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Optimizing assortment and pricing of multiple retail categories with cross-selling

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Abstract This paper investigates the joint optimization of assortment and pricing decisions for complementary retail categories. Each category comprises substitutable items (e.g., different coffee brands) and the categories are related by cross-selling considerations that are empirically observed in marketing studies to be asymmetric in nature. That is, a subset of customers who purchase a product from a primary category (e.g., coffee) can opt to also buy from one or several complementary categories (e.g., sugar and/or coffee creamer). We propose a mixed-integer nonlinear program that maximizes the retailer's profit by jointly optimizing assortment and pricing decisions for multiple categories under a classical deterministic maximum-surplus consumer choice model. A linear mixed-integer reformulation is developed which effectively enables an exact solution to relatively large problem instances using commercial optimization solvers. This is encouraging, because simpler product line optimization problems in the literature have posed significant computational challenges over the last decades and have been mostly tackled via heuristics. Moreover, our computational study indicates that overlooking cross-selling between retail categories can result in substantial profit losses, suboptimal (narrower) assortments, and inadequate prices.

Keywords Cross-selling · Assortment planning · Pricing · Retail · Mathematical programming

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1 Introduction and motivation

Retailers face the challenging problem of selecting a subset of products to carry in stores or online and setting prices over time in a manner that appeals to a variety of consumers and maximizes profit. This has motivated a rich literature on the so-called *product line optimization problem*; a difficult combinatorial optimization problem that seeks to determine product selection and pricing strategies under anticipated consumer behavior and choice rationales (e.g., [13]). With the advent of retail analytics, the focus of practitioners has gradually shifted from single-product, "brand management" to multi-product "category management" (CM), e.g. [4,18,40]. With growing interest in category management and market basket analysis [28,32], it has become apparent that certain retail categories are interdependent and, therefore, should not be planned in isolation. Despite this trend in retailing, the effect of *cross-selling* on the optimization of complementary categories remains understudied in the academic literature (see Maddah et al. [25] for a review of works on category optimization in retailing).

Marketing studies provide strong empirical support for the notion of "cross-category" shopping or cross-selling (e.g., [29,38,39]). For example, a price discount on a product category (e.g., spaghetti, cake mix, fabrics) can substantially stimulate sales for a complementary product category (e.g., pasta sauce, cake frosting, sewing tools). In addition, an *asymmetric* cross-selling effect is often observed, whereby the sales of one "primary" category (e.g., cake mix) drive the demand of another "secondary" complementary category (e.g., cake frosting), with the opposite effect (secondary product driving the primary product demand) being rather negligible. For example, Walters [39] argue that price promotions of spaghetti resulted in a significant increase in the sales of the spaghetti sauce, whereas the reverse phenomenon did not occur with price promotions on spaghetti sauce. Mulhern and Leone [29] also report similar results indicating an asymmetric cross-selling effect of cake mix over cake frosting (see also Manchanda et al. [26] and Shankar and Kannan [33] for further discussion).

Our research is prompted by the emerging topic of managing multiple retail categories using mathematical programming and makes the following conceptual and computational contributions. First, as far as the marketing literature is concerned, the paper contributes to the notion of multi-category management under cross-selling by proposing a mixed-integer nonlinear programing model. The developed model jointly optimizes assortment and pricing decisions for multiple complementary retail categories under a classical maximum-surplus consumer choice model. From a computational point of view, the proposed mixed-integer linear reformulation of the model is demonstrated to enable exact solutions to large-scale instances, which suggests that integrated optimization of multiple dependent categories is now computationally tractable. This is encouraging, because the extant literature on product line optimization mostly examines simpler settings (e.g. [12,13]) and, recognizing the difficulty of such mixed-integer (nonlinear) programs, resorts to using constructive and greedy heuristics.

The remainder of the paper is organized as follows. Section 2 briefly reviews the literature related to product line optimization and cross-selling. In Sect. 3, the problem is formally stated along with our notation, the proposed mixed-integer nonlinear programming model, and its linear reformulation. In Sect. 4, we present an illustrative example, followed by a computational study that involves relatively large problem instances. Section 5 concludes the paper with a discussion of our findings and directions for future research.

2 Literature review

This paper relates to the literature on retail category decision analysis and optimization. In particular, it involves the so-called *product line optimization problem* (simultaneous product *selection* and *pricing*) and cross-selling—a phenomenon that is identified by market basket analysis. These two aspects of the literature are discussed in the remainder of this section with greater focus on data-driven optimization-based approaches, as opposed to stylized models that investigate analytical results under simplified assumptions. The more inquisitive reader is referred to Maddah et al. [25].

2.1 Product line optimization

The product line optimization problem, which integrates product line selection and pricing decisions, lies at heart of our work. Specifically, product line selection is concerned with optimizing the assortment or variety of products or services offered in a product line by a seller. In the context of our work, the seller is a retailer, the buyer is a shopper, and product lines of interest are retail categories. Early discussions of product line pricing with substitutable and complementary products date back to Dean [10]. Seminal conceptualizations of product line optimization problems appeared later and continue to motivate a great deal of research due to their practical relevance. A key element of these studies is the adoption of an adequate consumer choice model which captures the behavior of consumers and their anticipated purchase decisions in reaction to assortments and prices set by a seller. In particular, Green and Krieger [15] and Zufryden [42] consider the “single-choice, deterministic, behavior” whereby a consumer chooses a single product that yields a greatest nonnegative *surplus* (if available). Furthermore, a consumer is assumed to refrain from buying if all products yield negative surpluses. Here, the consumer surplus for a product is defined as the difference between the consumer *reservation price* (or the maximum monetary value he/she is willing to pay for this product) and the price set by the seller. To incorporate this consumer choice model in a mathematical program for planning purposes, it is necessary to estimate reservation prices for different products across distinct customer segments. This can be achieved using different techniques, including conjoint analysis (see, for example, [12,13,16,17,19,34,35,42], and references therein).

As noted in Kraus and Yano [23], the deterministic single-choice consumer model is employed in “most articles on product line optimization.” Zufryden [42] discusses justification for this choice model and refers to earlier studies that provide empirical evidence in support of this consumer behavior [5,6,30]. In particular, Johnson [21] supports this behavioral assumption for applications with a high degree of sensitivity to the surplus; in this case, a customer commits to a product that yields a maximum surplus and would consider switching only if a new product is introduced with a better surplus. For other applications where the consumer choice may be less driven by surplus considerations, and more by brand image or quality etc., then a stochastic model may be more adequate. Ghoniem and Maddah [14] also provide empirical support for this deterministic consumer choice model based on Tuna data from a grocery store in the Northeast US. The data spans a year and a half of transactions for a light Tuna product line, whereby the demand distribution across substitutable products is largely due to price discounts introduced by the retailer and is well-approximated by the deterministic single-choice consumer model.

In Zufryden [43], a 0-1 integer program is formulated in order to tackle the *product design and selection* problem under the single-choice deterministic rule. Here, the decision-maker optimizes the design of a new product (by determining the levels of its attributes) with the expectation that certain customer segments would opt to switch to the new product when it is introduced into an existing product line/market. Beyond the modeling contribution, no solution methodology is delineated. Green and Krieger [15] examine two product line selection problem variants: The first is a *buyer-welfare* problem that optimizes assortment decisions in a fashion that maximizes the total consumer surplus, whereas the second is a *seller-welfare* problem that maximizes the seller's profit. Both variants are examined with the assumption of a single-choice deterministic consumer model and are solved using heuristics, with a dismissal of optimization-based approaches. McBride and Zufryden [27], however, re-examine the seller-welfare product line selection problem using integer programming and argue that optimal solutions are attainable for their simulated instances on mainframe and personal computers.

