

# Mizohata–Takeuchi estimates in the plane

**Bassam Shayya**

Department of Mathematics, American  
University of Beirut, Beirut, Lebanon

**Correspondence**

Bassam Shayya, Department of  
Mathematics, American University of  
Beirut, Beirut, Lebanon.  
Email: [bshayya@aub.edu.lb](mailto:bshayya@aub.edu.lb)

**Abstract**

Suppose that  $S$  is a smooth compact hypersurface in  $\mathbb{R}^n$  and  $\sigma$  is an appropriate measure on  $S$ . If  $Ef = \widehat{f}d\sigma$  is the extension operator associated with  $(S, \sigma)$ , then the Mizohata–Takeuchi conjecture asserts that  $\int |Ef(x)|^2 w(x) dx \lesssim (\sup_T w(T)) \|f\|_{L^2(\sigma)}^2$  for all functions  $f \in L^2(\sigma)$  and weights  $w : \mathbb{R}^n \rightarrow [0, \infty)$ , where the sup is taken over all tubes  $T$  in  $\mathbb{R}^n$  of cross-section 1, and  $w(T) = \int_T w(x) dx$ . This paper investigates how far we can go in proving the Mizohata–Takeuchi conjecture in  $\mathbb{R}^2$  if we only take the decay properties of  $\widehat{\sigma}$  into consideration. As a consequence of our results, we obtain new estimates for a class of convex curves that include exponentially flat ones such as  $(t, e^{-1/t^m})$ ,  $0 \leq t \leq c_m$ ,  $m \in \mathbb{N}$ .

**MSC 2020**

42B10, 42B20 (primary)

## 1 | INTRODUCTION

Let  $S$  be a smooth compact hypersurface in  $\mathbb{R}^n$ , and  $\sigma$  a finite Borel measure on  $S$ . The Fourier extension operator  $E$  associated with the pair  $(S, \sigma)$  is defined as

$$Ef(x) = \widehat{f}d\sigma(x) = \int e^{-2\pi i x \cdot \xi} f(\xi) d\sigma(\xi)$$

for  $f \in L^1(\sigma)$ .

Suppose that  $w$  is a nonnegative Lebesgue measurable function on  $\mathbb{R}^n$ . If  $T$  is the  $(1/2)$ -neighborhood of a line  $L$  in  $\mathbb{R}^n$ , we say that  $T$  is a 1-tube and call  $L$  the core line of  $T$ . For such

tubes, we define

$$w(T) = \int_T w(x)dx.$$

If  $\sigma$  is the surface measure on  $S$ , then the Mizohata–Takeuchi conjecture ([18, 23] and [24]) asserts that

$$\int |Ef(x)|^2 w(x)dx \lesssim \left( \sup_T w(T) \right) \|f\|_{L^2(\sigma)}^2$$

for all  $f \in L^2(\sigma)$ , where the sup is taken over all 1-tubes  $T$  in  $\mathbb{R}^n$ . This conjecture is open in all dimensions  $n \geq 2$ , but there are some known partial results. For example, the conjecture is true if  $S$  is the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$  and the weight  $w$  is radial. This result was shown, independently, in [3] and [10] (see also [9] and [11]).

There is also a known local version of the above estimate that was proved in [1] for the unit circle  $\mathbb{S}^1$ : to every  $\epsilon > 0$ , there is a constant  $C_\epsilon$  such that

$$\int_{|x| \leq R} |Ef(x)|^2 w(x)dx \leq C_\epsilon R^\epsilon R^{1/2} \left( \sup_T w(T) \right) \|f\|_{L^2(\sigma)}^2 \tag{1}$$

for all  $f \in L^2(\sigma)$ , where the sup is taken over all 1-tubes  $T$  in  $\mathbb{R}^2$ . (See (11) in the next section for a slightly better estimate than (1).)

For more results on the Mizohata–Takeuchi conjecture and its applications in Fourier analysis and PDE, we refer the reader to the papers [2, 4–6], and [7].

The main results of this paper are stated in the following two theorems. Both theorems are stated in dimension  $n = 2$ , but their method of proof applies and may very well lead to new results in higher dimensions; we restrict ourselves to  $\mathbb{R}^2$  to keep the technical details from distracting our attention from the main ideas. In both theorems, we think of  $\sigma$  as a finite Borel measure on  $\mathbb{R}^2$  rather than a measure on a curve  $S$ , consider that the extension operator  $E$  is associated with  $\sigma$  rather than  $S$ , and study the Mizohata–Takeuchi conjecture through the decay properties of the Fourier transform of  $\sigma$ . (When  $\sigma$  is supported on  $S$ , the decay of  $\hat{\sigma}$  often reflects some geometric property of  $S$ .)

For  $m \in \mathbb{R}$ , we denote by  $\mathbb{T}_m$  the set of all 1-tubes in  $\mathbb{R}^2$  whose core lines are parallel to either of the two lines  $\{x = (x_1, x_2) \in \mathbb{R}^2 : mx_1 + x_2 = 0 \text{ or } x_1 + mx_2 = 0\}$ . We note that if  $m \neq 0$ , then  $\mathbb{T}_m = \mathbb{T}_{1/m}$ .

**Theorem 1.1.** *Suppose  $0 < \delta \leq 1$  and  $\sigma$  is a finite Borel measure on  $\mathbb{R}^2$  whose Fourier transform obeys the decay estimate*

$$|\hat{\sigma}(x_1, x_2)| \leq \begin{cases} C|x_1|^{-\delta} & \text{if } |x_1| \geq 1 \text{ and } |x_2| \leq 1, \\ C|x_2|^{-\delta} & \text{if } |x_2| \geq 1 \text{ and } |x_1| \leq 1, \\ C|x_1 x_2|^{-\delta} & \text{if } |x_1| \geq 1 \text{ and } |x_2| \geq 1. \end{cases} \tag{2}$$

Also, suppose that the weight  $w$  is a tensor function of the form

$$w(x) = w(x_1, x_2) = \tilde{w}(ax_1)\tilde{w}(bx_2), \tag{3}$$

where  $\tilde{w} : \mathbb{R} \rightarrow [0, \infty)$  is Lebesgue measurable and  $a, b \in \mathbb{R}$  are positive constants. Then, for  $q > 1/\delta$ , we have

$$\int |Ef(x)|^q w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right) \|f\|_{L^2(\sigma)}^q \tag{4}$$

for all  $f \in L^2(\sigma)$ . (The implicit constant in (4) depends only on  $C, \delta, q$ , and  $\|\sigma\|$ .)

Condition (3) on  $w$  is perhaps the ‘‘opposite’’ of the radial condition of [3] and [10]. However, when  $\sigma$  is arc length measure on the unit circle  $\mathbb{S}^1$ , the decay exponent of  $\hat{\sigma}$  (as given in (2)) will be  $\delta = 1/4$ , and hence, Theorem 1.1 will only give us (4) for  $q > 4$  (see also (10) in the next section), whereas in the radial case, [3] and [10] tell us that (4) (with  $\sup_{T \in \mathbb{T}_{a/b}} w(T)$  replaced by  $\sup_T w(T)$ ) is true for  $q \geq 2$ .

For  $v \in \mathbb{R}^2$  with  $|v| = 1$ , we denote by  $\mathbb{T}_v$  the set of all 1-tubes in  $\mathbb{R}^2$  whose core lines are perpendicular to  $v$ .

