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RESEARCH

# Mimetic Horava and surface terms

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## Abstract

We consider mimetic Horava gravity, where the scalar field of mimetic gravity was used in the construction of diffeomorphism invariant models reducing to Horava gravity in the synchronous gauge. It will be shown that the gravitational action with the addition of the Gibbons–Hawking–York term and the mimetic Horava action are equivalent for manifolds whose topology is $R \times \Sigma$, where $\Sigma$ is a three-dimensional hypersurface; otherwise, the mimetic Horava action does not contain any surface terms.

**Keywords** Mimetic gravity · Horava gravity · Gibbons-Hawking-York term

Despite the huge success that GR has achieved, still it is unable to explain some challenges: dark matter, spacetime singularities including those of the big bang cosmology, and black holes. This issue has pushed physicists to modify GR in several ways. Some modified theories of gravity have aimed to quantize GR as a way to resolve the Big bang singularities or the black holes. Approaches to quantum gravity can be classified into two groups [1]. The first group starts from the classical theory of gravity and then applies the quantization steps. The covariant quantum gravity is a good example, where one starts from the path integral of the GR action and then applies the perturbation procedures for the metric around a background one. The resulting theory has been a nonrenormalizable one upon extending it to higher energies. The canonical quantum gravity is another model that belongs to the same group. In this approach, one starts constructing the Hamiltonian of the classical GR theory and then turns it into the quantum one by quantizing the constraints. Special attention should be taken upon dealing with surface terms [2]. The second group aims to construct a unified quantum theory of all the forces as in string theory. By adding higher-order derivative terms, such as $R_{\mu\nu}R^{\mu\nu}$, to the Einstein–Hilbert action, we obtain an improved UV behavior. However, it turned out that the modified theory contains ghosts [3]. The emergence of ghosts is closely related to the presence of orders of time derivatives higher than two. The most notable attempt to solve the problem of ghosts was to break Lorentz

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Invariance in the ultraviolet limit. This was done by Horava [4] where he included only high-order spatial derivative terms while keeping the time derivative terms to the second order.

Horava constructed a model of quantum gravity by choosing to break Lorentz Invariance. He considered anisotropic scaling between time and space. Terms dependent on the spatial Ricci tensor and curvature scalar and their space derivatives were added to the action ([5] and references therein). Although perturbative renormalizability of projectable Horava gravity was proved, upon making the model covariant by adding one new field, the property of renormalizability was lost [6]. In an attempt to restore Lorentz invariance while keeping the renormalizability, a dynamical scalar is added [7]. Such a theory propagates three degrees of freedom, two are related to the graviton and one is related to an additional propagating scalar mode.

In an attempt to explain dark matter, the mimetic dark model was proposed by Chamseddine and Mukhanov, as a modified theory of general relativity [8]. In this reformulation of GR, the conformal degree of freedom is isolated in a covariant way. The physical metric was defined in terms of an auxiliary metric and a scalar field ($\phi$). It was shown that the extra conformal degree of freedom becomes dynamical, and it can mimic cold dark matter. In addition to the prediction of dark matter, this model also predicted several cosmological solutions [9] and dark energy [10], resolving singularities [11, 12] and building a ghost-free massive gravity model [13].

Recently, a mimetic Horava gravity model, which regenerates the Horava gravity without any ghosts, was proposed by Chamseddine, Mukhanov, and Russ [14]. Using the scalar field of mimetic gravity, a diffeomorphism invariant model, without introducing new propagating degrees of freedom, was constructed that reduces to Horava gravity in the synchronous gauge. The quantities written are such that Horava gravity is formulated in a diffeomorphism invariant way without the introduction of ghost-like degrees of freedom.

The main purpose of this letter is to study the surface terms that appear upon varying the mimetic Horava action. Considering first the surface terms emerging in GR, upon variation of the Einstein–Hilbert action with respect to the metric, a surface integral over the boundary $\partial M$ is obtained. These surface terms gather into a total variation; therefore, to have a well-defined variational principle, a total divergence is added to the action. This is the Gibbons–Hawking–York surface (GHY) term [15, 16]. This guarantees that upon deriving the equations of motion, the boundary terms arising from the variation vanish. This manipulation was needed because a second-order derivative hides in the Ricci scalar $R$, which is not allowed in the path integral formalism. To get an action that depends only on the first derivatives of the metric, these second derivatives terms are removed by integration by parts. Then the resulting action is quadratic in the first derivatives of the metric. The boundary term that is obtained upon varying the Hilbert–Einstein action is given by

$$-\oint_{\partial M} \epsilon h^{\mu\nu} n^{\alpha} \delta g_{\mu\nu,\alpha} \sqrt{h}\, d^3x \tag{1}$$

which can be recast into a much more elegant form written as

$$2\oint_{\partial M} \epsilon K\sqrt{h}\, d^3x \tag{2}$$

where

$$\begin{aligned} g^{\mu\nu} &= \epsilon n^{\mu}n^{\nu} + h^{\mu\nu}, \\ n_{\mu}n^{\mu} &= \epsilon \end{aligned} \tag{3}$$