Dobson and Kalish [12, 13] consider the more challenging product line optimization problem with simultaneous product selection (assortment) and pricing decisions under the single-choice deterministic consumer model. The joint optimization of assortment and pricing decisions prompts a 0-1 mixed-integer *nonlinear* programming formulation that is shown to be NP-Complete. Due to the perceived intractability of this formulation, the authors resort to using constructive heuristics [13]. Shioda et al. [34] revisit the product line optimization problem in Dobson and Kalish [13] with the goal of enhancing its tractability via valid inequalities and refined heuristics. They refer to the single-choice deterministic consumer model as "the maximum utility or the envy-free pricing model." The latter designation is borrowed from the microeconomics literature (e.g., [37]).

Describing the single-choice deterministic consumer model as a "max-surplus choice rule," Burkart et al. [7] investigate product line pricing for services over a selling horizon with capacitated offerings. As consumers commit to products, these are depleted and may become unavailable, in which case consumers dynamically substitute and choose an available product that yields a maximum, nonnegative surplus. Ghoniem and Maddah [14] also examine an extension of the nonlinear MIP formulation of the product line optimization problem in Dobson and Kalish [13] whereby inventory considerations are integrated with assortment and pricing decisions over a multi-period horizon. The authors propose an effective linear reformulation of the model and develop several managerial and computational insights.

Although the deterministic consumer choice model has been widely used in the literature [23], several studies consider product line pricing problems under probabilistic consumer choice models. For example, Chen and Hausman [9] examine the product line optimization problem under the logit choice model, discrete price values, and lower and upper bounds on the size of the assortment. Kraus and Yano [23] investigate a similar problem under a so-called *share-of-surplus* choice model. Here, the fraction of a customer segment that buys a positive-surplus product is determined as the ratio of the surplus of this product over the total surplus across products having a positive surplus for that segment. This ratio determines the relative probabilities of customers buying products and involves positive-surplus products only. This contrasts with the multinomial logit choice model, where a customer can buy a negative-surplus product with a positive (albeit small) probability. Subramanian and Sherali [36] also model category pricing under logit demand and propose a reformulation of a nonlinear fractional program using an effective linearization scheme. They also take into account several common industry practices related to targets on volume and sales levels, discrete prices exhibiting a ladder structure, and relative pricing rules for store vs. national brands. In a recent paper, Keller et al. [22] investigate product line pricing problems under attraction demand models. The authors identify conditions under which non-convexities that arise in the formulation can be circumvented by recasting the model as a convex optimization problem, thereby significantly enhancing the problem tractability.

2.2 Cross-selling considerations

Although studies on using cross-selling to optimize product line planning are still scarce, two research streams are emerging: The first considers two-product settings, whereas the second addresses category optimization under cross-selling. Few works consider asymmetric cross-selling within a two-product context whereby demand for a primary product drives that for a secondary product, as in our present paper. Aydin and Ziya [3] analyze an *upselling* practice, where upon the purchase of a *regular* product, whose price is exogenous, the buyer is offered to buy a *promotional* product, possibly at a discount. They focus on utilizing dynamic pricing for clearing the inventory of the promotional product. Beyond certain similarities in the consumer choice model with Aydin and Ziya [3], our work has a distinctive focus on static pricing and assortment optimization for multiple complementary categories. Zhang et al. [41] consider the effect of cross-selling on inventory decisions within a joint replenishment model of two products, a *major* and a *minor* one, with a common ordering cycle. The authors capture the effect of reduction in the demand of the minor product as a result of the major product planned stock-out (in a backordering setting), with the classic economic order quantity (EOQ) setting.

Maddah and Bish [24] also investigate a stylized model for the notion of *locational tying* of two retail products, a *primary* and a *secondary* one, where the secondary product is offered in two distinct locations in a store, its own department and the department of the primary product. This leads to two demand streams for the secondary product; an indirect one (which depends on the primary product price) due to cross-selling at the primary product location and a direct one at its appropriate department. The demand model in the present paper also considers two demand streams for secondary products, even though secondary products are displayed in their own department only.

The second stream of literature, which relates to this paper, is on category optimization under cross-selling. Agrawal and Smith [1] consider joint assortment and inventory optimization under exogenous choice for substitutable sets. Each exogenous set is a combination of complementary products. This, however, introduces the hurdle of explicitly enumerating all possible combinations of complementary products for which consumer reservation prices need to be estimated. In contrast, we model complementarity across categories and fewer parameters that reflect the cross-selling potential of a customer segment need to be estimated (as detailed next in Sect. 3). Moreover, our focus is different than Agrawal and Smith [1] as we consider pricing and assortment decisions.

Also of interest is the work by Cachon and Kök [8] where the assortments of two categories offered simultaneously by two retailers are optimized under a competitive duopoly setting. According to a nested logit choice, customers choose a store first, and then choose to buy from one or both categories in the store. Three customer segments are considered pertaining to the two categories and to the “basket” composed of products from both categories. The distinctive feature of our model, with respect to Cachon and Kök [8], is that we consider multiple customer segments for each category with customer purchases being endogenously deduced from asymmetric cross-selling effects. In addition, whereas Cachon and Kök [8] consider assortment decisions only with a cost which is convex in the assortment size (akin to a newsvendor-type supply setting), we consider assortment and pricing decisions with a variable linear cost and a fixed cost for offering a product in the assortment.

Rodríguez and Aydin [31] consider assortment and pricing decisions for two complementary categories, involving a *required* and an *optional* product, respectively, in a newsvendor-type supply setting and under logit demand. The authors study a stylized model with a single customer segment with two purchase scenarios: (i) purchase with a combined utility for both products or (ii) a sequential purchase approach, where a customer first buys a required product and then considers buying an optional product. The demand model in the sequential setting bears certain similarities with our setting, with the difference that we also consider a direct demand stream for the secondary (optional) category. Our work focuses on developing an optimization model with a maximum-surplus choice model, multiple customer segments, and possibly more than two categories.

3 Problem statement and formulation

This section provides a formal problem statement for multiple category optimization with cross-selling and introduces our notation along with our proposed mixed-integer nonlinear formulation. The model is then recast as a mixed-integer linear reformulation and can, therefore, be solved using standard commercial optimization solvers such as CPLEX.

3.1 Mixed-integer nonlinear formulation

We examine the setting where a retailer seeks to jointly optimize assortment and pricing decisions for multiple retail categories under cross-selling. All retail categories are composed of substitutable products that reflect the same need for the consumer but differ in some minor attributes (e.g., different brands of coffee or sugar of the same size). A fraction of shoppers of the primary category consider buying from the secondary categories in a way that reflects asymmetric cross-selling effects. We adopt the classical assumption that any customer buys at most one product from a given category of substitutable products, as is common under the maximum-surplus choice rule.

Specifically, we consider $\mathcal{L}$ distinct categories, where the first category is referred to as the primary category, whereas the remaining $|\mathcal{L}| - 1$ categories are secondary in that each complements the primary category. The chosen assortment for any category, denoted by $\mathcal{P}_\ell$, shall comprise substitutable products that are selected from a broader set of candidate products Ω_ℓ, with $\mathcal{P}_\ell \subseteq \Omega_\ell,\ \forall \ell \in \mathcal{L}$, and $\Omega_{\ell_1} \cap \Omega_{\ell_2} = \emptyset,\ \forall \ell_1, \ell_2 \in \mathcal{L}, \ell_1 \neq \ell_2$. For clarity in the notation, we shall designate by j^ℓ the jth product in Ω_ℓ. For example, products 1^1 and 3^2 respectively refer to the first candidate product of the primary category and the third product of a secondary category. For any category ℓ, let $\mathcal{C}_\ell$ be the set of customer segments or *direct customers* interested in buying from category ℓ. It is assumed that customer segments in the same category and accross different categories are disjoint. For example, Fig. 1 represents a setting with two categories (laptops and printers), where the first category has three distinct customer segments and the second has two customer segments. For clarity in the notation, let i^ℓ be the ith customer segment of category ℓ. For example, customer segments 2^1 and 3^2 respectively refer to the second customer segment of the primary category and the third customer segment of the secondary category. Furthermore, the retailer can estimate from experience and historical data (or anticipates based on surveys and market analysis) that a fraction γ_i^k of customer segment $i \in \mathcal{C}_1$, upon purchasing a product from the primary category ($\ell = 1$), would also consider purchasing a product from a secondary category $k \in \mathcal{L}\backslash\{1\}$. Such customers will be referred to as *cross-selling customers*. For example, in Fig. 1, only a fraction of the three customer segments of the primary category would consider cross-selling.