**Theorem 1.2.** *Suppose  $0 < \delta \leq 1, v \in \mathbb{R}^2, |v| = 1$ , and  $\sigma$  is a finite Borel measure on  $\mathbb{R}^2$  whose Fourier transform obeys the decay estimate*

$$|\hat{\sigma}(x)| \leq \frac{C}{|x \cdot v|^\delta} \tag{5}$$

if  $|x| \geq 1$ . Also, suppose that  $w : \mathbb{R}^2 \rightarrow [0, \infty)$  is Lebesgue measurable. Then, for  $q > 2/\delta$ , we have

$$\int |Ef(x)|^q w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_v} w(T) \right) \|f\|_{L^2(\sigma)}^q \tag{6}$$

for all  $f \in L^2(\sigma)$ . (The implicit constant in (6) depends only on  $C, \delta, q$ , and  $\|\sigma\|$ .)

As a consequence of Theorem 1.2, we obtain the following result concerning the extension operator associated with convex curves in the plane.

**Corollary 1.1.** *Suppose  $\gamma : [0, c] \rightarrow \mathbb{R}$  satisfies the following properties:  $\gamma$  and  $\gamma'$  are convex with  $\gamma(0) = \gamma'(0) = 0, \gamma \in C^2((0, c]) \cap C^3((0, c)), (\gamma''^{1/2}/\gamma')' \leq 0$ , and  $\gamma(t)\gamma''(t)/\gamma'(t)^2 \leq C$  for all  $0 < t \leq c$ . Let  $\sigma$  be the measure on the curve  $(t, \gamma(t))$  given by  $d\sigma(t) = \gamma''(t)^{1/2} dt$ . Then, for  $q > 4$ , we have*

$$\int |Ef(x)|^q w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_{(0,1)}} w(T) \right) \|f\|_{L^2(\sigma)}^q \tag{7}$$

for all  $f \in L^2(\sigma)$  and Lebesgue measurable  $w : \mathbb{R}^2 \rightarrow [0, \infty)$ .

*Proof.* We know from [12, Proposition 3.1] that  $|\hat{\sigma}(x_1, x_2)| \lesssim |x_2|^{-1/2}$  for all  $x_2$  (uniformly in  $x_1$ ), so the corollary follows from applying Theorem 1.2 with  $\delta = 1/2$  and  $v = (0, 1)$ . □

We note that the convex curves  $(t, e^{-1/t^m}), 0 \leq t \leq c_m, m \in \mathbb{N}$ , which are exponentially flat at the origin, fall within the scope of Corollary 1.1, and hence satisfy (7). Since (7) implies (1), it follows that our convex curves also satisfy (1). Of course, the same argument that proves (1) for the unit circle  $\mathbb{S}^1$  also proves it for compact curves with nonvanishing curvature. The fact that (1) also holds for convex curves that are exponentially flat at some point indicates that the  $R^{1/2}$ -loss in (1) is not directly related to the curve’s second fundamental form, and hence suggests that further progress on the Mizohata–Takeuchi conjecture should be possible in the nonvanishing curvature case. Very recently, this turned out to indeed be true, and the  $R^{1/2}$ -loss was lowered to  $R^{1/3}$ -loss in [8] for compact  $C^2$  curves with nowhere vanishing Gaussian curvature.

We also note that if the curve is completely flat and  $\sigma$  is arc length measure on it, then the estimate of Corollary 1.1 is true for all  $q > 2$ , but with  $\mathbb{T}_{(0,1)}$  changed appropriately: if  $v$  is a unit vector that is parallel to the curve, then, for  $q > 2$ , we have

$$\int |Ef(x)|^q w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_v} w(T) \right) \|f\|_{L^2(\sigma)}^q \tag{8}$$

for all  $f \in L^2(\sigma)$  and  $w : \mathbb{R}^2 \rightarrow [0, \infty)$ . It is not hard to prove (8) (even for  $q \geq 2$  and for any compact flat hypersurface in  $\mathbb{R}^n, n \geq 2$ ) directly without needing any information about  $\hat{\sigma}$ , but it can also be easily obtained from Theorem 1.2 as follows. We can take  $\gamma(t) = 0, 0 \leq t \leq 1$ , so that  $S$  is the line segment  $[0, 1] \times \{0\} \subset \mathbb{R}^2$ . Then

$$|\hat{\sigma}(x_1, x_2)| = \left| \int_0^1 e^{-2\pi i x_1 t} dt \right| \lesssim \frac{1}{|x_1|}$$

for all  $(x_1, x_2) \in \mathbb{R}^2$ , and (8) follows by applying Theorem 1.2 with  $v = (1, 0)$  and  $\delta = 1$ .

So, while nonweighted restriction estimates (such as Stein–Tomas) require  $\hat{\sigma}$  to decay uniformly in all directions, Mizohata–Takeuchi estimates can hold by exploiting the decay of  $\hat{\sigma}$  in a single direction. (See also Subsection 2.2 below.)

Theorems 1.1 and 1.2 were motivated by the author’s recent paper [21]. This will be discussed in the next section. The subsequent sections will then be dedicated to the proofs of Theorems 1.1 and 1.2 by following the strategy of [21].

## 2 | THE MOTIVATION BEHIND THEOREMS 1.1 AND 1.2

Suppose  $n \geq 1$  and  $0 < \alpha \leq n$ . Following [20] and [21], for Lebesgue measurable functions  $H : \mathbb{R}^n \rightarrow [0, 1]$ , we define

$$A_\alpha(H) = \inf \left\{ C : \int_{B(x_0, R)} H(x) dx \leq CR^\alpha \text{ for all } x_0 \in \mathbb{R}^n \text{ and } R \geq 1 \right\},$$

where  $B(x_0, R)$  denotes the ball in  $\mathbb{R}^n$  of center  $x_0$  and radius  $R$ . We say that  $H$  is a *weight of fractal dimension*  $\alpha$  if  $A_\alpha(H) < \infty$ . We note that  $A_\beta(H) \leq A_\alpha(H)$  if  $\beta \geq \alpha$ , so the phrase “ $H$  is a weight of fractal dimension  $\alpha$ ” is not meant to assign a dimension to the function  $H$ ; rather, it is just another way for saying that  $A_\alpha(H) < \infty$ .

Suppose  $n \geq 2$ , the surface  $S \subset \mathbb{R}^n$  has a nowhere vanishing Gaussian curvature, and  $\sigma$  is surface measure on  $S$ . In [21] (see the  $\alpha = n/2$  case of [21, Theorem 2.1]), it was proved that the restriction estimate<sup>†</sup>

$$\int |Ef(x)|^q H(x) dx \lesssim A_{n/2}(H) \|f\|_{L^2(\sigma)}^q \tag{9}$$

holds for all functions  $f \in L^2(\sigma)$  and weights  $H$  on  $\mathbb{R}^n$  of dimension  $n/2$  whenever  $q > 2n/(n - 1)$ .

In dimension  $n = 2$ , and for exponents  $q > 4$ , (9) implies that

$$\int |Ef(x)|^q w(x) dx \lesssim \inf_{v \in \mathbb{S}^1} \left( \sup_{T \in \mathbb{T}_v} w(T) \right) \|f\|_{L^2(\sigma)}^q \tag{10}$$

for all  $f \in L^2(\sigma)$ , as we shall see in the next subsection.

