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Minimum Separators and Menger's Theorem

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Abstract

Menger's theorem implies simply the following property: A minimum separator S of non-adjacent vertices u and v in a graph G remains a minimum uv-separator in $G-e$, for every $e \in E(G[S])$. In this paper, we prove the equivalence between Menger's theorem and this property and so by given an elementary proof of it, we get a new proof of Menger's theorem.

Keywords Menger's theorem · Minimum separators · Internally disjoint paths · Connectivity

Mathematics Subject Classification 97K30

1 Introduction

All the graphs considered here are simple and finite. The subgraph of a graph G induced by a subset $A \subseteq V(G)$ will be denoted by $G[A]$. For two non-adjacent vertices u and v in a graph G, a subset S of $V(G) - \{u, v\}$ is said to be a uv-**separator** of G if u and v lie in two different connected components in $G - S$. That is, every uv-path in G contains at least one vertex from S. The minimal order of a uv-separator of G is called the uv-**connectivity** of G and is denoted by $\kappa_G(u, v)$, then a **minimum** uv-**separator** of G is of order $\kappa_G(u, v)$. A set of uv-paths is called **internally disjoint** if these paths are pairwise disjoint except for the vertices u and v. The maximal number of internally disjoint uv-paths in G is denoted by $\mu_G(u, v)$.

Menger's theorem is a classical result in graph theory that relates the connectivity of a graph to the existence of disjoint paths between pairs of vertices. It states that the minimum size of a set of vertices that separates two distinct non-adjacent vertices u

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and v in a graph G is equal to the maximum number of pairwise disjoint uv-paths in G; i.e. $\kappa_G(u, v) = \mu_G(u, v)$.

After Menger proved his theorem [8], it was formulated and generalized by many ways such as by the Max-flow Min-cut theorem in the field of network flows [4]. For shorter proofs, the first one was given by Dirac [2]. The second one is due to O'Neil [9] who took a different perspective. Then in [7] McCuaig gave a proof by induction based on some new ideas. Also, many other short proofs were presented by Göring et al. [1, 5, 6].

In this paper, we show the equivalence between Menger's theorem and the following crucial property concerning minimum separators: *A minimum separator S of non-adjacent vertices u and v in a graph G remains a minimum uv-separator in $G - e$, for every $e \in E(G[S])$, and so $\kappa_{G-e}(u, v) = \kappa_G(u, v)$.* Call this property the **minimum separator property**. The importance of this equivalence is displayed by giving an elementary proof of the minimum separator property which gives a new proof of Menger's theorem.

2 Minimum Separators and Menger's Theorem

It should be noted that the minimum separator property can be deduced from Menger's theorem by noticing that deleting any edge e joining two vertices in a minimum separator S of two non-adjacent vertices u and v in a graph G does not affect a set of internally disjoint uv-paths of order $\kappa_G(u, v)$ provided by Menger's theorem. The other sense of this equivalence proved in this section, together with the elementary proof of the minimum separator property will represent another new proof of Menger's theorem. This underscores the importance of this classic theorem, as there exists a multitude of proofs, each relying on different features of connected graphs and generalized over several decades.

The following inequalities about connectivity are well known [3].

Proposition 1 *Let u and v be two non-adjacent vertices in a graph G. Then, $\kappa_G(u, v) - 1 \leq \kappa_{G-a}(u, v) \leq \kappa_G(u, v)$ for all $a \in (V(G) - \{u, v\}) \cup E(G)$.*

Proof Let $a \in (V(G) - \{u, v\}) \cup E(G)$ and let $G' = G - a$. By simply remarking that any uv-separator in G is a uv-separator in G', then $\kappa_{G'}(u, v) \leq \kappa_G(u, v)$. For the first inequality, suppose by contradiction that $\kappa_{G'}(u, v) \leq \kappa_G(u, v) - 2$, and let S be a minimum uv-separator in G'. If a is a vertex, set $S' = S \cup \{a\}$, and if $a = xy$ is an edge, set $S' = S \cup \{x\}$. In both cases, S' is a uv-separator of G with $|S'| \leq \kappa_G(u, v) - 1$, a contradiction. Hence, $\kappa_G(u, v) - 1 \leq \kappa_{G'}(u, v)$. □

Let us establish few notations that will be utilized going forward. For two vertices x and y in a path P, we denote by $P_{[x,y]}$ the unique subpath of P with its ends are x and y. The **contracting** of an edge e in a graph G, is to replacing the edge e by a new vertex and connecting it to the neighboring vertices of both end-vertices of e in G.