We denote by α_{ij}^ℓ the reservation price (or valuation) of customer segment $i \in \mathcal{C}_\ell$ for product $j \in \Omega_\ell$. Likewise, let β_{ij}^k be the reservation price of a cross-selling customer $i \in \mathcal{C}_1$ for a secondary product $j \in \Omega_k,\ \forall k \in \mathcal{L}\backslash\{1\}$. Reservation prices are assumed to be known to the retailer and can be estimated using such techniques as those discussed in Sect. 2. The

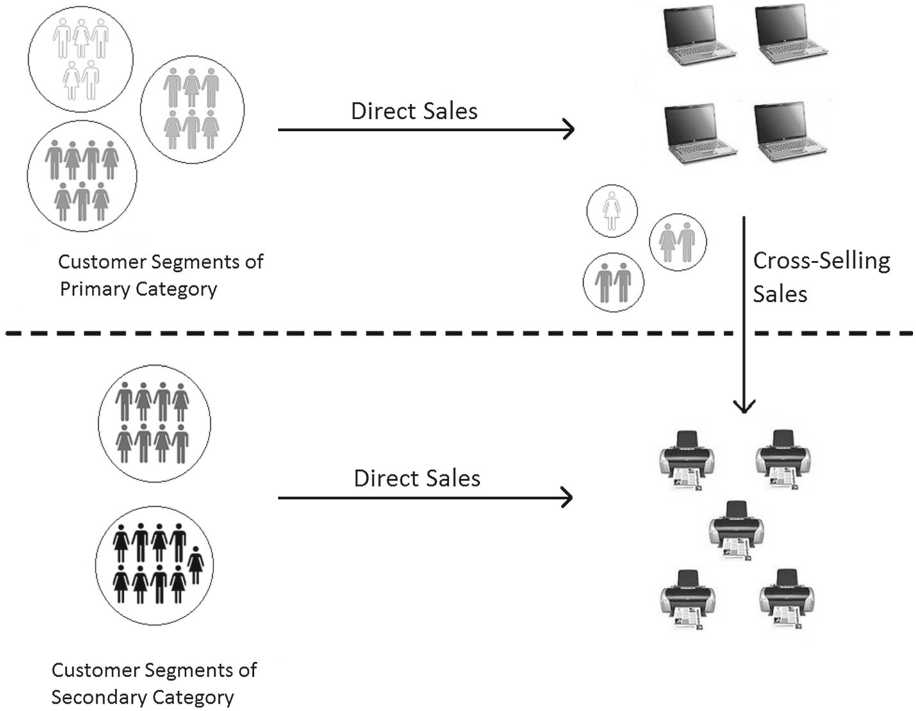

Fig. 1 Illustration of multiple categories with cross-selling

adopted maximum-surplus consumer choice model stipulates that a direct or a cross-selling customer would only buy a product that yields a maximum, nonnegative surplus. The latter is measured as the difference between the exogenous customer reservation prices and the endogenous prices set by the retailer. If prices are set so that all surplus values turn out to be negative for a given customer segment, the customer is priced out of the market and will opt not to buy from this retailer in this planning horizon. The cost structure we consider involves variable wholesale costs, as well as an additional fixed cost for offering a product in the assortment at the beginning of the selling season, which is also typical in the literature (e.g., [1,2,12,13]). The objective is to maximize the retailer's profit by selecting an optimal assortment for each category, along with optimal pricing decisions.

Consider the following notation:

Input parameters

- $\mathcal{L}$: Set of distinct product categories.
- Ω_ℓ : Set of all potential substitutable products in category ℓ, $\forall \ell \in \mathcal{L}$.
- $\mathcal{C}_\ell$: Set of customer segments that are interested in purchasing from category ℓ, $\forall \ell \in \mathcal{L}$. In Fig. 1, $\mathcal{C}_1$ and $\mathcal{C}_2$ comprise 3 and 2 distinct customer segments, respectively.
- s_i^ℓ : Number of customers in segment i for category ℓ, $\forall i \in \mathcal{C}_\ell, \ell \in \mathcal{L}$. For example, s_1^1 refers to the number of customers in segment 1 for category 1, whereas s_1^2 designates the number of customers in segment 1 for category 2. In Fig. 1, $s_1^1 = 5$ and $s_1^2 = 8$.
- α_{ij}^ℓ : Reservation price of customer segment i for product j in category ℓ, $\forall i \in \mathcal{C}_\ell$, $j \in \Omega_\ell, \ell \in \mathcal{L}$. For example, α_{13}^2 is the reservation price of customer segment 1, *direct customer of category 2*, for product 3.

- β_{ij}^k : Reservation price of customer segment i of the *primary* category for a *secondary* product j in category k, $\forall i \in \mathcal{C}_1$, $j \in \Omega_k$, $k \in \mathcal{L}\backslash\{1\}$. For example, β_{13}^2 is the reservation price of customer segment 1, *direct customer of the primary category*, for product 3 in category 2. This relates to a customer (of the primary category) who buys from a secondary category by cross-selling.
- γ_i^k : Fraction of customer segment i of the primary category who, upon purchasing a primary product from Ω_1, considers purchasing a complementary product from the secondary category set Ω_k, $\forall i \in \mathcal{C}_1$, $k \in \mathcal{L}\backslash\{1\}$. For example, referring to the first customer segment of the primary category in Fig. 1, only 1 out of 5 customers would consider cross-selling and, hence, $\gamma_1^2 = 0.2$.
- f_j^ℓ : Fixed cost for the inclusion of product j into $\mathcal{P}_\ell$, the chosen assortment from category ℓ, $\forall j \in \Omega_\ell$, $\ell \in \mathcal{L}$.
- c_j^ℓ : Unit ordering cost for product j in category ℓ, $\forall j \in \Omega_\ell$, $\ell \in \mathcal{L}$.
- u_j^1: Upper bound on the price of product j in the primary category; $u_j^1 \equiv \max_{i \in \mathcal{C}_1}\{\alpha_{ij}^1\}$, $\forall j \in \Omega_1$.
- u_j^k: Upper bound on the price of product j in a secondary category k; $u_j^k \equiv \max\{\max_{i \in \mathcal{C}_k}\{\alpha_{ij}^k\}, \max_{r \in \mathcal{C}_1}\{\beta_{rj}^k\}\}$, $k \in \mathcal{L}\backslash\{1\}$, $j \in \Omega_k$.

Decision variables

- $z_j^\ell \in \{0, 1\}$: $z_j^\ell = 1 \Leftrightarrow$ product $j \in \Omega_\ell$ is offered in the assortment $\mathcal{P}_\ell$, $\forall j \in \Omega_\ell$, $\ell \in \mathcal{L}$.
- $x_{ij}^\ell \in \{0, 1\}$: $x_{ij}^\ell = 1 \Leftrightarrow$ customer segment i of category ℓ purchases product $j \in \Omega_\ell$, $\forall i \in \mathcal{C}_\ell$, $j \in \Omega_\ell$, $\ell \in \mathcal{L}$. The x-variables are used to account for direct purchases.
- $y_{ij}^k \in \{0, 1\}$: $y_{ij}^k = 1 \Leftrightarrow$ A fraction γ_i^k of customer segment i of the primary category purchases product j in a secondary category k, $\forall i \in \mathcal{C}_1$, $j \in \Omega_k$, $k \in \mathcal{L}\backslash\{1\}$. The y-variables are used to represent cross-selling.
- p_j^ℓ : Price of product j in category ℓ, $\forall j \in \Omega_\ell$, $\ell \in \mathcal{L}$.
- d_j^ℓ : Demand for product j in category ℓ, $\forall j \in \Omega_\ell$, $\ell \in \mathcal{L}$, calculated endogenously as a function of purchase decisions and the size of customer segments.