By Hölder’s inequality and the inequality (12) of the next subsection, (10) gives the following improvement on (1):

$$\int_{|x| \leq R} |Ef(x)|^2 w(x) dx \leq C_\epsilon R^\epsilon R^{1/2} \inf_{v \in \mathbb{S}^1} \left( \sup_{T \in \mathbb{T}_v} w(T) \right) \|f\|_{L^2(\sigma)}^2. \tag{11}$$

### 2.1 | Proof that (9) implies (10)

We may assume that  $\|w\|_{L^\infty} \leq 1$ . Let  $v \in \mathbb{S}^1$ . If  $B_R$  is a ball in  $\mathbb{R}^2$  of radius  $R \geq 1$ , then covering  $B_R$  by tubes  $T_1, \dots, T_N \in \mathbb{T}_v$  having disjoint interiors, we see that

$$\int_{B_R} w(x) dx = \sum_{l=1}^N \int_{B_R \cap T_l} w(x) dx \leq N \sup_{T_l} w(T_l) \lesssim \left( \sup_{T \in \mathbb{T}_v} w(T) \right) R.$$

Thus,  $w$  is a weight on  $\mathbb{R}^2$  of fractal dimension  $\alpha = 1$ , and

$$A_1(w) \lesssim \sup_{T \in \mathbb{T}_v} w(T).$$

Since this is true for every unit vector  $v$ , it follows that

$$A_1(w) \lesssim \inf_{v \in \mathbb{S}^1} \left( \sup_{T \in \mathbb{T}_v} w(T) \right). \tag{12}$$

Applying (9) with  $H = w$  and  $n = 2$ , we obtain (10).

<sup>†</sup> In [21, Theorem 2.1],  $S$  was assumed to have a strictly positive second fundamental form, but the third paragraph following the statement of [21, Theorem 2.1] explains that it suffices to assume that  $S$  has a nowhere vanishing Gaussian curvature when  $\alpha \leq n/2$ .

## 2.2 | Proof of (10) via Theorem 1.2

Since everything in this section is done under the assumption that  $S$  has a nowhere vanishing Gaussian curvature, we have the following decay estimate on the Fourier transform of the arc length measure  $\sigma$  on  $S$ :

$$|\hat{\sigma}(x)| \lesssim \frac{1}{|x|^{1/2}} \lesssim \frac{1}{|x \cdot \nu|^{1/2}}$$

for all  $x \in \mathbb{R}^2$  and  $\nu \in \mathbb{S}^1$ . Therefore, we can apply Theorem 1.2 with  $\delta = 1/2$  to get (6) for all  $q > 4$  and  $\nu \in \mathbb{S}^1$ , and (10) immediately follows.

## 2.3 | Strategy of [21]

Most of the recent progress on the restriction problem in harmonic analysis relied on Guth’s polynomial partitioning method introduced in [14], and subsequently used in several papers (see [13, 15–17, 19, 20, 25] and [22]) to obtain new results on the restriction problem in both the weighted and nonweighted settings.

Guth’s polynomial partitioning method upgrades restriction estimates from low algebraic dimensions to higher ones. The main innovation of [21] is a method for upgrading restriction estimates from low fractal dimensions to higher ones.

For example, due to the decay estimate  $|\hat{\sigma}(x)| \lesssim |x|^{-(n-1)/2}$ , (9) is easy to prove in low fractal dimensions (more precisely, in the regime  $0 < \alpha < (n - 1)/2$ ). Since we are interested in the estimate at dimension  $\alpha = n/2$ , it becomes crucial to have a mechanism that will allow us to upgrade restriction estimates from low fractal dimensions to higher ones.

This was done in [21] through a certain dimensional maximal function that was defined as

$$MF(\alpha) = \left( \sup_H \frac{1}{A_\alpha(H)} \int F(x)^\alpha H(x) dx \right)^{1/\alpha},$$

where  $H$  ranges over all weights on  $\mathbb{R}^n$  that obey the condition  $0 < A_\alpha(H) < \infty$ , and which was shown to satisfy the following inequality.

**Theorem 2-A** [21, Theorem 1.1]. *Suppose  $n \geq 1$  and  $0 < \beta < \alpha \leq n$ . Then*

$$MF(\alpha) \leq MF(\beta)$$

for all nonnegative Lebesgue measurable functions  $F$  on  $\mathbb{R}^n$ .

To prove Theorem 1.1 of this paper, we first define a suitable notion of fractal dimension that will allow us to use the decay estimate (2) to establish a low-dimensional version of (4). We then define a suitable dimensional maximal function and prove an inequality (as in Theorem 2-A) that will allow us to upgrade the low-dimensional version of (4) to (4) itself.

We follow the same strategy to prove Theorem 1.2, which also needs its own notion of fractal dimension and dimensional maximal function.

### 3 | PROOF OF THEOREM 1.1

#### 3.1 | The dimensional maximal function

Given a point  $x = (x_1, x_2) \in \mathbb{R}^2$  and numbers  $R_1, R_2 \geq 1$ , we define the set  $\mathbb{B}(x, R_1, R_2)$  to be the following subset of  $\mathbb{R}^4$ :

$$\{(y, z) = (y_1, y_2, z_1, z_2) \in \mathbb{R}^4 : |y_1 + z_1 - x_1| \leq R_1 \text{ and } |y_2 + z_2 - x_2| \leq R_2\}.$$

Suppose  $0 < \alpha \leq 2$  and  $H : \mathbb{R}^2 \rightarrow [0, 1]$  is a Lebesgue measurable function. We define  $\mathbb{A}_\alpha(H)$  to be the square root of

$$\inf \left\{ C : \int_{\mathbb{B}(x, R_1, R_2)} H(y)H(z)d(y, z) \leq C(R_1R_2)^\alpha \text{ for all } x \in \mathbb{R}^2 \text{ and } R_1, R_2 \geq 1 \right\},$$

where  $d(y, z)$  is Lebesgue measure on  $\mathbb{R}^4$ . (To understand the motivation behind the definition of  $\mathbb{A}_\alpha(H)$ , it helps to read (13) below.) We say that  $H$  is a *weight of fractal dimension*  $\alpha$  if  $\mathbb{A}_\alpha(H) < \infty$ . Same as with the definition of  $A_\alpha(H)$  in the previous section, here, we also have  $\mathbb{A}_\beta(H) \leq \mathbb{A}_\alpha(H)$  if  $\beta \geq \alpha$ , and the phrase “ $H$  is a weight of fractal dimension  $\alpha$ ” only means that  $\mathbb{A}_\alpha(H) < \infty$ .

Also, for Lebesgue measurable  $F : \mathbb{R}^2 \rightarrow [0, \infty)$ , we define

$$\mathbb{M}F(\alpha) = \left( \sup \frac{1}{\mathbb{A}_\alpha(H)} \int F(x)^\alpha H(x)dx \right)^{1/\alpha},$$

where the sup is taken over all  $H$  that obey the condition  $0 < \mathbb{A}_\alpha(H) < \infty$ .

The following theorem and its corollary were motivated by [21, Theorem 1.1] and [21, Corollary 2.1].

**Theorem 3.1.** *Suppose  $0 < \beta < \alpha \leq 2$ . Then*

$$\mathbb{M}F(\alpha) \leq \mathbb{M}F(\beta)$$

for all nonnegative Lebesgue measurable functions  $F$  on  $\mathbb{R}^2$ .

