

Content Based Status Updates

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Abstract—Consider a stream of status updates generated by a source, where each update is of one of two types: high priority or ordinary (low priority). These updates are to be transmitted through a network to a monitor. However, the transmission policy of each packet depends on the type of stream it belongs to. For the low priority stream, we analyze and compare the performances of two transmission schemes: (i) Ordinary updates are served in a First-Come-First-Served (FCFS) fashion, whereas, in (ii), the ordinary updates are transmitted according to an M/G/1/1 with preemption policy. In both schemes, high priority updates are transmitted according to an M/G/1/1 with preemption policy and receive preferential treatment. An arriving high priority update discards and replaces any currently-in-service high priority update, and preempts (with eventual resume for scheme (i)) any ordinary update. We model the arrival processes of the two kinds of updates, in both schemes, as independent Poisson processes. For scheme (i), we find the arrival and service rates under which the system is stable and give closed-form expressions for average peak age and a lower bound on the average age of the ordinary stream. For scheme (ii), we derive closed-form expressions for the average age and average peak age of the high priority and low priority streams. We finally show that, if the service time is exponentially distributed, the M/M/1/1 with preemption policy leads to an average age of the low priority stream higher than the one achieved using the FCFS scheme. Therefore, the M/M/1/1 with preemption policy, when applied on the low priority stream of updates and in the presence of a higher priority scheme, is not anymore the optimal transmission policy from an age point of view.

Index Terms—Age of information, priority queues, queuing systems, Markov chains, Poisson arrivals, detour flow graph.

I. INTRODUCTION

WHILE the classical notion of delay is a measure of how long a packet spends in transit, the ‘Age of Information’ [1] is a receiver-centric notion that measures how fresh the data is at the receiver. Specifically, with $u(t)$ denoting the generation time of the last successfully received packet before time t , one defines $\Delta(t) = t - u(t)$ as the instantaneous age of the information at the receiver at time t . One can then consider

$$\Delta = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \Delta(t) dt, \quad (1)$$

as the (time) average age. Observe that $\Delta(t)$ increases linearly in the intervals between packet receptions, and when a

Manuscript received January 22, 2019; revised August 23, 2019; accepted October 24, 2019. Date of publication October 29, 2019; date of current version May 20, 2020. This work was supported in part by the Swiss National Science Foundation under Grant 200021_166106/1.

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Communicated by S. Watanabe, Associate Editor for Shannon Theory.

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Digital Object Identifier 10.1109/TIT.2019.2950212

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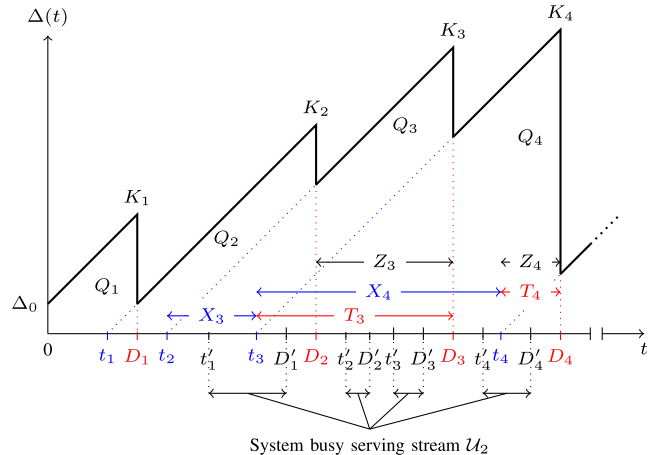


Fig. 1. Variation of the instantaneous age of stream \mathcal{U}_1 .

packet is received, $\Delta(t)$ jumps down to the delay experienced by this packet. This results in a sawtooth sample path as in Fig. 1. In [2]–[7] the properties of Δ were investigated under the assumption that the packets are generated by a Poisson process, and various transmission policies (M/M/1, M/M/∞, gamma service time,...).

A related metric, called average peak age, was introduced in [4] as the average of the value of the instantaneous age $\Delta(t)$ at times just before its downward jumps. In Fig. 1, K_j denotes the instantaneous age just before the reception of the j^{th} successfully transmitted packet, and hence, the average peak age is given by

$$\Delta_{peak} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N K_j. \quad (2)$$

Yates et al., in [8], studied the average age when considering multiple sources sending update through one queue. They computed the average age for three scenarios: all sources transmit according to an M/M/1 FCFS policy, all sources transmit according to an M/M/1/1 with preemption policy and all sources transmit according to an M/M/1/1 with preemption in waiting policy. In the M/M/1/1 with preemption policy, if a newly generated update finds the system busy, the transmitter preempts the one currently in service and starts sending the new packet. On the other hand, in the M/M/1/1 with preemption in waiting policy, the system has a buffer of size 1 and if the generated update finds the system busy, it replaces any waiting update in the buffer. In [9], Huang et al. also consider multiple sources transmitting through a single queue but in this case they assume a generally distributed service time. Moreover, they study two scenarios: all sources transmit according to an M/G/1 FCFS policy or all sources

transmit according to an M/G/1/1 with blocking policy. For each one of these policies, the authors give the expression of the average peak age of each source. In [10], Najm *et al.* also consider multiple sources sending through one queue. However, the authors derive the closed-form expressions of the average age and average peak age when assuming an M/G/1/1 with preemption, with the service time having a generic distribution. In their analysis, Najm *et al.* reintroduce the detour flow graph technique which we will use extensively in this paper.

In this paper, we assume updates are generated according to a Poisson process with rate λ , and that updates belong to two different streams where each stream i is chosen independently with probability p_i , $i = 1, 2$. So we have two independent Poisson streams with rates $\lambda_1 = \lambda p_1$ and $\lambda_2 = \lambda p_2$. However, unlike [8]–[10], we assume a different transmission policy for each stream. The two independent streams generated by the source can be used to model different types of content carried by the packets of each stream. For example, if the source is a sensor, one stream could carry emergency messages (fire alarm, high pressure, etc.) and thus it needs to be always as fresh as possible while the other stream will carry regular updates and hence is not age sensitive. Therefore, it stands to reason to transmit these two streams in a different manner. The paper is divided into three parts:

- In the first part, we assume a different transmission policy for each stream. The regular stream will be transmitted according to a FCFS policy, whereas the high priority stream will be sent by preemption; packets of the high priority stream preempt all packets including packets of their own stream. We further assume that the service time requirements of the two streams are different. Although packets from both streams spend an exponential time in service, a packet of the regular stream is served at rate μ_1 , while a packet of the high priority stream at rate μ_2 . This model was first presented by the authors of the current paper in [11]. In this part, we will answer the following questions: What should the relation between λ_1 , μ_1 , λ_2 and μ_2 be for the system to be stable? How does each stream affect the average age of the other one? What are the ages of each stream? To answer these questions, we give a necessary and sufficient condition for the system stability and find the steady-state distribution of the underlying state-space. We also give closed-form expressions for both the average peak-age and a lower bound on the average age of the regular stream, and compare them to the average age of the high-priority stream.
- In the second part, we assume the same transmission policy for both streams. We use an M/G/1/1 with preemption scheme. However, we consider that a packet from the low-priority (or regular) stream is served according to a service-time distribution similar to that of the random variable S_1 , whereas an update from the high-priority stream is served according to a service time distribution identical to that of the variable S_2 . We denote by $f_{S_1}(t)$ and $f_{S_2}(t)$ the respective probability density functions (p.d.f) of these service times. In this part, we generalize

part of the results presented in [12], relative to the preemption policy, and derive closed-form expressions for the average age and average peak age for any type of service-time distribution. Kaul *et al.*, in [12], address a similar problem. They consider multiple sources with different priorities with source 1 given the highest priority and source M the lowest priority. Two types of transmission schemes are investigated: (i) an M/M/1/1 with preemption where any new packet from source i preempts the packet currently in service if this update belongs to source j with $j \geq i$, and (ii) an M/M/1/2* where any new packet from source i that finds the server busy would be placed in a buffer of size 1. However, if the buffer is already occupied by an update from source j , $j \geq i$, then the waiting packet is dropped and replaced by the new one from source i .