The multi-category cross-selling (MCCS) problem can be stated as the following mixed-integer nonlinear program. The objective function (1a) maximizes the retailer's profit over the selling horizon, that is, the difference between the retailer's revenue and variable ordering costs and fixed costs for including products in the assortment. As is clear from the remainder of the model constraints, $d_j^\ell p_j^\ell$ is a nonlinear term that involves an endogenously predicted demand and retail prices.

$$\text{Maximize} \sum_{\ell \in \mathcal{L}} \sum_{j \in \Omega_\ell} \left(d_j^\ell p_j^\ell - c_j^\ell d_j^\ell - f_j^\ell z_j^\ell \right). \tag{1a}$$

Constraints (1b) ensure that a customer segment would choose, from amongst offered products, one that maximizes her surplus, provided that it yields a nonnegative surplus as enforced by Constraints (1c).

$$\sum_{k \in \Omega_\ell} (\alpha_{ik}^\ell - p_k^\ell) x_{ik}^\ell \geq (\alpha_{ij}^\ell - p_j^\ell) z_j^\ell, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{1b}$$

$$\sum_{j \in \Omega_\ell} (\alpha_{ij}^\ell - p_j^\ell) x_{ij}^\ell \geq 0, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell. \tag{1c}$$

Note that Constraints (1b) are equivalent to the following (more aggregate) constraints where the right-hand-side considers the max-surplus choice: $\sum_{k\in\Omega_\ell}(\alpha^\ell_{ik} - p^\ell_k)x^\ell_{ik} \geq \max_{j\in\Omega_\ell}\{(\alpha^\ell_{ij} - p^\ell_j)z^\ell_j\}, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell$.

Likewise, Constraints (1d)–(1e) stipulate that a customer segment $i \in \mathcal{C}_1$ would buy a complementary product $j \in \Omega_k$ provided that j yields a maximum, nonnegative surplus among all complementary products included in the assortment of this secondary category. The value for M in Constraints (1d) is set to $u^k_j \equiv \max\{\max_{i\in\mathcal{C}_k}\{\alpha^k_{ij}\}, \max_{r\in\mathcal{C}_1}\{\beta^k_{rj}\}\}, \forall k \in \mathcal{L}\backslash\{1\}, j \in \Omega_k$.

$$\sum_{h\in\Omega_k}(\beta^k_{ih} - p^k_h)y^k_{ih} \geq (\beta^k_{ij} - p^k_j)z^k_j - M(1 - \sum_{r\in\Omega_1} x^1_{ir}), \quad \forall k \in \mathcal{L}\backslash\{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{1d}$$

$$\sum_{j\in\Omega_k}(\beta^k_{ij} - p^k_j)y^k_{ij} \geq 0, \quad \forall k \in \mathcal{L}\backslash\{1\}, i \in \mathcal{C}_1. \tag{1e}$$

Constraints (1f) ensure that, for a certain category, any customer segment will purchase at most one product from amongst the substitutable products offered in the assortment. Constraints (1g) guarantee that a customer segment $i \in \mathcal{C}_1$ would buy some secondary product only if she is also purchasing a primary product. Constraints (1h)–(1i) ensure that any product cannot be purchased by a customer or cross-sold, unless it is included in the assortment.

$$\sum_{j\in\Omega_\ell} x^\ell_{ij} \leq 1, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell \tag{1f}$$

$$\sum_{h\in\Omega_k} y^k_{ih} \leq \sum_{j\in\Omega_1} x^1_{ij}, \quad \forall k \in \mathcal{L}\backslash\{1\}, i \in \mathcal{C}_1 \tag{1g}$$

$$x^\ell_{ij} \leq z^\ell_j, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{1h}$$

$$y^k_{ij} \leq z^k_j, \quad \forall k \in \mathcal{L}\backslash\{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{1i}$$

Constraints (1j) aggregate the demand for any product in the primary category based on customer direct purchases and the size of the different customer segments. Likewise, Constraints (1k) express the demand of any product in the secondary category by aggregating direct sales and sales due to cross-selling. Note that demand is price-sensitive in that it depends on the consumer choice variables (i.e., x- and y-variables) which, in turn, depend on the assortment and pricing decisions and are governed by the maximum-surplus consumer choice model.

$$d^1_j = \sum_{i\in\mathcal{C}_1} s^1_i x^1_{ij}, \quad \forall j \in \Omega_1 \tag{1j}$$

$$d^k_j = \sum_{i\in\mathcal{C}_k} s^k_i x^k_{ij} + \sum_{r\in C_1} \lfloor \gamma^k_r s^1_r \rfloor y^k_{rj}, \quad \forall k \in \mathcal{L}\backslash\{1\}, j \in \Omega_k. \tag{1k}$$

Constraints (1l) enforce upper bounds on prices based on the greatest reservation prices across customer segments, with $u^\ell_j \equiv \max\{\max_{i\in\mathcal{C}_\ell}\{\alpha^\ell_{ij}\}, \max_{r\in\mathcal{C}_1}\{\beta^\ell_{rj}\}\}, \quad \forall \ell \in \mathcal{L}\backslash\{1\}, j \in \Omega_\ell$, and logically relates the pricing and the assortment variables. Constraints (1m) introduce logical binary and non-negativity restrictions on decision variables.

$$p^\ell_j \leq u^\ell_j z^\ell_j, \quad \forall \ell \in \mathcal{L}, j \in \Omega_\ell \tag{1l}$$

$$\mathbf{x}, \mathbf{y}, \mathbf{z} \text{ binary}, \mathbf{p}, \mathbf{d} \geq 0. \tag{1m}$$

Model MCCS, which comprises (1a)–(1m), optimizes the retailer's assortment and pricing decisions, while predicting consumer decisions (maximizing their surplus) and the associated expected demand. To illustrate the rationale in the consumer choice model, we give a numerical example. Consider the primary category ($\ell = 1$). Suppose the model would like to introduce products 3 and 4 ($\in \Omega_1$), i.e., $z_3^1 = 1$, $z_4^1 = 1$, and $z_k^1 = 0$, for any product $k \in \Omega_1 \backslash \{3, 4\}$. Further, suppose the retailer would like to set the prices to $p_3^1 = 10$ and $p_4^1 = 25$. Consider now customer segment 1, with $\alpha_{13}^1 = 10$ and $\alpha_{14}^1 = 20$. Noting that $x_{1k}^1 = 0, \ \forall k \in \Omega_1 \setminus \{3, 4\}$ because of Constraints (1h), then Constraints (1b) reduce to:

$$(\alpha_{13}^1 - p_3^1)x_{13}^1 + (\alpha_{14}^1 - p_4^1)x_{14}^1 \geq \alpha_{13}^1 - p_3^1 \ (\equiv 0)$$
$$(\alpha_{13}^1 - p_3^1)x_{13}^1 + (\alpha_{14}^1 - p_4^1)x_{14}^1 \geq \alpha_{14}^1 - p_4^1 \ (\equiv -5)$$

Therefore, customer segment 1 is expected to buy product 3, i.e., $x_{13}^1 = 1$. Likewise, let customer segment 2 have $\alpha_{23}^1 = 12$ and $\alpha_{24}^1 = 30$, then Constraints (1b) for this segment enforce:

$$(\alpha_{23}^1 - p_3^1)x_{23}^1 + (\alpha_{24}^1 - p_4^1)x_{24}^1 \geq \alpha_{23}^1 - p_3^1 \ (\equiv 2)$$
$$(\alpha_{23}^1 - p_3^1)x_{23}^1 + (\alpha_{24}^1 - p_4^1)x_{24}^1 \geq \alpha_{24}^1 - p_4^1 \ (\equiv 5)$$

Therefore, customer segment 2 is expected to buy product 4, i.e., $x_{24}^1 = 1$. At last, suppose that customer segment 3 had reservation prices $\alpha_{33}^1 = 8$ and $\alpha_{34}^1 = 20$ with implied surpluses $\alpha_{33}^1 - p_3^1 \ (\equiv -2)$ and $\alpha_{34}^1 - p_4^1 \ (\equiv -5)$. In this case, segment 3 is simply priced out of the market, and does not buy anything, i.e., $x_{3j}^1 = 0, \ \forall j \in \Omega_1$.