*Proof.* Let  $F$  be a nonnegative Lebesgue measurable function on  $\mathbb{R}^2$ . For  $N \in \mathbb{N}$ , we let  $\chi_N$  be the characteristic function of the set

$$B(0, N) \cap \{x \in \mathbb{R}^2 : F(x) \leq N\}$$

and  $F_N = N^{-1}\chi_N F$ . We also let  $I = \int F_N(x)H(x)dx$ . Since the tensor function  $F_N(y)F_N(z)$  on  $\mathbb{R}^4$  is bounded by 1 and supported in the set

$$\{(y, z) = (y_1, y_2, z_1, z_2) \in \mathbb{R}^4 : |y_1|, |y_2|, |z_1|, |z_2| \leq N\},$$

which is, in turn, contained in  $\mathbb{B}(0, 2N, 2N)$ , it follows by the Fubini–Tonelli theorem that

$$\begin{aligned}
 I^2 &= \int F_N(y)F_N(z)H(y)H(z)d(y, z) \\
 &\leq \int_{\mathbb{B}(0,2N,2N)} H(y)H(z)d(y, z) \\
 &\leq \mathbb{A}_\alpha(H)^2(2N)^{2\alpha},
 \end{aligned}
 \tag{13}$$

so that  $I \leq 2^\alpha N^\alpha \mathbb{A}_\alpha(H)$ . Letting  $\beta_0 = 1$  and  $C_0 = 2^\alpha N^\alpha$ , we get

$$\int F_N(x)^{\beta_0}H(x)dx \leq C_0\mathbb{A}_\alpha(H).
 \tag{14}$$

For  $x \in \mathbb{R}^2$  and  $R_1, R_2 \geq 1$ , (14) and Hölder’s inequality (applied with respect to the measure  $H(y)H(z)d(y, z)$ ) tell us that

$$\begin{aligned}
 &\int_{\mathbb{B}(x,R_1,R_2)} F_N(y)^{\beta_0/p}F_N(z)^{\beta_0/p}H(y)H(z)d(y, z) \\
 &\leq \left( \int_{\mathbb{B}(x,R_1,R_2)} F_N(y)^{\beta_0}F_N(z)^{\beta_0}H(y)H(z)d(y, z) \right)^{1/p} \\
 &\quad \times \left( \int_{\mathbb{B}(x,R_1,R_2)} H(y)H(z)d(y, z) \right)^{1/p'} \\
 &\leq \left( \int F_N(x)^{\beta_0}H(x)dx \right)^{2/p} (\mathbb{A}_\alpha(H)^2(R_1R_2)^\alpha)^{1/p'} \\
 &\leq C_0^{2/p} \mathbb{A}_\alpha(H)^{(2/p)+(2/p')}(R_1R_2)^\alpha/p' \\
 &= C_0^{2/p} \mathbb{A}_\alpha(H)^2(R_1R_2)^\alpha/p',
 \end{aligned}$$

whenever  $p, p' > 1$  are conjugate exponents. Choosing  $p = \alpha/(\alpha - \beta)$ , so that  $\alpha/p' = \beta$ , we conclude that the function  $\mathcal{H} : \mathbb{R}^2 \rightarrow [0, 1]$  defined by  $\mathcal{H} = F_N^{\beta_0/p}H$  is a weight on  $\mathbb{R}^2$  of fractal dimension  $\beta$  with

$$\mathbb{A}_\beta(\mathcal{H}) \leq C_0^{1/p} \mathbb{A}_\alpha(H).$$

Therefore,

$$\begin{aligned}
 \int F_N(x)^\beta F_N(x)^{\beta_0/p}H(x)dx &= \int F_N(x)^\beta \mathcal{H}(x)dx \\
 &\leq (\mathbb{M}F_N(\beta))^\beta \mathbb{A}_\beta(\mathcal{H}) \\
 &\leq N^{-\beta}(\mathbb{M}F(\beta))^\beta C_0^{1/p} \mathbb{A}_\alpha(H),
 \end{aligned}$$

where we have used the fact that  $F_N \leq N^{-1}F$  to conclude that  $\mathbb{M}F_N(\beta) \leq N^{-1}\mathbb{M}F(\beta)$ . Letting  $M = N^{-\beta}(\mathbb{M}F(\beta))^\beta$ ,  $\beta_1 = \beta + (\beta_0/p)$ , and  $C_1 = MC_0^{1/p}$ , we therefore have

$$\int F_N(x)^{\beta_1}H(x)dx \leq C_1\mathbb{A}_\alpha(H). \tag{15}$$

Now, letting  $\beta_2 = \beta + (\beta_1/p)$  and  $C_2 = MC_1^{1/p}$ , and repeating the same procedure starting with (15) instead of (14), we arrive at

$$\int F_N(x)^{\beta_2}H(x)dx \leq C_2\mathbb{A}_\alpha(H).$$

Therefore, proceeding in this fashion and using mathematical induction, we obtain sequences  $\{\beta_k\}$  and  $\{C_k\}$ ,  $k \geq 1$ , so that  $\beta_k = \beta + (\beta_{k-1}/p)$  and  $C_k = MC_{k-1}^{1/p}$  and

$$\int F_N(x)^{\beta_k}H(x)dx \leq C_k\mathbb{A}_\alpha(H). \tag{16}$$

A simple calculation (see [21, §5] for more details) reveals that

$$\beta_k = \beta \frac{1 - (1/p)^k}{1 - (1/p)} + \frac{\beta_0}{p^k} \quad \text{and} \quad C_k = M^{(1-(1/p)^k)/(1-(1/p))}C_0^{(1/p)^k},$$

so that

$$\lim_{k \rightarrow \infty} \beta_k = \frac{\beta}{1 - (1/p)} = \alpha \quad \text{and} \quad \lim_{k \rightarrow \infty} C_k = M^{\alpha/\beta} = N^{-\alpha}(\mathbb{M}F(\beta))^\alpha$$

(recall that  $p = \alpha/(\alpha - \beta)$ ).

Therefore, letting  $k \rightarrow \infty$  in (16) and using Fatou’s lemma, we see that

$$\int F_N(x)^\alpha H(x)dx \leq N^{-\alpha}(\mathbb{M}F(\beta))^\alpha.$$

Recalling that  $F_N = N^{-1}\chi_N F$ , this becomes

$$\int \chi_N(x)F(x)^\alpha H(x)dx \leq (\mathbb{M}F(\beta))^\alpha.$$

Therefore, letting  $N \rightarrow \infty$ , we see that

$$\int F(x)^\alpha H(x)dx \leq (\mathbb{M}F(\beta))^\alpha.$$

Since this is true for all weights  $H$  of dimension  $\alpha$ , we arrive at our desired inequality  $\mathbb{M}F(\alpha) \leq \mathbb{M}F(\beta)$ . □

Returning to Fourier restriction, for  $0 < \alpha \leq 2$  and  $1 \leq p \leq \infty$ , we define  $Q(\alpha, p)$  to be the infimum of all exponents  $q > 0$  so that the following holds: there is a constant  $C$  such

$$\int |Ef(x)|^q H(x) dx \leq C A_\alpha(H) \|f\|_{L^p(\sigma)}^q$$

for all functions  $f \in L^p(\sigma)$  and weights  $H$  of fractal dimension  $\alpha$ .

**Corollary 3.1.** *Suppose  $0 < \beta < \alpha \leq 2$ . Then*

$$\frac{Q(\alpha, p)}{\alpha} \leq \frac{Q(\beta, p)}{\beta}.$$

*Proof.* The proof of this corollary is the same as the proof of [21, Corollary 2.1]. We include it here for the reader's convenience.