- In the third part, we compare, through simulations, the performance of the FCFS policy and that of the M/G/1/1 with preemption on the age of the low priority stream. In this part, we show, through numerical results, that preemption is not the optimal transmission scheme to adopt, in the presence of higher priority streams, even when the service time is exponential. This comes as a surprise since it was shown, by Bedewy *et al.* in [13], [14], that the LCFS with preemption policy is optimal when we consider a single source generating updates according to a Poisson process and the service time at each hop is exponentially distributed. In fact, we observe that the FCFS policy achieves a lower average age than the one achieved by the M/M/1/1 with preemption scheme. This means that if we are designing a system with a high and a low priority stream, and we have a choice between FCFS and M/G/1/1 with preemption as transmission schemes for the low priority stream, we should implement a FCFS transmission policy.

This paper is structured as follows: In Section II, we start by defining the update generation mechanism, common to both models and the different variables needed in our study. In Section III, we study our first model and derive the stability condition of the system and its stationary distribution. The closed-form expressions of the average peak-age and the lower bound on the average age of the regular stream are computed in Section III-C. In Section IV, we analyze the second model and compute the average age and average peak-age of both streams. Finally, in Section V, we present some numerical results and show through an example that the FCFS policy outperforms the M/M/1/1 with preemption, from the low priority stream point of view.

II. SYSTEM MODEL

We consider a sender that generates packets (or updates) according to a Poisson process of rate λ . Each packet, independently of the previous packets, is of type 1 with probability p_1 and of type 2 with probability $p_2 = 1 - p_1$. We can thus see our sender as consisting of two sources generating two independent Poisson streams \mathcal{U}_1 and \mathcal{U}_2 with rates $\lambda_1 = \lambda p_1$ and $\lambda_2 = \lambda p_2$ respectively, $\lambda = \lambda_1 + \lambda_2$ (see [15]). As noted in the introduction, the different streams can be used

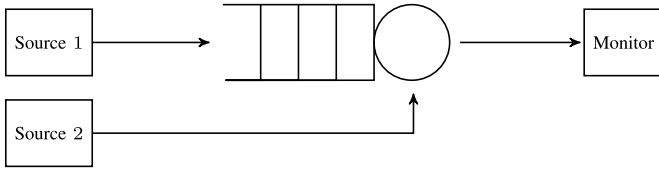


Fig. 2. Diagram representing the model with FCFS for the low priority stream.

to model packets of different types of content, for example, emergency messages, alerts, error messages, warnings, notices, etc.

We also assume that the updates are sent through a single server (or transmitter) to a monitor. The service times of packets from stream \mathcal{U}_1 are i.i.d according to $f_{S_1}(t)$, and those for stream \mathcal{U}_2 are i.i.d according to $f_{S_2}(t)$. The difference in service rates between the two streams accounts for the possible difference in compression, packet length, etc., between the two streams. In Section III, the service time of each packet is considered to be exponentially distributed, with rate μ_1 for stream \mathcal{U}_1 and rate μ_2 for stream \mathcal{U}_2 . However, in Section IV we keep the distributions general.

III. FCFS FOR THE LOW-PRIORITY STREAM

In this model, we constrain the transmitter so that all packets from stream \mathcal{U}_1 should be sent. Hence, the server applies a FCFS policy on the packets from stream \mathcal{U}_1 with a buffer to save waiting updates. Whereas, we assume that the information carried by stream \mathcal{U}_2 is more time sensitive (or has higher priority) hence we aim to minimize its average age. To this end, the transmitter is permitted to perform packet management: In this case, we assume the server applies a preemption policy whenever a packet from \mathcal{U}_2 is generated. This means that if a newly generated packet from stream \mathcal{U}_2 finds the system busy (serving a packet from \mathcal{U}_1 or \mathcal{U}_2), the server preempts the update currently in service and starts serving the new packet. On the one hand, if the preempted packet belongs to \mathcal{U}_1 , this packet is placed back at the head of the \mathcal{U}_1 -buffer so that it can be served once the system is idle again. On the other hand, if the preempted packet belongs to \mathcal{U}_2 then it is discarded. However, if a newly generated \mathcal{U}_1 -packet finds the system busy serving a \mathcal{U}_2 -packet, it is placed in the buffer and served when the system becomes idle. This choice of policy for the age sensitive stream is based on the conclusion reached in [14], that for exponentially distributed packet transmission times, the M/M/1/1 with preemption policy is the optimal policy among causal policies. Fig. 2 gives a graphical representation of this model.

These ideas are illustrated in part in Fig. 1 which also shows the variation of the instantaneous age of stream \mathcal{U}_1 . In this plot, t_i and D_i refer to the generation and delivery times of the i^{th} packet of stream \mathcal{U}_1 while t'_i and D'_i are the start and end times of the i^{th} period during which the system is busy serving packets from stream \mathcal{U}_2 only. Notice that for stream \mathcal{U}_1 none of the generated packets is discarded and all packets are received in the order of their generation.

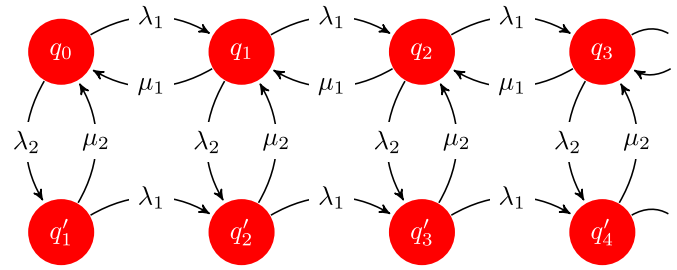


Fig. 3. Markov chain governing the number of packets in the system.

A. System Stability and Stationary Distribution

The fact that we seek to receive all of stream \mathcal{U}_1 updates and that stream \mathcal{U}_2 has a higher priority and preempts stream \mathcal{U}_1 might lead to an unstable system. In order to derive the necessary and sufficient condition for the stability of the system, we study the Markov chain of the number of packets in the system (in service and waiting) shown in Fig. 3. In this chain, q_0 is the idle state where the system is completely empty. States $q_i, i > 0$, in the upper row refer to states where the queue is serving a packet from stream \mathcal{U}_1 , whereas states $q'_i, i > 0$, in the lower row correspond to the queue serving a packet from stream \mathcal{U}_2 . In both cases, there are $i - 1$ stream \mathcal{U}_1 updates waiting in the buffer.

The system leaves state q_0 at rate λ_1 to state q_1 when a packet from stream \mathcal{U}_1 is generated first and it leaves q_0 at rate λ_2 to state q'_1 when a packet from stream \mathcal{U}_2 is generated first. However, when the system enters state $q_i, i > 0$, three exponential clocks start: (i) a clock with rate μ_1 , which corresponds to the service time of the stream \mathcal{U}_1 packet being served, (ii) a clock with rate λ_1 , which corresponds to the generation time of stream \mathcal{U}_1 packets and (iii) a clock with rate λ_2 , which corresponds to the generation time of stream \mathcal{U}_2 packets. If the μ_1 -clock ticks first, the system goes to state q_{i-1} : This means that the current stream \mathcal{U}_1 packet was delivered and the queue begins the service of the next one in the buffer (if there is any). However, if the λ_1 -clock ticks first, a new stream \mathcal{U}_1 update is generated and added to the buffer, hence the system goes to state q_{i+1} . Whereas, if the λ_2 -clock ticks first, the system preempts the packet currently in service and places it back at the head of the buffer and starts the service of the newly generated stream \mathcal{U}_2 update. Thus the system goes to state q'_{i+1} . When the system enters a state $q'_i, i > 0$, two exponential clocks start: the clock with rate λ_1 and a clock with rate μ_2 , which corresponds to the service time of a stream \mathcal{U}_2 packet. If the λ_1 -clock ticks first, the newly generated stream \mathcal{U}_1 packet is placed in the buffer and the stream \mathcal{U}_2 update is continued to be served. Hence the system goes to state q'_{i+1} . However, if the μ_2 -clock ticks first, the stream \mathcal{U}_2 packet has finished service and the system starts serving the first stream \mathcal{U}_1 packet in the buffer (if there is any). Hence the system goes to state q_{i-1} .