3.2 Mixed-integer linear reformulation

Model MCCS is a mixed-integer nonlinear formulation that jointly optimizes assortment and pricing decisions with cross-selling considerations. The simpler product line optimization problem under a maximum-surplus choice model is a special case of our problem and is shown to be NP-Complete in Dobson and Kalish [13]. This is indicative of the difficulty of our problem which poses computational challenges due to the discreteness of key decision variables (e.g., assortment and customer purchase decisions) and nonlinearities that arise in the expression of the revenue (with price-sensitive demand) and the customer choice and cross-selling constraints. The computational intractability can, however, be largely alleviated by developing a linear reformulation of MCCS, which we denote by **L-MCCS**.

We first linearize the objective function (1a). To this end, we introduce the following auxiliary nonnegative continuous variables in lieu of nonlinear terms in the objective function, as in (3a), along with the linearizing Constraints (3b)–(3g):

$$g_{ij}^\ell \equiv p_j^\ell x_{ij}^\ell, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{2a}$$
$$q_{ij}^k \equiv p_j^k y_{ij}^k, \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{2b}$$

$$\text{Maximize} \sum_{k \in \mathcal{L} \setminus \{1\}} \sum_{r \in C_1} \sum_{h \in \Omega_k} \lfloor \gamma_r^k s_r^1 \rfloor q_{rh}^k + \sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{C}_\ell} \sum_{j \in \Omega_\ell} s_i^\ell g_{ij}^\ell - \sum_{\ell \in \mathcal{L}} \sum_{j \in \Omega_\ell} \left(c_j^\ell d_j^\ell + f_j^\ell z_j^\ell \right) \tag{3a}$$

$$g_{ij}^\ell \leq u_j^\ell x_{ij}^\ell, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{3b}$$
$$g_{ij}^\ell \geq p_j^\ell - u_j^\ell (1 - x_{ij}^\ell), \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{3c}$$
$$g_{ij}^\ell \leq p_j^\ell, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{3d}$$
$$q_{ij}^k \leq u_j^k y_{ij}^k, \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{3e}$$
$$q_{ij}^k \geq p_j^k - u_j^k (1 - y_{ij}^k), \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{3f}$$

$$q_{ij}^k \le p_j^k, \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{3g}$$

We also note that in Constraints (1b), the nonlinear term $p_j^\ell z_j^\ell$ can be replaced by p_j^ℓ, because of Constraints (1l). The model linearization is completed by substituting Constraints (3h)–(3k) in lieu of Constraints (1b)–(1e) as follows:

$$\sum_{k \in \Omega_\ell} (\alpha_{ik}^\ell x_{ik}^\ell - g_{ik}^\ell) \ge \alpha_{ij}^\ell z_j^\ell - p_j^\ell, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{3h}$$

$$\sum_{j \in \Omega_\ell} (\alpha_{ij}^\ell x_{ij}^\ell - g_{ij}^\ell) \ge 0, \qquad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell \tag{3i}$$

$$\sum_{j \in \Omega_k} (\beta_{ij}^k y_{ij}^k - q_{ij}^k) \ge 0, \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1 \tag{3j}$$

$$\sum_{h \in \Omega_k} (\beta_{ih} y_{ih}^k - q_{ih}^k) \ge \beta_{ij}^k z_j^k - p_j^k - M(1 - \sum_{r \in \Omega_1} x_{ir}^1), \qquad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{3k}$$

Model L-MCCS is stated as follows:
L-MCCS: {Maximize (3a): (3b)–(3k), (1f)–(1l), and **x**, **y**, **z** binary, **p**, **d**, **g**, **q** ≥ 0}.

4 Computational study

In this section, we present an illustrative example followed by our computational results for large-scale instances. The illustrative example discusses the planning of a primary category and a single secondary category. The computational study demonstrates the tractability of the proposed model reformulation and the usefulness of adopting an integrated approach that incorporates cross-selling considerations. The larger instances considered are scaled with respect to different parameters, namely, the number of candidate products in each category, $|\Omega_\ell|$, $\forall \ell \in \mathcal{L}$, and the number of direct customer segments for each category, $|\mathcal{C}_\ell|, \forall \ell \in \mathcal{L}$. All runs were performed with AMPL/CPLEX 12.4.0.0 on Microsoft Windows 7 Professional with an Intel Core i7-2600, 3.40 GHz processor and 12 GB RAM.

4.1 Illustrative example: a single secondary category

This illustrative example involves optimizing assortment and pricing decisions for a primary category and a secondary category. For each of the two categories, the retailer may select from among three substitutable products, i.e., $|\Omega_1| = 3$ and $|\Omega_2| = 3$. Further, the retailer has identified two direct customer segments for each category, that is, $|\mathcal{C}_1| = 2$ and $|\mathcal{C}_2| = 2$. Table 1 summarizes other input parameter values pertaining to customer segment sizes, customer reservation prices (or valuations), cross-selling parameters, and fixed and variable costs for the different products. Table 2 reports the solution obtained under two policies: (i) Our proposed integrated approach that optimizes both categories under cross-selling as in Model MCCS and (ii) a disjoint approach where each category is planned in isolation, thereby ignoring cross-selling effects by setting all γ values to zero.

The results demonstrate the importance and usefulness of the proposed integrated model. Under the integrated approach, the optimal assortments for the primary and secondary categories, respectively, are $\mathcal{P}_1 = \{1^1, 3^1\}$ and $\mathcal{P}_2 = \{2^2, 3^2\}$ (where the superscript of a product identifies its category). In particular, product 3 in the primary category was introduced at an affordable price for customer segment 2 ($\in \mathcal{C}_1$), which resulted in profitable cross-selling

Table 1 Data for illustrative example with a single secondary category

Primary category						Secondary category								
$i \in \mathcal{C}_1$	s_i^1	α_{ij}^1			γ	$i \in \mathcal{C}_2$	s_i^2	α_{ij}^2			$i \in \mathcal{C}_1$	β_{ij}		
		$j=1$	$j=2$	$j=3$				$j=1$	$j=2$	$j=3$		$j=1$	$j=2$	$j=3$
$i=1$	880	95	97	90	0.2	$i=1$	600	115	120	125	$i=1$	115	120	125
$i=2$	1,020	80	83	85	0.4	$i=2$	800	110	115	120	$i=2$	110	115	115
f_j^1		620	1,082	1,000		f_j^2		1,035	748	633				
c_j^1		79	82	83		c_j^2		96	100	98				

Table 2 Solution for illustrative example with a single secondary category

Primary category				Secondary category							
$i \in \mathcal{C}_1$	x_{ij}^1			$i \in \mathcal{C}_2$	x_{ij}^2			$i \in \mathcal{C}_1$	y_{ij}^2		
$j \in \Omega_1$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$
Integrated solution with cross-selling effects											
$i=1$	1	0	0	$i=1$	0	0	1	$i=1$	0	0	1
$i=2$	0	0	1	$i=2$	0	0	1	$i=2$	0	1	0
z_j^1	1	0	1	z_j^2	0	1	1	Total profit = 48,234.6			
d_j^1	880	0	1,020	d_j^2	0	408	1,576				
p_j^1	90	0	85	p_j^2	0	115	120				
Disjoint solution without cross-selling effects											
$i=1$	1	0	0	$i=1$	0	0	1	$i=1$	0	0	1
$i=2$	0	0	0	$i=2$	0	0	1	$i=2$	0	0	0
z_j^1	1	0	0	z_j^2	0	0	1	Total profit = 42,872.6			
d_j^1	880	0	0	d_j^2	0	0	1,400				
p_j^1	95	0	0	p_j^2	0	0	120				

transactions and the inclusion of product 2 in the secondary category. When cross-selling was overlooked, the retailer did not perceive benefit in including product 3 in primary category and product 2 in the secondary category, thereby yielding suboptimal, narrower assortments denoted by $\tilde{\mathcal{P}}_1 = \{1^1\}$ and $\tilde{\mathcal{P}}_2 = \{3^2\}$. Such suboptimal assortment and/or pricing decisions are, of course, accompanied by a significant profit loss of around 13 %. Further, it results in a reduced business volume whereby 3,884 transactions are anticipated under the integrated approach as opposed 2,280 transactions under the disjoint approach. This can have two damaging consequences for the retailer. The first is the risk of underestimating demand and, therefore, having to lose or backorder certain transactions. The second, as a result of narrower assortments, can cause an overall reduction of customer footprint [11]—a major concern to retailers.