By the definition of  $Q(\beta, p)$ , for every  $q > Q(\beta, p)$ , there is a constant  $C_q$  such that

$$\int |Ef(x)|^q \mathcal{H}(x) dx \leq C_q A_\beta(\mathcal{H}) \|f\|_{L^p(\sigma)}^q$$

for all functions  $f \in L^p(\sigma)$  and weights  $\mathcal{H}$  of fractal dimension  $\beta$ . Letting  $F = |Ef|^{q/\beta}$ , we see that

$$\mathbb{M}F(\beta) \leq \left( C_q \|f\|_{L^p(\sigma)}^q \right)^{1/\beta}.$$

Applying Theorem 3.1, we get

$$\mathbb{M}F(\alpha) \leq \left( C_q \|f\|_{L^p(\sigma)}^q \right)^{1/\beta}.$$

Therefore,

$$\left( \frac{1}{A_\alpha(H)} \int |Ef(x)|^{(\alpha/\beta)q} H(x) dx \right)^{1/\alpha} \leq \left( C_q \|f\|_{L^p(\sigma)}^q \right)^{1/\beta}$$

for all functions  $f \in L^p(\sigma)$  and weights  $H$  of fractal dimension  $\alpha$ .

The definition of  $Q(\alpha, p)$  now tells us that  $(\alpha/\beta)q \geq Q(\alpha, p)$ . Therefore,

$$q \geq \frac{\beta}{\alpha} Q(\alpha, p)$$

for every  $q > Q(\beta, p)$ . Therefore,  $Q(\beta, p) \geq (\beta/\alpha)Q(\alpha, p)$ . □

### 3.2 | The low-dimensional estimate

The availability of favorable restriction estimates in low dimensions hangs on the fact that the Fourier transform of  $\sigma$  satisfies the decay condition (2), as we shall see during the proof of the following proposition.

**Proposition 3.1.** *Suppose  $0 < \alpha < \delta$ . Then, for  $q > 1$ , we have*

$$\int |Ef(x)|^q H(x) dx \lesssim \mathbb{A}_\alpha(H) \|f\|_{L^2(\sigma)}^q$$

for all functions  $f \in L^2(\sigma)$  and weights  $H$  of fractal dimension  $\alpha$ .

*Proof.* We may assume that  $H \in L^1(\mathbb{R}^2)$ . (Otherwise, we work with the weight  $\chi_{B(0,R)}H$ , obtain an estimate that is uniform in  $R$ , and then send  $R$  to infinity using the fact that  $\mathbb{A}_\alpha(\chi_{B(0,R)}H) \leq \mathbb{A}_\alpha(H)$ .) We define the measure  $\mu$  on  $\mathbb{R}^2$  by  $d\mu = Hdx$ , and observe that by the definition of  $\mathbb{A}_\alpha(H)$ , we have

$$(\mu \times \mu)(\mathbb{B}(x_0, R_1, R_2)) \leq \mathbb{A}_\alpha(H)^2 (R_1 R_2)^\alpha \tag{17}$$

for all  $x_0 \in \mathbb{R}^2$  and  $R_1, R_2 \geq 1$ .

Let  $f \in L^2(\sigma)$ . We need to estimate  $\|Ef\|_{L^q(\mu)}$ . We have

$$\|Ef\|_{L^q(\mu)}^q = \int_0^{\|f\|_{L^1(\sigma)}} q\lambda^{q-1} \mu(\{|Ef| \geq \lambda\}) d\lambda. \tag{18}$$

Also, the set  $\{|Ef| \geq \lambda\}$  is contained in

$$\left\{ (\operatorname{Re} Ef)_+ \geq \frac{\lambda}{4} \right\} \cup \left\{ (\operatorname{Re} Ef)_- \geq \frac{\lambda}{4} \right\} \cup \left\{ (\operatorname{Im} Ef)_+ \geq \frac{\lambda}{4} \right\} \cup \left\{ (\operatorname{Im} Ef)_- \geq \frac{\lambda}{4} \right\},$$

where  $(\operatorname{Re} Ef)_+$  and  $(\operatorname{Re} Ef)_-$  are, respectively, the positive and negative parts of  $\operatorname{Re} Ef$ ; and similarly for  $\operatorname{Im} Ef$ . So, it is enough to estimate the  $\mu$ -measure of the set  $\{(\operatorname{Re} Ef)_+ \geq \lambda/4\}$ , which we denote by  $G$ .

Since  $\lambda > 0$ , the set where  $(\operatorname{Re} Ef)_+ \geq \lambda/4$  is the same as the set where  $\operatorname{Re} Ef \geq \lambda/4$ , so

$$\frac{\lambda}{4} \mu(G) \leq \int_G (\operatorname{Re} Ef) d\mu = \operatorname{Re} \int_G Ef d\mu = \operatorname{Re} \int \chi_G \widehat{f d\sigma} d\mu = \operatorname{Re} \int \widehat{\chi_G d\mu} f d\sigma,$$

and so (by Cauchy-Schwarz),

$$\lambda^2 \mu(G)^2 \leq 16 \|f\|_{L^2(\sigma)}^2 \|\widehat{\chi_G d\mu}\|_{L^2(\sigma)}^2.$$

Now

$$\|\widehat{\chi_G d\mu}\|_{L^2(\sigma)}^2 = \int \widehat{\chi_G d\mu} \overline{\widehat{\chi_G d\mu}} d\sigma = \int (\overline{\widehat{\chi_G d\mu}} d\sigma) \widehat{\chi_G d\mu} = \int (\widehat{\sigma} * (\chi_G d\mu)) \chi_G d\mu,$$

so

$$\lambda^2 \mu(G)^2 \leq 16 \|f\|_{L^2(\sigma)}^2 \int (\widehat{\sigma} * (\chi_G d\mu)) \chi_G d\mu. \tag{19}$$

Let  $\psi_0$  be an even  $C_0^\infty$  function on  $\mathbb{R}$  satisfying  $0 \leq \psi_0 \leq 1$ ,  $\psi_0 = 1$  on the interval  $[-1, 1]$ , and  $\psi_0 = 0$  outside the interval  $(-2, 2)$ . Also, for  $l \in \mathbb{N}$ , define  $\psi_l(r) = \psi_0(r/2^l) - \psi_0(r/2^{l-1})$ . Then  $\psi_l$  is supported in the set  $2^{l-1} \leq |r| \leq 2^{l+1}$  for  $l \geq 1$ , and  $\sum_{l=0}^\infty \psi_l = 1$  on  $\mathbb{R}$ .

For  $x = (x_1, x_2) \in \mathbb{R}^2$ , define  $\Psi_{l,m}(x) = \psi_l(x_1)\psi_m(x_2)$ . Then

$$\hat{\sigma} * (\chi_G d\mu) = \sum_{l=0}^\infty \sum_{m=0}^\infty (\Psi_{l,m} \hat{\sigma}) * (\chi_G d\mu).$$

Since  $\Psi_{0,0}$  is supported in  $[-2, 2] \times [-2, 2]$ , and  $\Psi_{l,0}$  is supported in

$$\{(x_1, x_2) \in \mathbb{R}^2 : 2^{l-1} \leq |x_1| \leq 2^{l+1} \text{ and } |x_2| \leq 2\}$$

for  $l \in \mathbb{N}$ , and  $\Psi_{0,m}$  is supported in

$$\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 2 \text{ and } 2^{m-1} \leq |x_2| \leq 2^{m+1}\}$$

for  $m \in \mathbb{N}$ , and  $\Psi_{l,m}$  is supported in the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : 2^{l-1} \leq |x_1| \leq 2^{l+1} \text{ and } 2^{m-1} \leq |x_2| \leq 2^{m+1}\}$$

for  $(l, m) \in \mathbb{N}^2$ , it follows from (2) that  $|\Psi_{l,m} \hat{\sigma}| \lesssim 2^{-(l+m)\delta}$  for all  $(l, m) \in \mathbb{N}^2$ . Of course, we also have  $\|\hat{\sigma}\|_{L^\infty} \lesssim 1$ , so

$$|\Psi_{l,m} \hat{\sigma}| \lesssim 2^{-(l+m)\delta} \quad \text{for all } l, m \geq 0.$$