This next theorem gives the necessary and sufficient condition for the above system to be stable, as well as its stationary distribution.

Theorem 1. *The system described in Section III is stable, i.e. the average number of packets in the queue is finite, if and only if*

$$\mu_1 > \lambda_1 \left(1 + \frac{\lambda_2}{\mu_2} \right). \quad (3)$$

In this case the Markov chain shown in Fig. 3 has a stationary distribution $\Pi = [\pi_0, \pi_1, \dots, \pi_i, \dots, \pi'_1, \dots, \pi'_i, \dots]$, where π_i denotes the stationary probability of state q_i , $i \geq 0$, and π'_i denotes the stationary probability of state q'_i , $i > 0$. This stationary distribution is described by the following system of equations,

$$\pi_0 = \frac{\mu_2}{\mu_2 + \lambda_2} - \frac{\lambda_1}{\mu_1}, \quad (4)$$

$$\begin{bmatrix} \pi_i \\ \pi'_i \end{bmatrix} = [\mathbf{0} \quad \mathbf{I}_2] \mathbf{H}^i \begin{bmatrix} \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\lambda_1 + \mu_2)} \\ \frac{\lambda_2}{\lambda_1 + \mu_2} \\ 1 \\ 0 \end{bmatrix} \pi_0, \quad i \geq 1 \quad (5)$$

where $\lambda = \lambda_1 + \lambda_2$, $\mathbf{H} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{I}_2 & \mathbf{0} \end{bmatrix}$,

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\mu_2 + \lambda_1)} & -\frac{\mu_2 \lambda_1}{\mu_1(\mu_2 + \lambda_1)} \\ \frac{\lambda_2}{\mu_2 + \lambda_1} & \frac{\lambda_1}{\mu_2 + \lambda_1} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -\frac{\lambda_1}{\mu_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

\mathbf{I}_2 is the 2×2 identity matrix and $\mathbf{0}$ is the 2×2 zero matrix.

Corollary 1. *If we define $N(t)$ to be the number of stream \mathcal{U}_1 packets in the system at time t , then its moment generating function is $\phi_{N(t)}$*

$$\phi_{N(t)}(s) = \pi_0 \left(\frac{\mu_1(\lambda_1 + \lambda_2 + \mu_2 - \lambda_1 e^s)}{\mu_1 \mu_2 + \mu_1 \lambda_1 - e^s(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 \mu_1 + \lambda_1 \mu_2) + \lambda_1^2 e^{2s}} \right), \quad (6)$$

where π_0 is given by (4). Particularly, the expected value of $N(t)$ is

$$\mathbb{E}(N(t)) = \frac{\lambda_1(2\lambda_2\mu_2 + \lambda_2\mu_1 + \lambda_2^2 + \mu_2^2)}{(\mu_2 + \lambda_2)(\mu_1\mu_2 - \lambda_1(\mu_2 + \lambda_2))}. \quad (7)$$

Proof. The distribution given by (4) and (5) satisfy the detailed balance equations of the Markov chain shown in Fig. 3. Moreover, (3) is the condition needed to have $\pi_0 > 0$. As for the expression for $\phi_{N(t)}(s)$, it is a consequence of (4) and (5). The appendix in Section VII-A and Section VII-B presents a full technical version of the proof for Theorem 1 and Corollary 1. \square

B. Interpretation of the Stability Condition

The condition in (3) can be interpreted in two equivalent ways:

- 1) For an M/M/1 system with one source and an update rate λ_1 and service rate μ_1 , we need $\mu_1 > \lambda_1$ for the system to be stable. However, in the case of stream \mathcal{U}_1 we need to compensate for the amount of time the second stream occupies the system. This explains the additional $\frac{\lambda_1 \lambda_2}{\mu_2}$ term in (3) compared to an M/M/1 system.
- 2) Define the map f from the state-space of the chain as $f(s) = 0$ if s is in $\{q_0, q_1, \dots\}$ and $f(s) = 1$ if $s \in$

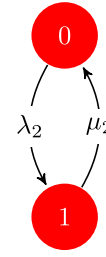


Fig. 4. Markov chain representing whether the system is serving \mathcal{U}_2 packets (state 1) or not (state 0).

$\{q'_1, q'_2, \dots\}$. For each s and s' for which $f(s) = 0$ and $f(s') = 1$ the transition rate from s to s' is the same (λ_2) and similarly for s and s' with $f(s) = 1$, $f(s') = 0$, (μ_2). Consequently $F(t) = f(s(t))$, with $s(t)$ being the state at time t , is Markov (which would not be the case for an arbitrary f), and it is easily seen that $F(t) = 0$ a fraction $\phi_0 = \mu_2/(\lambda_2 + \mu_2)$ amount of time, $F(t) = 1$ a fraction $\phi_1 = \lambda_2/(\lambda_2 + \mu_2)$ amount of time. This means that ϕ_0 is the fraction of time spent by the system serving \mathcal{U}_1 packets or being idle, and ϕ_1 is the fraction of time the system spends serving \mathcal{U}_2 packets. The Markov chain representing the process $F(t)$ is given by Fig. 4. Thus, while the Markov chain in Fig. 3 moves right at rate λ_1 , it moves left at a rate $\mu_1\phi_0$. The system is stable only if the rate of moving left is larger than the rate of moving right; which gives the condition (3).

C. Ages of Streams \mathcal{U}_1 and \mathcal{U}_2

1) *Preliminaries:* In this section, unless stated otherwise, all random variables correspond to stream \mathcal{U}_1 . We also follow the convention where a random variable U with no subscript corresponds to the steady-state version of U_j that refers to the random variable relative to the j^{th} received packet from stream \mathcal{U}_1 . To differentiate between streams, we will use superscripts, which means that $U^{(i)}$ corresponds to the steady-state variable U relative to stream \mathcal{U}_i , $i = 1, 2$.

In addition to this, we adopt the following notation:

- $X^{(i)}$ is the interarrival time between two consecutive generated updates from stream \mathcal{U}_i , so $f_{X^{(i)}}(x) = \lambda_i e^{-\lambda_i x}$, $i = 1, 2$
- $S^{(i)}$ is the service time random variable of stream \mathcal{U}_i updates, so $f_{S^{(i)}}(t) = \mu_i e^{-\mu_i t}$, $i = 1, 2$,
- T_j is the system time, or the total time spent by the j^{th} stream- \mathcal{U}_1 update in the queue (sum of its waiting time and its service time).

In our model, we assume the service time of the updates from the different streams to be independent of the interarrival time between consecutive packets (belonging to the same stream or not).

2) *Analysis of the System:* Given the aforementioned description of the model, we can define for each \mathcal{U}_1 packet j a “virtual” service time Z_j that could be different from its “physical” service time $S_j^{(1)}$. We define the “virtual” service time Z_j as follows:

$$Z_j = D_j - \max(D_{j-1}, t_j), \quad (8)$$

where D_j is the delivery time of the j^{th} packet and t_j is its generation time. Fig. 1 shows the “virtual” service time for packets 3 and 4.

For stream \mathcal{U}_1 , given that the average age calculations seem to be intractable, we compute its average peak age and give a lower bound on its average age. To this end, we first study the steady state “virtual” service time Z .

We define the event

$$\Psi_j = \{\text{packet } j \text{ finds the system in state } q'_1\}$$

and its complement $\overline{\Psi_j}$. Then, we need the following lemmas.