$\partial M$ is the boundary of M, $h_{ab}$ is the induced metric on $\partial M$, and K is the trace of the second fundamental form on $\partial M$. If **n** is a time-like unit vector, then $\epsilon = 1$.
Accordingly, the amended gravitational action, which is fully consistent with the Einstein field equations, becomes [17, 18]

$$I = -\int_M \sqrt{g}R\, d^4x - 2\oint_{\partial M} \epsilon K\sqrt{h}\, d^3x$$

This modification is interesting because the variation of the GHY term cancels the terms involving the variations of the derivatives of the metric, and therefore, setting the variation of the metric equal to zero becomes sufficient to make the action stationary. In canonical formulation, some ignore the presence of the surface term. As a result, certain boundary terms must be added to the constraint to give a well-defined equation of motion [19].
Based on [14], mimetic Horava gravity, we consider a Horava action in mimetic gravity given by

$$I = \int \sqrt{-g}(-R - 2\nabla_{\mu}(\Box\phi\nabla^{\mu}\phi) + 2\nabla_{\sigma}(\nabla_{\mu}\nabla^{\sigma}\phi\nabla^{\mu}\phi))d^4x \tag{4}$$

In this letter, our aim is to prove that this action is equivalent to the Einstein–Hilbert action plus Gibbon's Hawking boundary term for manifolds that have topology $R \times \Sigma$. The main goal behind canceling the surface terms is to perform the Hamiltonian analysis which is the base for the canonical quantization of GR. It has been years and the inclusion of boundary terms in the action, in the Hamiltonian quantization of gravity, is considered a must. Here, the remarkable point is that, in mimetic Horava gravity, the cancellation occurs without the addition of the Gibbons–Hawking–York surface (GHY) term.

We start by writing the action 4 as a sum of three parts

$$I = I_H + I_1 + I_2$$

$I_H$ is the Einstein–Hilbert action for general relativity. It is given by

$$I_H = -\int \sqrt{-g}R\, d^4x \tag{5}$$

$I_1$ and $I_2$ are given by

$$I_1 = \int \sqrt{-g}(-2\nabla_\mu(\Box\phi\nabla^\mu\phi)d^4x)$$

and

$$I_2 = \int \sqrt{-g}(2\nabla_\sigma(\nabla_\mu\nabla^\sigma\phi\nabla^\mu\phi)d^4x)$$

Upon varying the Einstein–Hilbert action (5), we get the surface contributions given in Eq. (1) which must vanish if the action is to be stationary. To check if the overall surface terms obtained upon varying the mimetic Horava action (4) are going to cancel out, we need to study the variation of the other terms ($I_1$ & $I_2$). The variation of each of $I_1$ and $I_2$ will be written as a sum of two terms, one that contains the $\phi$ variation and the other that contains the metric variation. These are given by

$$\delta I_1 = I_1'(\delta\phi) + 2\oint_{\partial\gamma} \frac{1}{2} g^{\rho\sigma} g^{\alpha\lambda}(2\delta g_{\rho\lambda,\sigma} - \delta g_{\rho\sigma,\lambda})\nabla_\mu\phi\ \nabla_\alpha\phi\ d\Sigma^\mu$$

and

$$\delta I_2 = I_2'(\delta\phi) - \oint_{\partial\gamma} g^{\rho\tau}(\delta g_{\mu\tau,\sigma} + \delta g_{\sigma\tau,\mu} - \delta g_{\mu\sigma,\tau})\nabla_\rho\phi\ \nabla^\mu\phi\ d\Sigma^\sigma.$$

Studying first the variation in the field $\phi$, the two terms, $I_1'(\delta\phi)$ and $I_2'(\delta\phi)$, are given by

$$I_1'(\delta\phi) = 2\oint_{\partial\gamma} \nabla_\mu\phi\ \Box\delta\phi\ d\Sigma^\mu + 2\oint_{\partial\gamma} \nabla_\mu\delta\phi\ \Box\phi\ d\Sigma^\mu$$

and

$$I_2'(\delta\phi) = 2\oint_{\partial\gamma} \nabla_\mu\nabla_\sigma\delta\phi\ \nabla^\mu\phi\ d\Sigma^\sigma + 2\oint_{\partial\gamma} \nabla_\mu\nabla_\sigma\phi\ \nabla^\mu\delta\phi\ d\Sigma^\sigma.$$

It is clear that these two terms could be integrated out (surface of a surface) and will vanish upon setting $\delta\phi = 0$ on the boundaries.