Therefore,

$$\begin{aligned} |(\Psi_{l,m} \hat{\sigma}) * (\chi_G d\mu)(x)| &\leq \int |\Psi_{l,m}(x - y) \hat{\sigma}(x - y)| \chi_G(y) d\mu(y) \\ &\lesssim 2^{-(l+m)\delta} \int_{|x_1 - y_1| \leq 2^{l+1}, |x_2 - y_2| \leq 2^{m+1}} \chi_G(y) d\mu(y), \end{aligned}$$

and so,

$$\begin{aligned} &\int |(\Psi_{l,m} \hat{\sigma}) * (\chi_G d\mu)(x)| \chi_G(x) d\mu(x) \\ &\lesssim 2^{-(l+m)\delta} \int_{\mathbb{B}(0, 2^{l+1}, 2^{m+1})} \chi_G(x) \chi_G(y) d\mu(x) d\mu(y) \\ &\leq 2^{-(l+m)\delta} (\mu \times \mu)(\mathbb{B}(0, 2^{l+1}, 2^{m+1})) \end{aligned}$$

for all  $l, m \geq 0$ . Invoking (17), this becomes

$$\int |(\Psi_{l,m} \hat{\sigma}) * (\chi_G d\mu)| \chi_G d\mu \lesssim \mathbb{A}_\alpha(H)^2 2^{-(l+m)(\delta-\alpha)}.$$

Therefore,

$$\int (\hat{\sigma} * (\chi_G d\mu)) \chi_G d\mu \lesssim A_\alpha(H)^2 \sum_{l=0}^\infty \sum_{m=0}^\infty 2^{-(l+m)(\delta-\alpha)} \lesssim A_\alpha(H)^2,$$

where we have used the assumption that  $0 < \alpha < \delta$ .

Returning to (19), we now have  $\lambda^2 \mu(G)^2 \lesssim \|f\|_{L^2(\sigma)}^2 A_\alpha(H)^2$ , so that

$$\mu(G) \lesssim \|f\|_{L^2(\sigma)} A_\alpha(H) \lambda^{-1}.$$

Therefore, by (18),

$$\|Ef\|_{L^q(\mu)}^q \lesssim A_\alpha(H) \|f\|_{L^2(\sigma)} \int_0^{\|f\|_{L^1(\sigma)}} \lambda^{q-2} d\lambda \lesssim A_\alpha(H) \|f\|_{L^2(\sigma)}^q$$

provided  $q > 1$ . □

### 3.3 | The Mizohata–Takeuchi estimate (4)

In the language of Corollary 3.1, Proposition 3.1 says that  $Q(\beta, 2) \leq 1$  for all  $0 < \beta < \delta$ . So, applying Corollary 3.1 with  $\alpha = 1$  and recalling that  $\delta \leq 1$ , we get

$$\frac{Q(1, 2)}{1} \leq \frac{Q(\beta, 2)}{\beta} \leq \frac{1}{\beta}$$

for all  $0 < \beta < \delta$ . Therefore,  $Q(1, 2) \leq 1/\delta$ . Therefore,

$$\int |Ef(x)|^q H(x) dx \lesssim A_1(H) \|f\|_{L^2(\sigma)}^q \tag{20}$$

whenever  $q > 1/\delta$ ,  $H$  is a weight of fractal dimension 1, and  $f \in L^2(\sigma)$ .

We are now ready to return to our Mizohata–Takeuchi estimate. We let  $w$  be a nonnegative function on  $\mathbb{R}^2$  and apply (20) with  $H_w = \|w\|_{L^\infty}^{-1} w$  to get

$$\int |Ef(x)|^q w(x) dx \lesssim \|w\|_{L^\infty} A_1(H_w) \|f\|_{L^2(\sigma)}^q, \tag{21}$$

and it remains to estimate  $A_1(H_w)$ . We will be able to do this only when  $w$  is of the form (3):

$$w(x) = w(x_1, x_2) = \tilde{w}(ax_1) \tilde{w}(bx_2)$$

with  $\tilde{w} : \mathbb{R} \rightarrow [0, \infty)$  and  $a, b > 0$ .

Recall that, for  $m \in \mathbb{R}$ ,  $\mathbb{T}_m$  was defined as follows: a 1-tube in  $\mathbb{R}^2$  belongs to  $\mathbb{T}_m$  if its core line is parallel to the line  $\{x = (x_1, x_2) \in \mathbb{R}^2 : mx_1 + x_2 = 0\}$  or the line  $\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + mx_2 = 0\}$ .

For  $x \in \mathbb{R}^2$  and  $R_1, R_2 \geq 1$ , we have

$$\int_{\mathbb{B}(x, R_1, R_2)} H_w(y)H_w(z)d(y, z) = \|w\|_{L^\infty}^{-2}J(a)J(b),$$

where

$$J(a) = \int_{|y_1+z_1-x_1| \leq R_1} \tilde{w}(ay_1)\tilde{w}(az_1)d(y_1, z_1)$$

and

$$J(b) = \int_{|y_2+z_2-x_2| \leq R_2} \tilde{w}(by_2)\tilde{w}(bz_2)d(y_2, z_2).$$

Since both integrals  $J(a)$  and  $J(b)$  are the same, it is enough to bound one of them.

We write

$$J(b) = \int_{-\infty}^{\infty} \tilde{w}(bz_2)I(z_2)dz_2 \quad \text{with} \quad I(z_2) = \int_{x_2-z_2-R_2}^{x_2-z_2+R_2} \tilde{w}(by_2)dy_2.$$

Applying the change of variables  $v = (b/a)y_2$ , the second integral becomes

$$I(z_2) = \frac{a}{b} \int_{(b/a)(x_2-z_2-R_2)}^{(b/a)(x_2-z_2+R_2)} \tilde{w}(av)dv.$$

Plugging this back in  $J(b)$ , we see that

$$J(b) = \frac{a}{b} \int_{\tilde{T}} \tilde{w}(av)\tilde{w}(bz_2)d(v, z_2) = \frac{a}{b} \int_{\tilde{T}} w(v, z_2)d(v, z_2)$$

where  $\tilde{T}$  is the tube

$$\{(v, z_2) \in \mathbb{R}^2 : -(a/b)v + x_2 - R_2 \leq z_2 \leq -(a/b)v + x_2 + R_2\}.$$

Since  $\tilde{T}$  has cross-section  $2bR_2/\sqrt{a^2 + b^2}$ , it follows that

$$J(b) \leq \frac{a}{b} \frac{2bR_2}{\sqrt{a^2 + b^2}} \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right) = \frac{2aR_2}{\sqrt{a^2 + b^2}} \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right).$$

Therefore,

$$J(a)J(b) \leq \frac{2bR_1}{\sqrt{a^2 + b^2}} \frac{2aR_2}{\sqrt{a^2 + b^2}} \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right)^2 \leq 2R_1R_2 \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right)^2.$$

We have proved that

$$\int_{\mathbb{B}(x,R_1,R_2)} H_w(y)H_w(z)d(y,z) \leq 2\|w\|_{L^\infty}^{-2} \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right)^2 R_1 R_2$$

for all points  $x \in \mathbb{R}^2$  and numbers  $R_1, R_2 \geq 1$ . By the definition of the functional  $A_\alpha$ , this implies that

$$A_1(H_w) \leq \sqrt{2} \|w\|_{L^\infty}^{-1} \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right).$$

Inserting this back in (21), we get

$$\int |Ef(x)|^q w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_{a/b}} w(T) \right) \|f\|_{L^2(\sigma)}^q$$

for  $q > \delta^{-1}$ , as promised.