Lemma 1. *Let Y_j be the “virtual” service time of packet j given that this packet does not find the system in state q'_1 , i.e. $\mathbb{P}(Y_j > t) = \mathbb{P}(Z_j > t | \overline{\Psi_j})$. Then, in steady state,*

$$\phi_Y(s) = \mathbb{E}(e^{sY}) = \frac{\mu_1(\mu_2 - s)}{s^2 - s(\mu_2 + \mu_1 + \lambda_2) + \mu_1\mu_2}. \quad (9)$$

Similarly, let Y'_j be the “virtual” service time of packet j given that this packet finds the system in state q'_1 , i.e. $\mathbb{P}(Y'_j > t) = \mathbb{P}(Z_j > t | \Psi_j)$. Then, in steady state,

$$\phi_{Y'}(s) = \mathbb{E}(e^{sY'}) = \frac{\mu_1\mu_2}{s^2 - s(\mu_2 + \mu_1 + \lambda_2) + \mu_1\mu_2}. \quad (10)$$

Proof. This proof is based on the detour flow graph (or signal flow graph) method. An overview of this method as well as the complete proof are presented in Section VII-C. \square

3) *Average Peak-Age of Stream \mathcal{U}_1 :* It is worth noting that the system under consideration cannot be seen as an M/G/1 queue with service time distributed as Z , because the “virtual” service times of different packets are correlated. Indeed, if we know that the “virtual” service time of packet j , Z_j , is big, then with very high probability the $(j+1)^{\text{th}}$ packet will be generated during the service of the j^{th} packet. Hence, with high probability, Z_{j+1} will be distributed as Y . Whereas, if Z_j is small, then there is a non-negligible probability with which the $(j+1)^{\text{th}}$ packet will find the system serving stream \mathcal{U}_2 . Hence, Z_{j+1} will be distributed as Y' .

Theorem 2. *The average peak age of stream \mathcal{U}_1 is given by*

$$\Delta_{\text{peak},1} = \frac{1}{\lambda_1} + \frac{2\lambda_2\mu_2 + \lambda_2\mu_1 + \lambda_2^2 + \mu_2^2}{(\mu_2 + \lambda_2)(\mu_1\mu_2 - \lambda_1(\mu_2 + \lambda_2))}. \quad (11)$$

Proof. As we can deduce from Fig. 1, the j^{th} peak $K_j = X_j^{(1)} + T_j$ where $X_j^{(1)}$ is the j^{th} interarrival time for stream \mathcal{U}_1 and T_j is the system time of the j^{th} stream \mathcal{U}_1 update. At steady state, we get $\Delta_{\text{peak},1} = \mathbb{E}(K) = \mathbb{E}(X^{(1)}) + \mathbb{E}(T)$. From Little’s law we know that $\mathbb{E}(T) = \mathbb{E}(N(t)) \mathbb{E}(X^{(1)})$, with the expected number of stream \mathcal{U}_1 packets $\mathbb{E}(N(t))$ given by (7) and $\mathbb{E}(X^{(1)}) = 1/\lambda_1$. \square

4) *Lower Bound on the Average Age of Stream \mathcal{U}_1 :* We now compute a lower bound of the average age.

Consider a fictitious system where if a stream \mathcal{U}_1 arrival finds the system in state q'_1 , then the stream \mathcal{U}_2 packet that is being served is discarded (and the stream \mathcal{U}_1 packet enters service immediately). The instantaneous age process of this fictitious system is pointwise less than the instantaneous age

of the true system, consequently its average age lower bounds the true average age. Note that the fictitious system from the point of view of the stream \mathcal{U}_1 is M/G/1, with service time distributed like Y in (9).

Lemma 2. *Assume an M/G/1 queue with interarrival time $X^{(1)}$ exponentially distributed with rate λ_1 and service time Y whose moment generating function is given by (9). The service time and the interarrival time are assumed to be independent. Then the distribution of the system time T is*

$$f_T(t) = C_1 e^{-\alpha_1 t} (\mu_2 - \alpha_1) - C_1 e^{-\alpha_2 t} (\mu_2 - \alpha_2), \quad t \geq 0, \quad (12)$$

where $\alpha_1, \alpha_2 > 0$ are the roots of the quadratic expression

$$s^2 - s(\mu_1 + \mu_2 + \lambda_2 - \lambda_1) + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_1\lambda_2,$$

$$C_1 = \frac{(1 - \rho)\mu_1}{\alpha_2 - \alpha_1},$$

and $\rho = \lambda_1 \mathbb{E}(Y) = \frac{\lambda_1(\mu_2 + \lambda_2)}{\mu_1\mu_2}$.

Proof. See Appendix VII-D. \square

From [2], we know that the average age of the M/G/1 queue with interarrival time $X^{(1)}$ and service time Y is

$$\Delta_{LB} = \lambda_1 \left(\frac{1}{2} \mathbb{E}(X_j^{(1)2}) + \mathbb{E}(T_j X_j^{(1)}) \right), \quad (13)$$

where for the j^{th} packet we have $T_j = (T_{j-1} - X_j^{(1)})^+ + Y_j$, $f(x) = (x)^+ = x \mathbb{1}_{\{x \geq 0\}}$ and $\mathbb{1}_{\{ \cdot \}}$ is the indicator function. So $\mathbb{E}(T_j X_j^{(1)})$ becomes

$$\mathbb{E}(T_j X_j^{(1)}) = \mathbb{E}(X_j^{(1)}(T_{j-1} - X_j^{(1)})^+) + \mathbb{E}(Y_j) \mathbb{E}(X_j^{(1)}), \quad (14)$$

where the second term is due to the independence between Y_j and $X_j^{(1)}$.

Proposition 1.

$$\begin{aligned} & \mathbb{E}(X_j^{(1)}(T_{j-1} - X_j^{(1)})^+) \\ &= \frac{\lambda_1\mu_2 + 2\lambda_1\lambda_2}{\mu_1^2(\mu_1\mu_2 - \lambda_1(\mu_2 + \lambda_2))} \\ &+ \frac{\lambda_2\lambda_1}{\mu_2} \left(\frac{(\mu_2 + \mu_1 + \lambda_2)^2 - 2\mu_1\mu_2}{\mu_1^2(\mu_2 + \lambda_1)(\mu_1\mu_2 - \lambda_1(\mu_2 + \lambda_2))} \right) \\ &+ \frac{\lambda_2\lambda_1}{\mu_2} \left(\frac{2\mu_2\lambda_1(\mu_1 + \lambda_2) + \lambda_2(\lambda_1^2 + \mu_2)}{\mu_1^2(\mu_2 + \lambda_1)^2(\mu_1\mu_2 - \lambda_1(\mu_2 + \lambda_2))} \right). \end{aligned} \quad (15)$$

Proof. Given that T_{j-1} and $X_j^{(1)}$ are independent then

$$\begin{aligned} & \mathbb{E}(X_j^{(1)}(T_{j-1} - X_j^{(1)})^+) \\ &= \int_0^\infty \int_x^\infty x(t-x) f_T(t) \lambda_1 e^{-\lambda_1 x} dt dx \end{aligned}$$

Replacing $f_T(t)$ by its value in (12) and using the fact that

$$\alpha_1 + \alpha_2 = \mu_1 + \mu_2 + \lambda_2 - \lambda_1,$$

$$\alpha_1\alpha_2 = \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_1\lambda_2,$$

we get (15) after some computations. \square

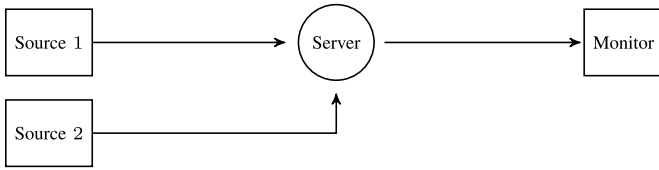


Fig. 5. Diagram representing the model with preemption for the low priority stream.