Theorem 1 *Let u and v be two non-adjacent vertices in a graph G. Then $\kappa_G(u, v) = \mu_G(u, v)$ if and only if $\kappa_{G-e}(u, v) = \kappa_G(u, v)$ for all $e \in E(G[S])$, where S is a minimum uv-separator of G.*

Proof As noted earlier, we only need to deal with the sufficient condition. Since every uv-separator of G must contain an internal vertex from each path in any set of internally disjoint uv-paths in G, then $\kappa_G(u,v) \geq \mu_G(u,v)$. Supposing the theorem is false, and assume that G is selected with the minimum number of vertices and edges, such that $\kappa_G(u,v) = k$ and there are no k internally disjoint uv-paths in G. Obviously, $\kappa_{G-a}(u,v) = k-1$ for all $a \in (V(G) - \{u,v\}) \cup E(G)$. Thus, since G satisfies the minimum separator property, any minimum uv-separator in G must be a stable set in G. Certainly, there exist x and $y \in V(G) - \{u,v\}$ such that $xy \in E(G)$, since otherwise G contains k internally disjoint uv-paths of length 2 each one, a contradiction. Let G_{xy} be the graph obtained from G by contracting the edge xy. We may consider G_{xy} to be the graph obtained from G by deleting y and then joining x to the neighbors of y that are not adjacent to x in G. In this sense, $G - y$ is a subgraph of G_{xy}. Set $N(y) = \{x_0, \ldots, x_t\}$ with $x_0 = x$. □

Claim $\kappa_{G_{xy}}(u,v) = k$.

Proof Let $H = G - xy$. We have $\kappa_H(u,v) = k-1$, then there exists a uv-separator S of H such that $|S| = k-1$. Note that $S \cap \{x,y\} = \phi$, since otherwise we have $G - S = H - S$, but S is a uv-separator of H and so S is a uv-separator of G with $|S| = k-1$, a contradiction. Set $S_x = S \cup \{x\}$. It is clear that S_x is a uv-separator of G. Since $G - y \subseteq G_{xy}$, then $\kappa_{G_{xy}}(u,v) \geq k-1$ as $\kappa_{G-y}(u,v) = k-1$. On the other hand, $G_{xy} - S_x \subseteq G - S_x$, then S_x is a uv-separator of G_{xy} with $|S_x| = k$. Therefore, $k-1 \leq \kappa_{G_{xy}}(u,v) \leq k$. Suppose to the contrary that $\kappa_{G_{xy}}(u,v) = k-1$. Let S' be a minimum uv-separator of G_{xy}, then $|S'| = k-1$. Note that $x \notin S'$, since otherwise we get $G_{xy} - S' = G - (S' \cup \{y\})$ with $|S' \cup \{y\}| = k$ as $y \notin G_{xy}$, and so $S' \cup \{y\}$ is a minimum uv-separator of G with it is not a stable set in G due to $xy \in E(G[S' \cup \{y\}])$, which contradicts our earlier assertion. Let P be a uv-path in G. If $y \notin V(P)$, then $P \subseteq G - y \subseteq G_{xy}$ and so $P \cap S' \neq \phi$. If $y \in V(P)$, set x_i and x_j be the neighbors of y on P where $x_i \in P_{[u,y]}$ and $x_j \in P_{[y,v]}$. If $x \in V(P)$, then without loss of generality suppose that $P_{[u,x]} \subseteq P_{[u,y]}$. Consider the uv-path $Q = P_{[u,x]} \cup xx_j \cup P_{[x_j,v]}$ in G_{xy}, we have $Q \cap S' \neq \phi$ and so $P \cap S' \neq \phi$. If $x \notin V(P)$, consider the uv-path $R = P_{[u,x_i]} \cup \{x\} \cup x_i x \cup xx_j \cup P_{[x_j,v]}$ in G_{xy}, similarly since $R \cap S' \neq \phi$, then $P \cap S' \neq \phi$ as $x \notin S'$. Therefore, S' is a uv-separator of G with $|S'| = k-1$, which gives a contradiction. □

Since $v(G_{xy}) < v(G)$, then G_{xy} contains k internally disjoint uv-paths $P_1, P_2, \ldots, P_k$. Since $\mu_G(u,v) < k$ then one of these paths should be contains an edge xx_i such that $xx_i \notin E(G)$. Suppose without loss of generality that $xx_i \in E(P_1)$, since $P_1, P_2, \ldots, P_k$ are internally disjoint then $x \notin V(P_j)$ for every $2 \leq j \leq k$, and so $P_2, \ldots, P_k$ are uv-paths in G. Let w be the another neighbor of x in P_1. If $w \notin N_G(y)$, then set $P_1' = (P_1 - xx_i) \cup \{y\} \cup xy \cup yx_i$, if $w \in N_G(y)$ set $P_1' = (P_1 - \{x\}) \cup \{y\} \cup wy \cup yx_i$. In both cases, P_1' is a uv-path in G such that $V(P_1') \subseteq V(P_1) \cup \{y\}$, thus $P_1', P_2, \ldots, P_k$ are k internally disjoint uv-paths in G, a contradiction.