## 4 | PROOF OF THEOREM 1.2

### 4.1 | The dimensional maximal function

Let  $v$  be the unit vector given in the statement of Theorem 1.2. We start by introducing some notation. For  $x \in \mathbb{R}^2$  and  $r > 0$ , we let  $T(x, r)$  be the tube of cross-section  $2r$  that contains the ball  $B(x, r)$  and is parallel to  $v$  (more precisely, the core line of  $T(x, r)$  is parallel to the vector  $v$ ).

Suppose  $0 < \alpha \leq 2$  and  $H : \mathbb{R}^2 \rightarrow [0, 1]$  is a Lebesgue measurable function. We define

$$A_\alpha(H) = \inf \left\{ C : \int_{T(x_0,R)} H(x) dx \leq CR^\alpha \text{ for all } x_0 \in \mathbb{R}^2 \text{ and } R \geq 1 \right\}.$$

We say  $H$  is a *weight of fractal dimension  $\alpha$*  if  $A_\alpha(H) < \infty$ . Also, for Lebesgue measurable  $F : \mathbb{R}^2 \rightarrow [0, \infty)$ , we define

$$\mathcal{M}F(\alpha) = \left( \sup \frac{1}{A_\alpha(H)} \int F(x)^\alpha H(x) dx \right)^{1/\alpha},$$

where the sup is taken over all  $H$  that obey the condition  $0 < A_\alpha(H) < \infty$ .

**Theorem 4.1.** *Suppose  $0 < \beta < \alpha \leq 2$ . Then*

$$\mathcal{M}F(\alpha) \leq \mathcal{M}F(\beta)$$

for all nonnegative Lebesgue measurable functions  $F$  on  $\mathbb{R}^2$ .

The proof of Theorem 4.1 is much closer to the proof of [21, Theorem 1.1] than the proof of Theorem 3.1 was, so we omit the details.

For  $0 < \alpha \leq 2$  and  $1 \leq p \leq \infty$ , we define  $Q(\alpha, p)$  to be the infimum of all numbers  $q > 0$  so that the following holds: there is a constant  $C$  such

$$\int |Ef(x)|^q H(x) dx \leq C A_\alpha(H) \|f\|_{L^p(\sigma)}^q$$

for all functions  $f \in L^p(\sigma)$  and weights  $H$  of fractal dimension  $\alpha$ . Theorem 4.1 has the following corollary whose proof is the same as that of Corollary 3.1 and [21, Corollary 2.1].

**Corollary 4.1.** *Suppose  $0 < \beta < \alpha \leq 2$ . Then*

$$\frac{Q(\alpha, p)}{\alpha} \leq \frac{Q(\beta, p)}{\beta}.$$

### 4.2 | The low-dimensional estimate

In this subsection, we prove the following result, which, like Proposition 3.1, was motivated by [21, Proposition 6.1].

**Proposition 4.1.** *Suppose  $0 < \alpha < \delta$ . Then, for  $q > 2$ , we have*

$$\int |Ef(x)|^q H(x) dx \lesssim A_\alpha(H) \|f\|_{L^2(\sigma)}^q$$

for all functions  $f \in L^2(\sigma)$  and weights  $H$  of fractal dimension  $\alpha$ .

*Proof.* We begin, as in the proof of Proposition 3.1, by assuming that  $H \in L^1(\mathbb{R}^2)$ , defining the measure  $\mu$  on  $\mathbb{R}^2$  by  $d\mu = H dx$ , and observing that

$$\mu(T(x_0, R)) \leq A_\alpha(H) R^\alpha \tag{22}$$

for all  $x_0 \in \mathbb{R}^2$  and  $R \geq 1$ .

Next, for  $f \in L^2(\sigma)$ , we write

$$\|Ef\|_{L^q(\mu)}^q = \int_0^{\|f\|_{L^1(\sigma)}} q \lambda^{q-1} \mu(\{|Ef| \geq \lambda\}) d\lambda, \tag{23}$$

and observe that we will be done as soon as we derive a suitable bound on the  $\mu$  measure of the set where  $\text{Re } Ef \geq \lambda/4$ , which we again denote by the letter  $G$ .

As in the proof of Proposition 3.1, we have

$$\lambda^2 \mu(G)^2 \leq 16 \|f\|_{L^2(\sigma)}^2 \int (\hat{\sigma} * (\chi_G d\mu)) \chi_G d\mu. \tag{24}$$

However, instead of showing that

$$\int (\hat{\sigma} * (\chi_G d\mu)) \chi_G d\mu \lesssim \mathcal{A}_\alpha(H)^2$$

as we did in Proposition 3.1, here, we will only be able to show that

$$|\hat{\sigma} * (\chi_G d\mu)(x)| \lesssim \mathcal{A}_\alpha(H) \tag{25}$$

for all  $x \in \mathbb{R}^2$ .

We let  $\psi_0$  be an even  $C_0^\infty$  function on  $\mathbb{R}$  satisfying  $0 \leq \psi_0 \leq 1$ ,  $\psi_0 = 1$  on the interval  $[-1, 1]$ , and  $\psi_0 = 0$  outside the interval  $(-2, 2)$ . For  $l \in \mathbb{N}$ , we then define  $\psi_l(r) = \psi_0(r/2^l) - \psi_0(r/2^{l-1})$ . The function  $\psi_l$  is supported in the set  $2^{l-1} \leq |r| \leq 2^{l+1}$  for  $l \geq 1$ , and  $\sum_{l=0}^\infty \psi_l = 1$  everywhere on  $\mathbb{R}$ .

For  $x = (x_1, x_2) \in \mathbb{R}^2$ , we also define  $\Psi_l(x) = \psi_l(x \cdot v)$ . Then

$$\hat{\sigma} * (\chi_G d\mu) = \sum_{l=0}^\infty (\Psi_l \hat{\sigma}) * (\chi_G d\mu).$$

Since  $\Psi_0$  is supported in the tube  $|x \cdot v| \leq 2$ , and  $\Psi_l$  is supported in the tube  $2^{l-1} \leq |x \cdot v| \leq 2^{l+1}$  for  $l \in \mathbb{N}$ , it follows from (5) that  $|\Psi_l \hat{\sigma}| \lesssim 2^{-l\delta}$  for all  $l \in \mathbb{N}$ . Since  $\|\hat{\sigma}\|_{L^\infty} \lesssim 1$ , this is also true for  $l = 0$ . So,

$$|\Psi_l \hat{\sigma}| \lesssim 2^{-l\delta} \quad \text{for all } l \geq 0,$$

and so

$$\begin{aligned} |(\Psi_l \hat{\sigma}) * (\chi_G d\mu)(x)| &\leq \int |\Psi_l(x - y) \hat{\sigma}(x - y)| \chi_G(y) d\mu(y) \\ &\lesssim 2^{-l\delta} \int_{|(x-y) \cdot v| \leq 2^{l+1}} \chi_G(y) d\mu(y) \\ &\leq 2^{-l\delta} \mu(T(x, 2^{l+1})). \end{aligned}$$

Invoking (22), this becomes

$$|(\Psi_l \hat{\sigma}) * (\chi_G d\mu)(x)| \lesssim \mathcal{A}_\alpha(H) 2^{-l(\delta-\alpha)}$$

for all  $x \in \mathbb{R}^2$  and  $l \geq 0$ . Summing over all  $l$  using the assumption  $\alpha < \delta$ , this proves (25).