Theorem 3.

$$\Delta_{LB} = \frac{1}{\lambda_1} + \frac{\mu_2 + \lambda_2}{\mu_1 \mu_2} + \frac{\lambda_1^2 \mu_2 + 2\lambda_1^2 \lambda_2}{\mu_1^2 (\mu_1 \mu_2 - \lambda_1 (\mu_2 + \lambda_2))} + \frac{\lambda_2 \lambda_1^2}{\mu_2} \left(\frac{(\mu_2 + \mu_1 + \lambda_2)^2 - 2\mu_1 \mu_2}{\mu_1^2 (\mu_2 + \lambda_1) (\mu_1 \mu_2 - \lambda_1 (\mu_2 + \lambda_2))} \right) + \frac{\lambda_2 \lambda_1^2}{\mu_2} \left(\frac{2\mu_2 \lambda_1 (\mu_1 + \lambda_2) + \lambda_2 (\lambda_1^2 + \mu_2)}{\mu_1^2 (\mu_2 + \lambda_1)^2 (\mu_1 \mu_2 - \lambda_1 (\mu_2 + \lambda_2))} \right). \tag{16}$$

This is also a lower bound on the true average age of stream U_1 packets.

Proof. Using (15), $\mathbb{E}(Y_j) = \mathbb{E}(Y) = \frac{\mu_2 + \lambda_2}{\mu_1 \mu_2}$ and $\mathbb{E}(X_j^{(1)}) = \mathbb{E}(X^{(1)}) = \frac{1}{\lambda_1}$, we can find a closed-form expression for $\mathbb{E}(T_j X_j^{(1)})$. Replacing this expression in (13) and using the fact that $\mathbb{E}(X_j^{(1)^2}) = \frac{2}{\lambda_1^2}$, we obtain a closed-form expression of the average age Δ_{LB} of an M/G/1 queue with interarrival time $X^{(1)}$ and service time Y . \square

5) *Average Age of Stream U_2 :* By design, stream U_2 is not interfered at all by stream U_1 hence behaves like a traditional M/M/1 with preemption queue with generation rate λ_2 and service rate μ_2 . The average age of this stream was computed in [3] to be

$$\Delta_{U_2} = \frac{1}{\mu_2} + \frac{1}{\lambda_2}. \tag{17}$$

IV. M/G/1/1 WITH PREEMPTION FOR THE LOW-PRIORITY STREAM

Fig. 5 presents an illustration of the model. In this model, we assume we have no memory, hence packets from stream U_1 preempt each other. However, if an arriving U_1 packet finds the system busy serving a U_2 packet, the server discards the stream U_1 packet because stream U_2 packets are given higher priority. Furthermore, the server applies a preemption policy whenever a packet from U_2 is generated. This means that if a newly generated packet from stream U_2 finds the system busy (serving a packet from U_1 or U_2), the server preempts the update currently in service and starts serving the new packet. Moreover, if the preempted packet belongs to U_1 or U_2 , this packet is discarded.

These ideas are illustrated in part in Fig. 6, which also shows the variation of the instantaneous age of stream U_1 . In this plot, t_j refers to the generation time of the j^{th} packet, and D_i corresponds to the delivery time of the i^{th} successfully received packet of stream U_1 . As in this case not all the packets generated by source U_1 are received, we distinguish between *generated* packets and *successful* packets. Moreover,

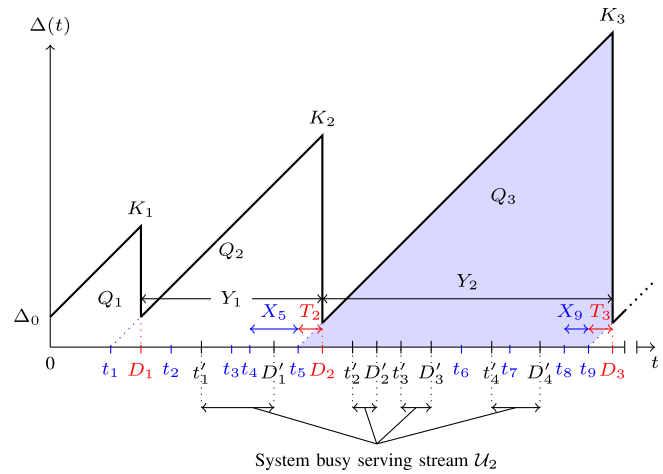


Fig. 6. Variation of the instantaneous age of stream U_1 .

t'_i and D'_i are the start and end times of the i^{th} period during which the system is busy serving packets only from stream U_2 .

A. Ages of Streams U_1 and U_2

1) *Preliminaries:* In this section also, unless stated otherwise, all random variables correspond to stream U_1 . We also follow the convention where a random variable U with no subscript corresponds to the steady-state version of U_j that refers to the random variable relative to the j^{th} received packet from stream U_1 . To differentiate between streams, we use superscripts, so that $U^{(i)}$ corresponds to the steady-state variable U relative to stream U_i , $i = 1, 2$.

In contrast to Section III, here we follow a slightly different notation:

- $X^{(i)}$ is the interarrival time between two consecutive generated updates from stream U_i , so $f_{X^{(i)}}(x) = \lambda_i e^{-\lambda_i x}$, $i = 1, 2$,
- $S^{(i)}$ is the service time random variable of stream U_i updates with p.d.f $f_{S^{(i)}}(t)$, $i = 1, 2$,
- T_j is the system time, or the time spent by the j^{th} successfully received stream U_1 update in the queue,
- Y_j to be the interdeparture time between the j^{th} and $j + 1^{th}$ successfully received stream U_1 updates.
- $R(\tau) = \max \{n : D_n \leq \tau\}$ is the number of successfully received updates from stream 1 in the interval $[0, \tau]$.

Given that in this model there is no waiting in the queue, the system time of a received packet is equal to its service time. In our model, we assume the service time of the updates from the different streams to be independent of the interarrival time between consecutive packets (regardless if they belong to the same stream).

Finally, two important quantities that we will use extensively are

- $P_\lambda = \mathbb{E}(e^{-\lambda S^{(1)}}) = \int f_{S^{(1)}}(t) e^{-\lambda t} dt$,
- $L_{\lambda_2} = \mathbb{E}(e^{-\lambda_2 S^{(2)}}) = \int f_{S^{(2)}}(t) e^{-\lambda_2 t} dt$.

These are the Laplace transform of $f_{S^{(1)}}(t)$ and $f_{S^{(2)}}(t)$ evaluated at $\lambda = \lambda_1 + \lambda_2$ and λ_2 , respectively.

2) Average Age and Average Peak Age of Stream \mathcal{U}_1 :

Lemma 3. For the priority preemption system described above, the moment generating function of the system time T corresponding to stream \mathcal{U}_1 is given by

$$\phi_T(s) = \frac{P_{\lambda-s}}{P_\lambda}. \quad (18)$$

Proof. All variables in this proof corresponds to stream \mathcal{U}_1 . The system time T_j of the j^{th} successfully received packet corresponds to the service time of the j^{th} received packet given that service was completed before any new arrival (because any new packet from any stream will preempt the current update being served). Therefore, in steady state, $\mathbb{P}(T > t) = \mathbb{P}(S^{(1)} > t | S^{(1)} < \min(X^{(1)}, X^{(2)}))$. Hence, for $L = \min(X^{(1)}, X^{(2)})$,

$$\begin{aligned} f_T(t) &= \lim_{\epsilon \rightarrow 0} \frac{\mathbb{P}(T \in [t, t + \epsilon])}{\mathbb{P}(S^{(1)} \in [t, t + \epsilon] | S^{(1)} < L)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mathbb{P}(S^{(1)} \in [t, t + \epsilon]) \mathbb{P}(S^{(1)} < L | S^{(1)} \in [t, t + \epsilon])}{\mathbb{P}(S^{(1)} < L)} \\ &= \frac{f_{S^{(1)}}(t) \mathbb{P}(L > t)}{\mathbb{P}(S^{(1)} < L)} = \frac{f_{S^{(1)}}(t) e^{-\lambda t}}{\mathbb{P}(S^{(1)} < L)}, \end{aligned}$$

where the last equality is due to the fact that L is exponentially distributed with rate $\lambda = \lambda_1 + \lambda_2$. Thus,

$$\begin{aligned} \phi_T(s) &= \mathbb{E}(e^{sT}) = \int_0^\infty \frac{f_{S^{(1)}}(t)}{\mathbb{P}(S^{(1)} < L)} e^{-(\lambda-s)t} dt \\ &= \frac{P_{\lambda-s}}{\mathbb{P}(S^{(1)} < L)}. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbb{P}(S^{(1)} < L) &= \int_0^\infty f_{S^{(1)}}(t) \mathbb{P}(L > t) dt \\ &= \int_0^\infty f_{S^{(1)}}(t) e^{-\lambda t} dt = P_\lambda. \end{aligned}$$