4.2 Results for larger instances

In this section, we report in Table 3 results for large-scale instances that we randomly generated using the data generation scheme in the appendix. Central to our computational study is a comparison between our proposed integrated approach which accounts for cross-selling and a disjoint approach that overlooks cross-selling and optimizes each category in isolation, as explained in Sect. 4.1. Each of the 18 instances reported in Table 3 is identified by

Table 3 Computational analysis of MCCS for single-period instances

Inst.	$\lvert\Omega_1\rvert$	$\lvert\Omega_2\rvert$	$\lvert\Omega_3\rvert$	Integrated approach					Disjoint approach							
				$\lvert\mathcal{P}_1\rvert$	$\lvert\mathcal{P}_2\rvert$	$\lvert\mathcal{P}_3\rvert$	Profit (%)	CPU	$\lvert\tilde{\mathcal{P}}_1\rvert$	$\lvert\mathcal{P}_1 \cap \tilde{\mathcal{P}}_1\rvert$	$\lvert\tilde{\mathcal{P}}_2\rvert$	$\lvert\mathcal{P}_2 \cap \tilde{\mathcal{P}}_2\rvert$	$\lvert\tilde{\mathcal{P}}_3\rvert$	$\lvert\mathcal{P}_3 \cap \tilde{\mathcal{P}}_3\rvert$	Prof. Loss (%)	CPU
Instances with $\lvert\mathcal{C}_1\rvert = \lvert\mathcal{C}_2\rvert = \lvert\mathcal{C}_3\rvert = 3$																
1	25	25	25	3	2	3	5.1	1.0	3	3	2	2	2	2	65.4	0.6
2	25	50	75	2	2	2	2.0	3.9	1	1	2	2	2	1	11.8	0.7
3	50	50	50	2	3	3	2.8	3.4	1	1	2	2	2	2	50.8	0.8
4	50	75	100	1	2	2	2.4	3.3	1	1	1	1	2	0	11.0	1.0
5	75	50	25	1	1	1	2.3	1.9	1	1	1	1	1	1	0.0	0.7
6	75	100	150	2	2	1	2.2	10.4	1	0	2	1	1	1	7.9	2.1
7	100	75	50	2	1	2	2.2	2.1	2	2	1	0	2	2	19.4	0.8
Instances with $\lvert\mathcal{C}_1\rvert = \lvert\mathcal{C}_2\rvert = \lvert\mathcal{C}_3\rvert = 4$																
8	25	50	75	2	2	2	2.5	6.4	2	2	2	2	1	1	5.59	0.9
9	50	75	100	2	3	2	2.2	28.6	2	2	3	1	2	1	17.2	1.4
10	75	50	25	2	2	3	2.0	4.8	2	2	2	0	3	2	31.7	1.2
11	75	75	75	3	2	4	4.2	37.1	3	3	2	0	3	3	57.9	1.5
12	75	100	150	2	3	2	2.4	113.3	2	2	2	1	2	1	6.5	2.1
13	100	75	50	1	3	2	2.2	51.3	1	1	2	1	2	1	9.9	1.4
Instances with $\lvert\mathcal{C}_1\rvert = \lvert\mathcal{C}_2\rvert = \lvert\mathcal{C}_3\rvert = 5$																
14	25	50	75	2	2	2	2.3	15.5	2	2	2	2	2	1	5.6	1.2
15	50	75	100	3	3	4	2.3	155.3	3	3	2	1	3	0	20.9	2.3
16	75	50	25	2	3	2	2.2	7.6	1	1	2	2	2	2	22.0	1.1
17	75	100	150	2	3	3	2.2	652.9	2	2	2	0	3	1	18.1	10.2
18	100	75	50	2	3	4	2.1	333.1	1	0	3	0	4	4	8.9	3.0

its number and is characterized by the number of candidate, substitutable products in each category. All instances in our computational study involve one primary category and two secondary categories (i.e., $|\mathcal{L}| = 3$). For the integrated approach, Table 3 reports $|\mathcal{P}_1|$, $|\mathcal{P}_2|$, and $|\mathcal{P}_3|$—the size of the optimal assortments for the primary category and the two secondary categories. It also reports the profit as a percentage of the total revenue and the CPU time (seconds) to solve the instance to optimality. For the disjoint approach, we also report the size of selected assortments, $|\tilde{\mathcal{P}}_1|$, $|\tilde{\mathcal{P}}_2|$, and $|\tilde{\mathcal{P}}_3|$. For each category, we also report the number of products under the disjoint approach that are common to the optimal assortment, i.e., $|\mathcal{P}_\ell \cap \tilde{\mathcal{P}}_\ell|$, $\forall \ell \in \mathcal{L}$. The last two columns report the profit loss and the CPU time (s) under the disjoint approach.

From a computational viewpoint, it is worthwhile to note that the linear MIP reformulation, L-MCCS, solved all instances to optimality, with up to 5 customer segments for each category and over 75 substitutable products in each category. For most instances in our test-bed, the solution effort required less than one CPU minute. For the larger and more difficult instances, the CPU time ranged between 2 and 11 CPU minutes. This empirically observed computational tractability of the proposed MIP reformulation is encouraging and bears the potential of benefiting retailers for large-scale, industry-sized problem instances. The disjoint approach confirms that optimizing single-category decisions, when pertinent, is computationally manageable with the available computing power.

From a managerial point of view, the following observations and insights are in order.

1. *Profit reduction.* Over the 18 instances in our test-bed, the disjoint approach coincidentally yielded an optimal solution for only one instance (Instance 5). This atypical situation arises when direct customers are more profitable than cross-selling customers and it is optimal for the retailer to plan assortment and pricing decisions without consideration for cross-selling. For all the other instances, the profit reduction caused by the disjoint approach ranged from 5.6 to 65.4 %.
2. *Suboptimal, narrower assortments*. One recurrent disadvantage of the disjoint approach is that it tends to yield suboptimal (and often narrower) assortments. For Instance 18, the primary category comprises two products under the integrated approach, whereas only a single product forms the primary category under the disjoint approach. Further, the latter product is not part of the pair of products chosen in the integrated approach. Likewise, the three products selected in the first secondary category ($\ell = 2$) do not overlap at all with the three products selected under the integrated approach. The larger assortments observed under the integrated approach are often due to the introduction of a primary product as an incentive for attractive cross-selling customers. This, in turn, may result in the inclusion of additional secondary products that can secure profitable cross-selling transactions, e.g., Product 15^3 in Instance 1 (see our detailed discussion of Instance 1 below). A more aggressive version of this phenomenon relates to the concept of "loss-leaders" [20] whereby a retailer would sell a product at loss with the anticipation that customers who purchase it would also buy secondary products that are more lucrative.
3. *Suboptimal pricing*. The disjoint approach is also observed to yield suboptimal prices. Of particular interest are the prices of products that are selected under both the integrated and disjoint approaches. This comparison is pertinent when the entire assortment of a category is common to both approaches.
   - *Over-pricing products*. When the product valuations by cross-selling customers are relatively lower than those by direct customers for secondary categories, there is a risk of over-pricing under the disjoint approach. In fact, here, the retailer overlooks cross-selling and chooses higher prices based solely on direct customers. When cross-