The more interesting part is studying the other surface terms, over $\delta g_{\mu\nu}$, of $\delta I_H$, $\delta I_1$ and $\delta I_2$ and checking under what condition (if any) these cancel out upon addition. The resulting boundary terms from $I_1$ and $I_2$ are given by

$$\begin{aligned} &2\oint_{\partial\gamma} \frac{1}{2} g^{\rho\sigma} g^{\alpha\lambda}(2\delta g_{\rho\lambda,\sigma} - \delta g_{\rho\sigma,\lambda})\nabla_\mu\phi\ \nabla_\alpha\phi\ d\Sigma^\mu \\ &- \oint_{\partial\gamma} g^{\rho\tau}(\delta g_{\mu\tau,\sigma} + \delta g_{\sigma\tau,\mu} - \delta g_{\mu\sigma,\tau})\nabla_\rho\phi\ \nabla^\mu\phi\ d\Sigma^\sigma \end{aligned} \tag{6}$$

which can be written as

$$\oint_{\partial\gamma} 2g^{\rho\sigma}\delta g_{\rho\lambda,\sigma}\nabla_\mu\phi\ \nabla^\lambda\phi\ d\Sigma^\mu - \oint_{\partial\gamma} g^{\rho\sigma}\delta g_{\rho\sigma,\lambda}\nabla_\mu\phi\ \nabla^\lambda\phi\ d\Sigma^\mu$$
$$-\oint_{\partial\gamma}\delta g_{\mu\tau,\sigma}\nabla^\tau\phi\ \nabla^\mu\phi\ d\Sigma^\sigma \tag{7}$$

where $d\Sigma^\mu = n^\mu\epsilon\sqrt{h}\ d^3y$.

If the Lorentzian spacetime manifold has topology $R\times\Sigma$ and can be foliated by a succession of three-dimensional space-like hypersurfaces, the normal **n** is time-like unit vector and satisfies $n_\mu n^\mu = 1$ [20, 21]. This is like in ADM canonical formulation where the sign of $\epsilon = n_\mu n^\mu$ in Eq. (2) is taken to be plus one. With the constraint governing the evolution of $\phi$ from [14], we can identify

$$n_\mu = \partial_\mu\phi. \tag{8}$$

Upon this choice and with the use of equations 3, the surface terms 7 will cancel the surface term from the Hilbert–Einstein action. We say that the gravitational action with the added Gibbons–Hawking–York term becomes equivalent to the mimetic Horava action. Otherwise, the mimetic Horava action is good on its own without adding any surface terms because the second-time derivatives cancel. The elimination of the surface terms in the latter will occur in the synchronous gauge, where the time-like unit vector $n_\mu$ is given by $\partial_\mu\phi$. This means that there is no need to integrate by part to compute the Hamiltonian; hence, the momenta is defined without ambiguity. This can be easily seen by writing the mimetic Horava action in $3+1$ formalism. The $\phi$-part of the mimetic Horava action is given by

$$I_\phi = \int\sqrt{-g}(-2\nabla_\alpha(n^\alpha\nabla_\mu n^\mu) + 2\nabla_\alpha(n^\mu\nabla_\mu n^\alpha))d^4x \tag{9}$$

and the scalar curvature $R$ (including the total derivative) can be written as

$${}^4R = (R - K^2 + K_{ij}K^{ij}) - 2\nabla_\alpha(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu) \tag{10}$$

The cancellation of the surface terms is manifested.

It is worth noting that $\phi$ in mimetic Horava gravity, in the synchronous gauge, can always be used as one particular synchronous time coordinate. Furthermore, the mimetic constraint, $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$ is the same as the condition needed to define synchronous time [22]. This implies that the vector $n_\mu$ is always time-like because if it is time-like at one event, then it is everywhere time-like [23]. Making use of this time-like unit vector, $n_\mu = \partial_\mu\phi$, any Lagrangian can be projected to have higher space derivative terms and no higher time derivative ones. Here, we have shown that the variation of the resulting higher space derivative surface terms is capable of canceling the surface term resulting from the variation of Einstein–Hilbert action. The remarkable point is that starting from the mimetic Horava action, the signs and the coefficients of the surface terms are just right to cancel out.