□

Lemma 4. The moment generating function of the interdeparture time of stream \mathcal{U}_1 , Y , is

$$\phi_Y(s) = \frac{\lambda_1 P_{\lambda-s} (\lambda_2 L_{\lambda_2-s} - s)}{\lambda_1 P_{\lambda-s} (\lambda_2 L_{\lambda_2-s} - s) - s(\lambda_2 - s)}. \quad (19)$$

Proof. We use again the detour flow graph method. The detailed proof is presented in Section VII-E. □

Before presenting the main theorem of this section, we need the following lemma that proves that the studied system is ergodic.

Lemma 5. Consider stream \mathcal{U}_1 . For any $j \geq 1$, the random variables T_j and Y_j relative to the j^{th} successful packet are independent. Moreover the process $(Y_j)_{j \geq 1}$ is i.i.d, with its distribution given by Lemma 4, and the process $R(\tau) = \sup\{n \in \mathbb{N}; D_n \leq \tau\}$ is a renewal process.

Proof. Let $L_j = \min(X_j^{(1)}, X^{(2)})$. Since the interarrival times for both streams are exponential and independent, L_j is

also exponential with rate $\lambda = \lambda_1 + \lambda_2$. Except L_j , all other variables are relative to stream \mathcal{U}_1 . The j^{th} successful packet leaves the queue empty hence $Y_j = \hat{X}_j + Z_j$. $\hat{X}_j = L_j - T_j$ is the remaining time between the departure of the stream- \mathcal{U}_1 j^{th} successful packet, and the generation time of the next packet to be transmitted (it can belong to stream \mathcal{U}_1 or stream \mathcal{U}_2). Z_j is the time for a new stream- \mathcal{U}_1 packet to be successfully delivered. Z_j does not overlap with T_j and thus is independent from it. As for \hat{X}_j , we also obtain that it is independent of T_j . Intuitively, since L_j is exponentially distributed with rate λ and thus memoryless, then the distribution of the remainder \hat{X}_j is independent of the value of T_j . Indeed, \hat{X}_j is also exponentially distributed with rate λ . A more formal proof can be found in Appendix VII-F.

Furthermore, since $Y_{j-1} = \hat{X}_{j-1} + Z_{j-1}$, \hat{X}_j is independent from T_j and the interarrival process is i.i.d and independent from the i.i.d service process, then \hat{X}_j and Z_j are independent of Y_{j-1} . This implies that for any $j \geq 1$, Y_{j-1} and Y_j are independent. Moreover, it is clear that the Z_j 's have the same distribution. Since the \hat{X}_j 's are exponential with rate λ then the $(Y_j)_{j \geq 1}$ is an i.i.d process. Given that Y_j is the interval of time between the receptions of two consecutive successful stream- \mathcal{U}_1 packets, then the number of successfully received packets in the interval $[0, \tau]$, $R(\tau)$, is a renewal process. □

Now we can state the main theorem of this section.

Theorem 4. Assume an M/G/1/1 queue with preemption and a sender consisting of two sources generating packets according to two independent Poisson processes with rates λ_i , $i = 1, 2$, such that $\lambda = \lambda_1 + \lambda_2$. Moreover, packets belonging to stream i are served according to $S^{(i)}$. If stream \mathcal{U}_2 is given higher priority over stream \mathcal{U}_1 , then

1) the average age of stream \mathcal{U}_1 is given by

$$\Delta_1 = \frac{1}{\lambda_1 P_\lambda L_{\lambda_2}} + \frac{1 - L_{\lambda_2} - \lambda_2 \mathbb{E}(S^{(2)} e^{-\lambda_2 S^{(2)}})}{\lambda_2 L_{\lambda_2}} \quad (20)$$

2) and the average peak age of stream \mathcal{U}_1 is given by

$$\Delta_{peak,1} = \frac{1}{\lambda_1 P_\lambda L_{\lambda_2}} + \frac{\mathbb{E}(S^{(1)} e^{-\lambda S^{(1)}})}{P_\lambda}. \quad (21)$$

Proof. By Lemma 5, $R(\tau)$ forms a renewal process. By [15], $\lim_{\tau \rightarrow \infty} \frac{R(\tau) - 1}{\tau} = \frac{1}{\mathbb{E}(Y)}$, where Y is the steady-state interdeparture random variable. Introducing the quantity $C_j = \int_{D_j}^{D_{j+1}} \Delta(t) dt$ to be the reward function over the renewal period Y_j , we obtain using renewal reward theory [15], [16] that

$$\Delta_1 = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \Delta(t) dt = \frac{\mathbb{E}(C_j)}{\mathbb{E}(Y_j)} = \frac{\mathbb{E}(Q_j)}{\mathbb{E}(Y_j)} = \frac{\mathbb{E}(Q)}{\mathbb{E}(Y)} < \infty,$$

where Q is the steady-state counterpart of Q_j , and the last equality stems from the fact that the average age for Stream 1 can also be expressed as the sum of the geometric areas Q_j under the instantaneous age curve of Fig. 6.

It was shown in [3], [4], [17] that, using Fig. 6,

$$\mathbb{E}(Q) = \frac{1}{2}\mathbb{E}(Y^2) + \mathbb{E}(TY).$$

Since, by Lemma 5, the variables T_j and Y_j are independent for any $j \geq 1$, then

$$\mathbb{E}(Q) = \frac{1}{2}\mathbb{E}(Y^2) + \mathbb{E}(T)\mathbb{E}(Y).$$

Therefore,

$$\Delta_1 = \mathbb{E}(T) + \frac{\mathbb{E}(Y^2)}{2\mathbb{E}(Y)} \quad (22)$$

Moreover, from Fig. 6 we see that the peak age at the instant before receiving the j^{th} packet is given by

$$K_j = T_{j-1} + Y_{j-1}.$$

Hence, at steady state we get

$$\Delta_{peak,1} = \mathbb{E}(K) = \mathbb{E}(T) + \mathbb{E}(Y). \quad (23)$$

Using Lemma 3, we obtain $\mathbb{E}(T) = P_\lambda^{-1}\mathbb{E}(S^{(1)}e^{-\lambda S^{(1)}})$.