Table 4 Analysis of Instance 1

Assortment[a]	Fixed cost	Sales	Unit price	Unit cost	Unit profit	Gross profit
Integrated approach						
11^1	539	376	137.13	132.6	4.53	1,703.28
15^1	639	259	135.35	130.6	4.75	1,230.25
24^1	594	397	133.12	128.1	5.02	1,992.94
15^2	796	639	142.29	138.7	3.59	2,294.01
23^2	588	(383,112)[b]	127.51	123.7	3.81	1,885.95
3^3	657	607	137.96	133.1	4.86	2,950.02
15^3	1149	(0,486)[b]	139.57	104.2	35.37	17,189.82
24^3	830	288	116.67	112.6	4.07	1,172.16
Total	5,792	(2949, 598)				30,418.43
Disjoint approach						
11^1	539	376	137.13	132.6	4.53	1,703.28
15^1	639	259	135.35	130.6	4.75	1,230.25
24^1	594	397	133.12	128.1	5.02	1,992.94
15^2	796	639	142.29	138.7	3.59	2,294.01
23^2	588	383	128.44	123.7	4.74	1,815.42
3^3	657	607	137.96	133.1	4.86	2,950.02
24^3	830	288	116.67	112.6	4.07	1,172.16
Total	4,643	2949				13,158.08

[a] Product j^ℓ corresponds to the jth product in category ℓ
[b] Provides the breakdown between direct and cross-selling sales when cross-selling occurs

selling customers visit the store, they may find the assortment relatively interesting, but would perceive the prices of secondary products as expensive (yielding a negative surplus). This would result in lost sales for the retailer. This is the case of Product 23^2 in Instance 1 (see our detailed discussion of Instance 1 below).

- *Under-pricing products*. When, on the contrary, cross-selling customers have relatively high product valuations for secondary categories, they would perceive the prices set by the retailer under the disjoint approach as quite attractive. This will generate a substantial stream of cross-selling purchases which will accelerate the depletion of the secondary products ordered by the retailer and are likely to cause stock-outs.

Except for Instance 5 for which the integrated and disjoint approaches yielded the same optimal solution, all other instances exhibited solutions where assortment and pricing decisions were adjusted under the integrated approach to take advantage of cross-selling opportunities. To illustrate, we discuss in some detail Instance 1 where a substantial profit loss of 65.4 % is observed under the disjoint approach despite many commonalities in the assortments of both approaches. In this Instance, the size of the customer segments are: (i) for the primary category $\ell = 1$, $s_1^1 = 397$, $s_2^1 = 259$, $s_3^1 = 376$; (ii) for the secondary category $\ell = 2$, $s_1^2 = 383$, $s_2^2 = 293$, and $s_3^2 = 346$; and (iii) for the third category $\ell = 3$, $s_1^3 = 288$, $s_2^3 = 244$, and $s_3^3 = 363$. Key elements of the optimal integrated and disjoint solutions obtained are summarized in Table 4. The optimal integrated assortment is $\mathcal{P}_1 = \{11^1, 15^1, 24^1\}$, $\mathcal{P}_2 = \{15^2, 23^2\}$, and $\mathcal{P}_3 = \{3^3, 15^3, 24^3\}$. The optimal disjoint assortment consists of $\tilde{\mathcal{P}}_1 = \{11^1, 15^1, 24^1\}$, $\tilde{\mathcal{P}}_2 = \{15^2, 23^2\}$, and $\tilde{\mathcal{P}}_3 = \{3^3, 24^3\}$. The only difference between assortments under both approaches is that product 15^3 is not included in the assortment of the category $\ell = 3$. Product prices are also identical under both approaches, except for product 15^3 which is included in the assortment only under the integrated approach. The following observations are in order:

1. Product 15^3 was introduced in the third category only because customers for the primary category had high cross-selling reservations prices (β-values) for this product. The product was therefore introduced in the assortment of the third category and its price was set to yield a nonnegative surplus to cross-selling customers, yielding a high unit profit. This was particularly lucrative, yielding a total profit for product 15^3 of around 17,190 which explains the 65.4 % loss in profit in the disjoint approach. Here, the customer segments of the primary category had high cross-selling potentials for the third category, reflected by their associated γ-values: $\gamma_1^3 = 0.52$, $\gamma_2^3 = 0.46$, and $\gamma_3^3 = 0.43$.
2. Cross-selling for product 23^2: Under the integrated approach, the price of product 23^2, $p_{23}^2 = 127.51$, is unchanged and yields an additional stream of revenue by attracting (112) cross-selling customers that are unaccounted for under the disjoint solution. This indicates that the retailer would underestimate demand and runs the risk of demand shortfalls if the ordered quantities were based on a disjoint approach that overlooks cross-selling.

The profit was particularly high under the disjoint approach for Instance 1, because the data therein presented a greatly lucrative product for customers with a high cross-selling potential (as a combination of the reservation prices β and the high cross-selling parameter γ) that yielded high profits under the integrated approach. Similar results are observed for Instances 3 and 11. This pattern of altered assortments and prices that favor cross-selling transactions are observed in most other instances with varying effects on the overall profit, depending on the reservation prices of cross-selling customers and their relative sizes.

5 Conclusions and directions for future research

We have examined the multi-category cross-selling (MCCS) problem, where a retailer seeks to jointly optimize assortment and pricing decisions for a primary category and several related secondary categories—each of which is composed of substitutable products. We developed a novel mixed-integer nonlinear formulation that maximizes the retailer profit under a maximum utility consumer choice model. We highlight that the nonlinearity of this model can be circumvented by introducing auxiliary variables and accompanying linearization constraints. The linear MIP reformulation is empirically observed to afford exact solutions to large-scale, industry-sized problem instances in manageable times (ranging from a few CPU seconds to a few CPU minutes). We have demonstrated the importance of jointly planning retail categories that are related by cross-selling. In fact, failing to do so results in substantial profit losses (ranging from 5 to 65 % in our computational experience), suboptimal (and often narrower) assortments, and inadequate prices. When such retail categories are planned in isolation, price inadequacy is evidenced by over-pricing certain secondary products, which can cannibalize cross-selling transactions, or under-pricing which stimulates cross-selling purchases that were unaccounted for to extent of causing stock-outs.

The approach articulated in the paper can help overcome computational difficulties noted in the literature, e.g. in the work by Dobson and Kalish [13]. In the latter, only heuristics approaches were devised for a single product selection and pricing problem under a maximum surplus consumer choice model. Our work can also serve as a cornerstone for future research on the integration of additional decisions related to inventory holding and shelf space allocation. Another direction that we recommend for future research is to analyze the effect of promotional campaigns. Finally, we recommend examining product line optimization problems with probabilistic consumer choice models, especially to address applications for which the deterministic consumer choice model may not be adequate.

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6 Appendix: Data generation scheme

A set of simulated problem instances are generated using the following data generation scheme:

- Number of customers in each customer segment (s_i^ℓ):
  Randomly set $s_i^\ell \leftarrow$ floor(Uniform(150, 400)).
- Fixed assortment cost (f_j^ℓ):
  Randomly set $f_j^\ell \leftarrow$ floor(Uniform(500, 2000)).
- Unit ordering cost (c_{jt}^ℓ):
  Randomly set $c_j^\ell \leftarrow$ round(Uniform(100, 140), 1). (Rounded to 1 digit past the decimal)
- Customer reservation prices:
  Randomly set $\alpha_{ij}^\ell \leftarrow$ round(c_j^ℓ * Uniform(0.99, 1.04), 2). (Rounded to 2 digits past the decimal)
  Randomly set $\beta_{ij}^\ell \leftarrow$ round(c_j^ℓ * Uniform(0.99, 1.04), 2) (for $\ell \in \mathcal{L} \setminus \{1\}$).
- Cross-selling matrix (γ_i^k):
  Randomly set $\gamma_i^k \leftarrow$ round(Uniform(0,0.6), 2).