In summary, quantizing gravity helps to explain several unsolved problems. Most quantum gravity theory depends on the Hamiltonian analysis of general relativity. The presence of a surface term in the variation of the Einstein–Hilbert action generates problems in the equations of motion. GHY has been widely accepted as the correct modification of the action, but this raises the question of the uniqueness of the modification. In this letter, we proved that the Einstein–Hilbert action plus Gibbon's Hawking boundary term is equivalent to the mimetic Horava action for manifolds that has topology $R \times \Sigma$; otherwise, the mimetic action is good on its own without adding any surface terms because the second time derivatives cancel.

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## Declarations

**Conflict of interest** The authors declare that they have no competing interests.

## References

1. Kiefer, C.: Conceptual problems in quantum gravity and quantum cosmology. ISRN Math. Phys. **2013**, 509316 (2013)
2. Chamseddine, A., Connes, A.: Quantum gravity boundary terms from the spectral action of noncommutative space. Phys. Rev. Lett. **99**, 071302 (2007)
3. Stelle, K.S.: Renormalization of higher-derivative quantum gravity. Phys. Rev. D **16**, 953 (1977)
4. Horava, P.: Quantum gravity at a Lifshitz point. Phys. Rev. D **79**, 084008 (2009)
5. Wang, A.: Horava gravity at a Lifshitz point a progress report. Int. J. Mod. Phys. D **26**(07), 1730014 (2017)
6. Barvinsky, A., Blas, D., Herrero-Valea, M., Sibiryakov, S., Steinwachs, C.: Renormalization of Horava gravity. Phys. Rev. D **93**, 064022 (2016)
7. Germani, C., Kehagias, A., Sfetsos, K.: Relativistic quantum gravity at a Lifshitz point. JHEP **009**, 060 (2009)
8. Chamseddine, A.H., Mukhanov, V.: Mimetic dark matter. JHEP **1311**, 135 (2013)
9. Chamseddine, A.H., Mukhanov, V., Vikman, A.: Cosmology with mimetic matter. JCAP **14**(6), 017 (2014)
10. Chamseddine, A.H., Mukhanov, V.: Inhomogeneous dark energy. JCAP **1602**(2), 040 (2016)
11. Chamseddine, A.H., Mukhanov, V.: Resolving cosmological singularities. JCAP **1703**(3), 009 (2017)
12. Chamseddine, A.H., Mukhanov, V.: Nonsingular black hole. Eur. Phys. J. C **77**(3), 183 (2017)
13. Chamseddine, A.H., Mukhanov, V.: Ghost free mimetic massive gravity. JHEP **1806**, 060 (2018)
14. Chamseddine, A., Mukhanov, V.R., Tobias, B.: Mimetic Hořava gravity. Phys. Lett. B **798**, 66 (2019)
15. Gibbons, G.W., Hawking, S.W.: Action integrals and partition functions in quantum gravity. Phys. Rev. D **15**, 2752 (1977)
16. York, J.: Role of conformal three-geometry in the dynamics of gravitation. Phys. Rev. Lett. **28**, 1082 (1972)
17. Hawking, S., Horowitz, G.: The gravitational Hamiltonian, action, entropy, and surface terms. Class. Quantum Gravity **13**, 1487–1498 (1996)

18. Hawking, S.: Euclidean Quantum Gravity. World Scientific, Singapore (1993)
19. Regge, T.: Role of surface integrals in the Hamiltonian formulation of general relativity. Ann. Phys. **88**, 286–318 (1974)
20. Charap, J.M., Nelson, J.E.: Canonical general relativity: the primary constraint algebra. J. Phys. A Math. Gen. **16**(14), 3355 (1983)
21. Charap, J.M., Nelson, J.E.: The canonical Hamiltonian for vierbein general relativity. Class. Quantum Gravity **3**(6), 1061 (1986)
22. Landau, L., Lifshitz, E.: The Classical Theory of Fields Butterworth–Heinemann, section 97, 4th ed (1980)
23. Misner, C., Thorne, K., Wheeler, J.: The Classical Theory of Fields: Volume 2, section 13.4. Freeman, New York (1973)