Using Lemma 4, we get that $\mathbb{E}(Y) = (\lambda_1 P_\lambda L_{\lambda_2})^{-1}$ and

$$\begin{aligned} \frac{\mathbb{E}(Y^2)}{2\mathbb{E}(Y)} &= \frac{1}{\lambda_2} - \frac{\mathbb{E}(S^{(1)}e^{-\lambda S^{(1)}})}{P_\lambda} - \frac{\mathbb{E}(S^{(2)}e^{-\lambda_2 S^{(2)}})}{L_{\lambda_2}} \\ &+ \frac{1}{\lambda_1 P_\lambda L_{\lambda_2}} + \frac{1}{\lambda_2 L_{\lambda_2}}. \end{aligned}$$

Using these expressions in (22) and (23), we achieve our result for stream \mathcal{U}_1 . \square

3) *Average Age of Stream \mathcal{U}_2* : By design, stream \mathcal{U}_2 is not at all interfered by stream \mathcal{U}_1 hence behaves like a traditional M/G/1/1 with preemption queue with generation rate λ_2 and service time $S^{(2)}$. The average age of this stream was computed in [10], [18] to be

$$\Delta_2 = \frac{1}{\lambda_2 L_{\lambda_2}}. \quad (24)$$

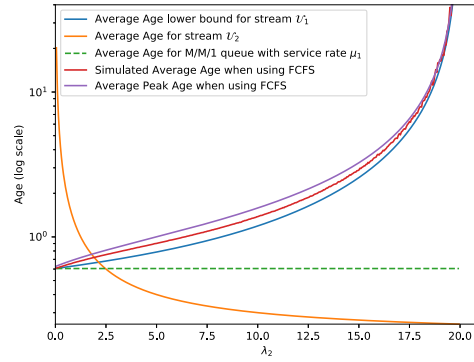
V. DISCUSSION ON THE AGE OF THE LOW PRIORITY STREAM

Having analyzed the FCFS and the M/G/1/1/ with preemption transmission schemes for the low priority streams, we can now compare their performances.

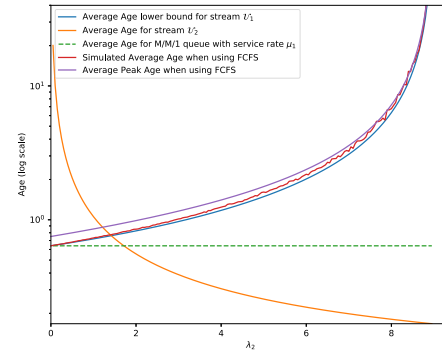
A. Relative to the FCFS Policy

Fig. 7a shows the simulated average age, the average peak-age ($\Delta_{peak,1}$) and the lower bound on the average age (Δ_{LB}), as computed in the previous section for stream \mathcal{U}_1 , and the average age ($\Delta_{\mathcal{U}_2}$) of stream \mathcal{U}_2 . In this plot, we fix $\mu_1 = 10$, $\mu_2 = 5$, $\lambda_1 = 2$ and vary λ_2 . As we can see, for stream \mathcal{U}_1 the average age, the lower bound, and the average peak-age grow without bounds when λ_2 gets close to $\frac{\mu_1\mu_2}{\lambda_1} - \mu_2$. This observation is in line with our result in Theorem 1 and the stability condition (3). In this simulation, we also notice that the average peak age and the lower bound appear to be good bounds on the average age, especially for small λ_2 and for values of $\lambda_2 \sim 0$ close to the limit $\frac{\mu_1\mu_2}{\lambda_1} - \mu_2$.

It is easy to see via a coupling argument that if we increase λ_2 , the age process $\Delta_{\mathcal{U}_1}(t)$ of the \mathcal{U}_1 stream will stochastically



(a) $\mu_1 = 10$, $\mu_2 = 5$, $\lambda_1 = 2$ and $\lambda_2 < \frac{\mu_1\mu_2}{\lambda_1} - \mu_2 = 20$.



(b) $\mu_1 = 6$, $\mu_2 = 18$, $\lambda_1 = 4$ and $\lambda_2 < 9$.

Fig. 7. Plot of the average age for stream \mathcal{U}_2 and average peak age and lower bound on the average age for stream \mathcal{U}_1 .

increase. We see from the plots that the lower bound on $\Delta_{\mathcal{U}_1}$ and that its average peak-age exhibit the same behavior. However, the average age of stream \mathcal{U}_2 is decreasing in λ_2 (from (17)). Consequently, minimizing $\Delta_{\mathcal{U}_2}$ and minimizing $\Delta_{\mathcal{U}_1}$ are conflicting goals.

We have seen that the average age of stream \mathcal{U}_2 is not affected by the presence of the other stream. However, Fig. 7a shows the effect of stream \mathcal{U}_2 on the average age of stream \mathcal{U}_1 (Δ_1). For this, we plot the average age (Δ_{ref}) of an M/M/1 queue with generation rate $\lambda_1 = 2$ and service rate $\mu_1 = 10$ (given in [2]). We observe an expected behavior: for very low values of λ_2 , the two average ages and the lower bound Δ_{LB} are close (they are all equal at $\lambda_2 = 0$). However, as λ_2 increases the presence of stream \mathcal{U}_2 quickly leads to an increase in Δ_1 . In fact, for $\lambda_2 = 5$, Δ_1 is already 50% higher than Δ_{ref} . This shows that the presence of the priority stream \mathcal{U}_2 takes a heavy toll on the stream \mathcal{U}_1 age. Another observation is that the average age curve of stream \mathcal{U}_2 crosses the average age of stream \mathcal{U}_1 at a value of λ_2 , denoted $\lambda_2^* = 1.9$. This means that for $\lambda_2 \leq \lambda_2^*$, stream \mathcal{U}_2 has an average age higher than stream \mathcal{U}_1 . These observations show that not all values of λ_2 are suitable for our system. A small λ_2 will not ensure for stream \mathcal{U}_2 the priority it needs, whereas a large λ_2 will make the average age of stream \mathcal{U}_1 large and the system unstable.

Fig. 7b plots the same quantities as Fig. 7a but under different settings: in this case, $\mu_1 = 6$, $\mu_2 = 18$, $\lambda_1 = 4$ and $\lambda_2 < 9$. In this particular scenario, we notice that the lower bound is a tight bound on the simulated average age

for all values of λ_2 , and it is tighter than the average peak age.

B. Relative to the M/G/1/1 With Preemption Policy

A close observation of Equations (20) and (21) leads to the following remarks:

- If the service time for stream \mathcal{U}_2 is 0, $\Delta_1 = \frac{1}{\lambda_1 P_{\lambda_1}} \geq \frac{1}{\lambda_1 P_{\lambda_1}}$, where $\Delta = \frac{1}{\lambda_1 P_{\lambda_1}}$ is the value of the average of stream \mathcal{U}_1 if stream \mathcal{U}_2 is not present. This result is due to the fact that whenever a stream \mathcal{U}_2 packet is generated, it immediately preempts the stream \mathcal{U}_1 packet being served hence increases the instantaneous age of the latter stream.
- By using L'Hopital's rule, we can show that

$$\lim_{\lambda_2 \rightarrow 0} \Delta_1 = \Delta = \frac{1}{\lambda_1 P_{\lambda_1}}$$

as it is expected. The average peak-age also converges to its value when no stream \mathcal{U}_2 exists.

- *Special case:* assume $S^{(1)} \sim \text{Exp}(\mu_1)$ and $S^{(2)} \sim \text{Exp}(\mu_2)$. Then

$$\Delta_1 = \frac{(\mu_1 + \lambda_1)}{\lambda_1 \mu_1} \left(\frac{\mu_2 + \lambda_2}{\mu_2} \right) + \frac{\lambda_2}{\mu_2} \left(\frac{\mu_2 + \lambda_2}{\lambda_1 \mu_1} + \frac{1}{\mu_2 + \lambda_2} \right) \quad (25)$$

and

$$\Delta_{peak,1} = \frac{1}{\mu_1 + \lambda_1 + \lambda_2} + \frac{(\mu_1 + \lambda_1 + \lambda_2)(\mu_2 + \lambda_2)}{\lambda_1 \mu_1 \mu_2}. \quad (26)$$

Equation (25) coincides exactly to the result obtained by Kaul et al. in [12] for the stream with lowest priority and when we have two sources.