Although the data generator is not based on a specific application, certain relationships were enforced between input parameters to yield meaningful instances. In additional, the reservation prices (α- and β-parameters) were randomly generated by multiplying the unit ordering cost by a scalar randomly generated using a uniform distribution over the interval (0.99, 1.04). As such, reservations prices in our instances tend to be within 4 % above the ordering cost and occasionally slightly below the ordering cost in order to reflect exigent customer segment in a highly competitive market. Finally, the fraction of customers of a given segment of the primary category that considers cross-selling from a secondary category (the γ parameter) is chosen between 0 and 60 %. This covers a wide range of customer behavior from not being interested in cross-selling to extensively exercising it.

References

1. Agrawal, N., Smith, S.A.: Optimal retail assortments for substitutable items purchased in sets. Nav. Res. Logist. **50**(7), 793–822 (2003)
2. Anupindi, R., Gupta, S., Venkataramanan, M.: Managing variety on the retail shelf: using household scanner panel data to rationalize assortments. In: Retail Supply Chain Management, International Series in Operations Research Management Science, Springer, US **122**, pp. 155–182 (2009)
3. Aydin, G., Ziya, S.: Pricing promotional products under upselling. Manuf. Serv. Oper. Manag. **10**(3), 360–376 (2008)
4. Basuroy, S., Mantrala, M., Walters, R.: The impact of category management on retail prices and performance: theory and evidence. J. Mark. **65**, 16–32 (2001)
5. Best, R.J.: The predictive aspects of a joint-space theory of stochastic choice. J. Mark. Res. **13**, 198–206 (1976)
6. Braun, M.A., Srinivasan, V.: Amount of information as a determinant of consumer behavior toward new products. In: Proceedings of the American Marketing Association, pp. 373–378 (1975)

7. Burkart, W.R., Klein, R., Mayer, S.: Product line pricing for services with capacity constraints and dynamic substitution. Eur. J. Oper. Res. **219**, 347–359 (2012)
8. Cachon, G.P., Kök, A.G.: Category management and coordination in retail assortment planning in the presence of basket shopping consumers. Manag. Sci. **53**(6), 934–951 (2007)
9. Chen, K., Hausman, W.H.: Mathematical properties of the optimal product line selection problem using choice-based conjoint analysis. Manag. Sci. **46**(2), 327–332 (2000)
10. Dean, J.: Pricing policies for new products. Harv. Bus. Rev. **28**, 45–53 (1950)
11. DeGraba, P.: Volume discounts, loss leaders, and competition for more profitable customers. Working paper # 260—Federal Trade Commission, Bureau of Economics (2003)
12. Dobson, G., Kalish, S.: Positioning and pricing of a product-line: formulation and heuristics. Mark. Sci. **7**, 107–125 (1988)
13. Dobson, G., Kalish, S.: Heuristics for pricing and positioning a product-line using conjoint and cost data. Manag. Sci. **39**(2), 160–175 (1993)
14. Ghoniem, A., Maddah, B.: Integrated retail decisions with multiple selling periods and customer segments: optimization and insights. Working paper, University of Massachusetts, Amherst (2013)
15. Green, P.E., Krieger, A.M.: Models and heuristics for product line selection. Mark. Sci. **4**(1), 1–19 (1985)
16. Green, P.E., Srinivasan, V.: Conjoint analysis in consumer research: issues and outlook. J. Consum. Res. **5**, 103–123 (1978)
17. Green, P.E., Srinivasan, V.: Conjoint analysis in marketing: new developments with implications for research and practice. J. Mark. **54**(4), 3–19 (1990)
18. Hall, J.M., Kopalle, P.K., Krishna, A.: Retailer dynamic pricing and ordering decisions: category management versus brand-by-brand approaches. J. Retail. **86**(2), 172–183 (2010)
19. Hanson, W., Martin, R.K.: Optimal bundle pricing. Manag. Sci. **36**, 155–174 (1990)
20. Hess, J.D., Gerstner, D.: Loss leader pricing and rain check policy. Mark. Sci. **6**(4), 358–374 (1987)
21. Johnson, R.M.: Trade-off analysis of consumer values. J. Mark. Res. **11**, 121–127 (1976)
22. Keller, P.W., Levi, R., Perakis, G.: Efficient formulations for pricing under attraction demand models. Math. Program. **145**(1–2), 223–261 (2014)
23. Kraus, U.G., Yano, C.A.: Product line selection and pricing under a share-of-surplus choice model. Eur. J. Oper. Res. **150**, 653–671 (2003)
24. Maddah, B., Bish, E.K.: Locational tying of complementary retail items. Nav. Res. Logist. **56**, 421–438 (2009)
25. Maddah, B., Bish, E.K., Munreo, B.: Pricing, variety, and inventory decisions for categories of substitutable items. In: Kemph, K., Keskinocak, P., Uzsoy, R. (eds.) Planning Production and Inventories in the Extended Enterprise: A State of the Art Handbook. Kluwer Academic Publishers, Dordrecht (2011)
26. Manchanda, P., Ansari, A., Gupta, S.: The shopping basket: a model for multicategory purchase incidence decisions. Mark. Sci. **18**(2), 95–114 (1999)
27. McBride, R.D., Zufryden, F.S.: An integer programming approach to the optimal product line selection problem. Mark. Sci. **7**(2), 126–140 (1988)
28. Mild, A., Reutterer, T.: An improved collaborative filtering approach for predicting cross-category purchase based on binary market basket data. J. Retail Consum. Serv. **10**, 123–133 (2003)
29. Mulhern, F.J., Leone, R.P.: Implicit price bundling of retail products: a multiproduct approach to maximizing store profitability. J. Mark. **55**(4), 63–76 (1991)
30. Pessemier, E.A., Burger, P., Teach, R., Tigert, D.: Using laboratory brand preference scales to predict consumer brand purchases. Manag. Sci. **17**, 371–385 (1971)
31. Rodríguez, B., Aydin, G.: Assortment selection and pricing for configurable products under demand uncertainty. Eur. J. Oper. Res. **210**(3), 635–646 (2011)
32. Russel, G.J., Petersen, A.: Analysis of cross category dependence in market basket selection. J. Retail. **76**, 367–392 (2000)
33. Shankar, V., Kannan, P.K.: An across-store analysis of intrinsic and extrinsic cross-category effects. Cust. Needs Solut. J. **1**(2), 143–153 (2014)
34. Shioda, R., Tuncel, L., Myklebust, T.G.J.: Maximum utility product pricing models and algorithms based on reservation price. Comput. Optim. Appl. **48**, 157–198 (2011)
35. Smith, J.C., Lim, C., Alptekinoğlu, A.: New product introduction against a predator: a bilevel mixed-integer programming approach. Nav. Res. Logist. **56**, 714–729 (2009)
36. Subramanian, S., Sherali, H.: A fractional programming approach for retail category price optimization. J. Glob. Optim. **48**(2), 263–277 (2010)
37. Walras, L.: Elements of Pure Economics or the Theory of Social Wealth. Allen and Unwin, London (Sydney) (1954)
38. Walters, R.G.: Retail promotions and retail store performance: a test of some key hypotheses. J. Retail. **64**, 153–180 (1988)

39. Walters, R.G.: Assessing the impact of retail price promotions on product substitution, complementary purchase, and interstore sales displacement. J. Mark. **55**, 16–28 (1991)
40. Zenor, M.J.: The profit benefits of category management. J. Mark. Res. **31**, 202–213 (1994)
41. Zhang, R., Kaku, I., Xiao, Y.: Deterministic EOQ with partial backordering and correlated demand caused by cross-selling. Eur. J. Oper. Res. **210**(3), 537–551 (2011)
42. Zufryden, F.S.: A conjoint measurement-based approach for optimal new product design and market segmentation. In: Shocker, A.D. (ed.) Analytic Approaches to Product and Marketing Planning. Cambridge, MA, pp. 100–114 (1977)
43. Zufryden, F.S.: Product line optimization by integer programming. In: Proceedings of the Annual Meeting of ORSA/TIMS, San Diego, CA (1982)