Inserting the bound (25) in (24), we get

$$\lambda^2 \mu(G)^2 \lesssim \|f\|_{L^2(\sigma)}^2 \int \mathcal{A}_\alpha(H) \chi_G d\mu = \|f\|_{L^2(\sigma)}^2 \mathcal{A}_\alpha(H) \mu(G),$$

which gives

$$\mu(G) \lesssim \mathcal{A}_\alpha(H) \|f\|_{L^2(\sigma)}^2 \lambda^{-2}.$$

Therefore, by (23),

$$\|Ef\|_{L^q(\mu)}^q \lesssim \mathcal{A}_\alpha(H)\|f\|_{L^2(\sigma)}^2 \int_0^{\|f\|_{L^1(\sigma)}} \lambda^{q-3} d\lambda \lesssim \mathcal{A}_\alpha(H)\|f\|_{L^2(\sigma)}^q$$

provided  $q > 2$ . □

### 4.3 | The Mizohata–Takeuchi estimate (6)

In the language of Corollary 4.1, Proposition 4.1 says that  $Q(\beta, 2) \leq 2$  for all  $0 < \beta < \delta$ . So, applying Corollary 4.1 with  $\alpha = 1$  and recalling that  $\delta \leq 1$ , we get

$$\frac{Q(1, 2)}{1} \leq \frac{Q(\beta, 2)}{\beta} \leq \frac{2}{\beta}$$

for all  $0 < \beta < \delta$ . Therefore,  $Q(1, 2) \leq 2/\delta$ , which means that

$$\int |Ef(x)|^q H(x) dx \lesssim \mathcal{A}_1(H)\|f\|_{L^2(\sigma)}^q \tag{26}$$

whenever  $q > 2/\delta$ ,  $H$  is a weight of fractal dimension one, and  $f \in L^2(\sigma)$ .

If the function  $w$  is as given in the statement of Theorem 1.2, then it is easy to see that

$$\int_{T(x_0, R)} w(x) dx \lesssim \left( \sup_{T \in \mathbb{T}_v} w(T) \right) R$$

for all  $x_0 \in \mathbb{R}^2$  and  $R \geq 1$ . Thus,

$$\mathcal{A}_1(\|w\|_{L^\infty}^{-1} w) \lesssim \|w\|_{L^\infty}^{-1} \left( \sup_{T \in \mathbb{T}_v} w(T) \right).$$

Inserting this bound back in (26) (with  $H$  replaced by  $\|w\|_{L^\infty}^{-1} w$ ), we arrive at our Mizohata–Takeuchi estimate (6).

### JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

### REFERENCES

1. J. A. Barceló, J. M. Bennett, and A. Carbery, *A note on localised weighted inequalities for the extension operator*, *J. Aust. Math. Soc.* **84** (2008), 289–299.
2. J. A. Barceló, J. M. Bennett, A. Carbery, A. Ruiz, and M. C. Vilela, *Some special solutions of the Schrödinger equation*, *Indiana Univ. Math. J.* **56** (2007), 1581–1593.

3. J. A. Barceló, A. Ruiz, and L. Vega, *Weighted estimates for the Helmholtz equation and some applications*, J. Funct. Anal. **150** (1997), 356–382.
4. J. Bennett, A. Carbery, F. Soria, and A. Vargas, *A Stein conjecture for the circle*, Math. Ann. **336** (2006), 671–695.
5. J. Bennett and S. Nakamura, *Tomography bounds for the Fourier extension operator and applications*, Math. Ann. **380** (2021), 119–159.
6. A. Carbery, *Large sets with limited tube occupancy*, J. Lond. Math. Soc. (2) **79** (2009), 529–543.
7. A. Carbery, T. S. Hänninen, and S. I. Valdimarsson, *Disentanglement, multilinear duality and factorisation for non-positive operators*, Anal. PDE **16** (2023), 511–543.
8. A. Carbery, M. Iliopoulou, and H. Wang, *Some sharp inequalities of Mizohata–Takeuchi-type*, Preprint, arXiv:2302.11877.
9. A. Carbery, E. Romera, and F. Soria, *Radial weights and mixed norm inequalities for the disc multiplier*, J. Funct. Anal. **109** (1992), 52–75.
10. A. Carbery and F. Soria, *Pointwise Fourier inversion and localisation in  $\mathbb{R}^n$* , J. Fourier Anal. Appl. **3** (1997), 847–858.
11. A. Carbery, F. Soria, and A. Vargas, *Localisation and weighted inequalities for spherical Fourier means*, J. Anal. Math. **103** (2007), 133–156.
12. A. Carbery and S. Ziesler, *Restriction and decay for flat hypersurfaces*, Publ. Mat. **46** (2002), 405–434.
13. X. Du, L. Guth, Y. Ou, H. Wang, B. Wilson, and R. Zhang, *Weighted restriction estimates and application to Falconer distance set problem*, Amer. J. Math. **143** (2021), 175–211.
14. L. Guth, *A restriction estimate using polynomial partitioning*, J. Amer. Math. Soc. **29** (2016), 371–413.
15. L. Guth, *Restriction estimates using polynomial partitioning II*, Acta Math. **221** (2018), 81–142.
16. L. Guth, J. Hickman, and M. Iliopoulou, *Sharp estimates for oscillatory integral operators via polynomial partitioning*, Acta Math. **223** (2019), 251–376.
17. J. Hickman and K. Rogers, *Improved Fourier restriction estimates in higher dimensions*, Camb. J. Math. **7** (2019), 219–282.
18. S. Mizohata, *On the Cauchy problem*, Notes and Reports in Mathematics in Science and Engineering, vol. 3, Academic Press, Inc., Orlando, FL; Science Press, Beijing, 1985.
19. Y. Ou and H. Wang, *A cone restriction estimate using polynomial partitioning*, J. Eur. Math. Soc. **24** (2022), 3557–3595.
20. B. Shayya, *Weighted restriction estimates using polynomial partitioning*, Proc. Lond. Math. Soc. (3) **115** (2017), 545–598.
21. B. Shayya, *Fourier restriction in low fractal dimensions*, Proc. Edinb. Math. Soc. **64** (2021), 373–407.
22. B. Shayya, *Improved weighted restriction estimates in  $\mathbb{R}^3$* , Preprint, arXiv:2206.06325.
23. J. Takeuchi, *A necessary condition for the well-posedness of the Cauchy problem for a certain class of evolution equations*, Proc. Japan Acad. **50** (1974), 133–137.
24. J. Takeuchi, *Some remarks on my paper: “On the Cauchy problem for some non-Kowalewskian equations with distinct characteristic roots”*, J. Math. Kyoto Univ. **20** (1980), 105–124, J. Math. Kyoto Univ. **24** (1984), 741–754.
25. H. Wang, *A restriction estimate in  $\mathbb{R}^3$  using brooms*, Duke Math. J. **171** (2022), 1749–1822.