We give now an elementary proof of the minimum separator property.

Theorem 2 *Let u and v be two non-adjacent vertices in a graph G. Then, $\kappa_{G-e}(u,v) = \kappa_G(u,v)$ for all $e \in E(G[S])$, where S is a minimum uv-separator of G.*

Proof We will proceed by induction on $\kappa_G(u,v)$. The case $\kappa_G(u,v)=1$ is trivial. Now, for $\kappa_G(u,v)=k$, $(k\geq 2)$. Let $S=\{x_1,x_2,\ldots,x_k\}$ be a minimum uv-separator of G. Suppose to the contrary that there exists an edge $e\in G[S]$ such that $\kappa_{G-e}(u,v)=k-1$. Without loss of generality, we may suppose that $e=x_1x_2$. Let $G'=G-e$ and S' be a minimum uv-separator of G'. We first prove that $S\cap S'=\phi$. In fact, suppose to the contrary that $x_i\in S\cap S'$ for some i, $1\leq i\leq k$. Let $G_i=G-x_i$, $G'_i=G_i-e$ and $S_i=S-x_i$. It is clear that $\kappa_{G_i}(u,v)=k-1$ and S_i is a minimum uv-separator of G_i. If $i\in\{1,2\}$, then $G'_i=G_i$ and so $\kappa_{G'_i}(u,v)=k-1$. If $i\in\{3,\ldots,k\}$, then by induction, we get $\kappa_{G'_i}(u,v)=\kappa_{G_i}(u,v)=k-1$. Note that $S'-\{x_i\}$ is a uv-separator in G'_i as $G'_i\subseteq G'$ and S' is a uv-separator of G'. A contradiction is raced since $\kappa_{G'_i}(u,v)=k-1$ and $|S'-\{x_i\}|=k-2$.

Let C_u and C_v be two distinct connected components in $G'-S'$ such that $u\in C_u$ and $v\in C_v$. Since $|S'|=k-1$, then $G-S'$ contains a uv-path. Then the edge e should have one of its ends in C_u and the other in C_v. Without loss of generality, we may assume that $x_1\in C_u$ and $x_2\in C_v$. Set $S_1=S\cap C_u$ and $S_2=S-S_1$. By remarking that $x_1\in S_1$ then we may find a uv-path in $G-S_2$. Let P be such a path, then $P\cap S_1\neq\phi$. Let $x_1(P)\in V(P)$ such that $P_{[x_1(P),v]}\cap S_1=\{x_1(P)\}$. Note that, $x_1(P)\in C_u$, and so $x_1(P)\notin C_v$, which means that $P_{[x_1(P),v]}$ doesn't lie in $G'-S'$, and thus $P_{[x_1(P),v]}\cap S'\neq\phi$ knowing that $e\notin E(P)$ and $S\cap S'=\phi$. Consequently we may find $S'_1\subseteq S'$ such that $S'_1\cup S_2$ is a uv-separator in G and for every $x\in S'_1$ there exist an xv-path in $G-S$. Remark that one can easily get $|S'_1|\geq|S_1|$. Similarly, for P is a uv-path in $G-S_1$, we define $x_2(P)\in V(P)$ such that $P_{[u,x_2(P)]}\cap S_2=\{x_2(P)\}$. Note that $x_2(P)\notin C_u$, since otherwise $x_2(P)\in S\cap C_u=S_1$, a contradiction. As above, $P_{[u,x_2(P)]}\cap S'\neq\phi$. We may deduce here also that there exists $S'_2\subseteq S'$ such that $S_1\cup S'_2$ is a uv-separator in G and for every $x\in S'_2$ there exist an ux-path in $G-S$ and $|S'_2|\geq|S_2|$. We have $S'_1\cap S'_2\neq\phi$, since otherwise $k-1=|S'|\geq|S'_1\cup S'_2|=|S'_1|+|S'_2|\geq|S_1|+|S_2|=|S|=k$, a contradiction. Let $a\in S'_1\cap S'_2$, there exist a ua-path R and an av-path Q in $G-S$. Thus $R\cup Q$ contains a uv-path not intersecting S, a contradiction. □

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