathcal{C}^4$ generic real submanifold strongly Levi non-degenerate of the form (1.1), and admitting a non-defective stationary disc passing through 0, then any germ at 0 of CR automorphism of M of class $\mathcal{C}^3$ is uniquely determined by its 2-jet at p. The key point in the conjecture is to show that the 1-jet map $\mathfrak{j}_1$ is a local diffeomorphism. In [5], we proved that if M_H is strongly Levi non-degenerate quadric and the map $\mathfrak{j}_1$ is a local diffeomorphism, then M_H is stationary minimal. At the moment, it is not clear to us how to obtain the converse.

Questions *Consider a $\mathcal{C}^4$ generic real submanifold $M \subset \mathbb{C}^{n+d}$. Assume that M is (strongly) $\mathfrak{D}(a)$-non-degenerate at* 0.

i. *Is the* 1*-jet map* $\mathfrak{j}_1$ *a local diffeomorphism at* (a, V) *for some* $V \in \mathbb{C}^n$?
ii. *Are germs at* 0 *of biholomorphisms sending M into itself uniquely determined by their* 2*-jet at* 0*?*
iii. *Are germs at* 0 *of CR automorphisms of M of class $\mathcal{C}^3$ uniquely determined by their* 2-jet at 0?

Note that although i. implies ii. and iii., it may be possible to prove the second and third points ii. and iii. without using the stationary disc method via different techniques.

Data availability Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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