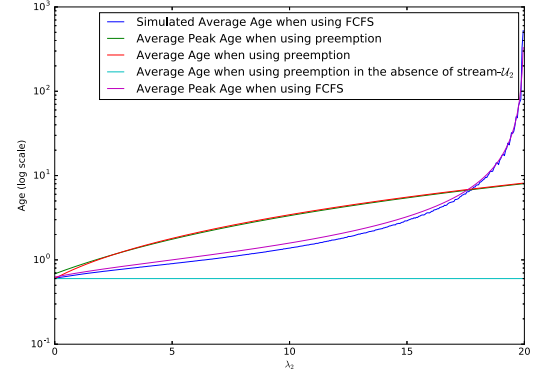
Denoting $\Delta_{Norm} = \frac{\mu_1 + \lambda_1}{\mu_1 \lambda_1}$ to be the average age of stream \mathcal{U}_1 when stream \mathcal{U}_2 does not exist (see [3]), we can compute the additional age the presence of stream \mathcal{U}_2 costs to stream \mathcal{U}_1 :

$$\begin{aligned} \Delta_{diff} &= \Delta_1 - \Delta_{Norm} \\ &= \frac{\lambda_2}{\lambda_1 \mu_1} + \frac{\lambda_2}{\lambda_1 \mu_2} + \frac{\lambda_2}{\mu_1 \mu_2} + \frac{\lambda_2^2}{\lambda_1 \mu_1 \mu_2} + \frac{\lambda_2}{\mu_2(\mu_2 + \lambda_2)}. \end{aligned}$$

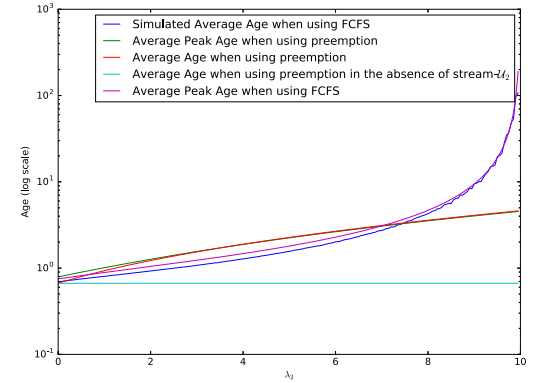
By letting $\mu_2 \rightarrow \infty$ we obtain $\Delta_{diff} \rightarrow \frac{\lambda_2}{\lambda_1 \mu_1} > 0$, and by taking $\lambda_2 = 0$ we obtain $\Delta_{diff} = 0$ as predicted by the previous two remarks.

C. Comparing the Two Policies

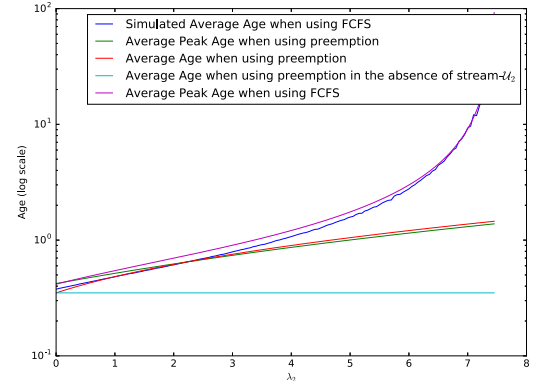
Using (11), (25) and (26), we compare the performance of the preemption policy on stream \mathcal{U}_1 with that of the FCFS scheme from an age point of view when the service times corresponding to both sources are exponential. Fig. 8a plots the average ages and average peak-ages relative to stream \mathcal{U}_1 for the preemption, as well as for the FCFS schemes. In both cases, we assume stream \mathcal{U}_1 packets are generated according to a Poisson process of rate $\lambda_1 = 2$ and served according to an exponential service time with rate $\mu_1 = 10$. As for stream \mathcal{U}_2 updates, they are generated according to a Poisson process with rate λ_2 and served according to an exponential service time with rate $\mu_2 = 5$. We observe from



(a) $\mu_1 = 10, \mu_2 = 5, \lambda_1 = 2$.



(b) $\mu_1 = 6, \mu_2 = 5, \lambda_1 = 2$.



(c) $\mu_1 = 10, \mu_2 = 5, \lambda_1 = 4$.

Fig. 8. Comparison between the average peak ages of the low priority source \mathcal{U}_1 when using the FCFS and the preemption schemes and assuming exponential service times.

Fig. 8a that the preemption scheme performs worse than the FCFS except when λ_2 is close to the FCFS stability condition. This observation comes as a surprise because we would think that the constraint of delivering all generated packets imposed by a FCFS system would pull the age up, compared to the more flexible preemptive scheme. However, we can explain this result in the following way: When using the preemptive scheme and not storing any updates, the system incurs a substantial idle time (from the source \mathcal{U}_1 point of view) during which it waits for a new stream- \mathcal{U}_1 update to be generated. In fact, this is a direct consequence of the

first remark in Section V-B. Moreover, Bedewy et al. in [14] show that for a single source and exponential service time, the optimal policy to adopt is the preemptive scheme. Fig. 8a proves that the introduction of an additional source with higher priority has a significant impact on the performances of the different transmission schemes. For instance, the preemption scheme is not optimal anymore even for exponential service times. However, we can notice that for λ_2 very close to 0, the M/G/1/1 scheme has a lower average age than the FCFS scheme, which is to be expected.

Fig. 8b and Fig. 8c also plot the average age and average peak-age relative to stream \mathcal{U}_1 for both schemes (preemption and FCFS). Compared to Fig. 8a, in Fig. 8b we keep $\lambda_1 = 2$ and $\mu_2 = 5$ but decrease the service rate of stream \mathcal{U}_1 to $\mu_1 = 6$. Whereas for Fig. 8c, we keep $\mu_1 = 10$ and $\mu_2 = 5$ but increase the generation rate of stream- \mathcal{U}_1 packets to $\lambda_1 = 4$. In Fig. 8b, we notice that by decreasing the service rate of stream \mathcal{U}_1 , the performances of the FCFS scheme and the M/G/1/1 scheme get closer compared to Fig. 8a. We can explain this behavior in the following way: the performances exhibited by the two transmission schemes in Fig. 8b are worse than their respective counterparts in Fig. 8a because every transmitted packet needs more time on average to be serviced. However, the FCFS scheme is more affected by this degradation in service than the M/G/1/1 scheme due to the compound effect of a slower service time on the waiting time of the packets in the queue. This means that the packets waiting in the queue will sustain a higher waiting time on average which will affect the age.

In Fig. 8c, we notice that by increasing the generation rate of stream \mathcal{U}_1 , the performances of the M/G/1/1 scheme becomes better than that of the FCFS scheme for $\lambda_2 \leq 1$ and $\lambda_2 \geq 2.5$, while the two performances are very close in the interval $1 \leq \lambda_2 \leq 2.5$. This shows that λ_1 increases, preemption performs better than FCFS because of two simultaneous effects:

- 1) The idle time incurred by the M/G/1/1/ system waiting for a new packet is reduced. Hence the impact of the presence of stream \mathcal{U}_2 , as explained in the first remark of Section V-B, is decreased.
- 2) As λ_1 increases, the queue for the FCFS system becomes more congested. This leads to an increase of the average waiting time sustained by the packets.

These two effects explain why we notice a paradigm shift and why in this case it is better to adopt an M/G/1/1 scheme.

To sum up, Fig. 8 shows that, for Poisson generation process and exponential service time, the M/G/1/1 scheme is not optimal anymore and that the FCFS scheme might perform better depending on the values of λ_1 , λ_2 , μ_1 and μ_2 .

VI. CONCLUSION

In this paper, we have studied the effect of implementing content-dependent policies on the average age of the packets. We have considered a sender that generates two independent Poisson streams with one stream having higher priority than the other stream. The “high priority” stream is sent using a preemption policy, whereas at first the “regular” stream is transmitted using a FCFS policy and then it is transmitted

using preemption. We derived the stability condition for the former system, as well as closed-form expressions for the average peak-age and a lower bound on the average age of the “regular” stream. For the latter system we have given exact expressions for the average age and average peak-age of the “regular” stream and we have shown through simulations that, even if the service times relative to both streams are exponential, preemption is not the optimal strategy to adopt for the “regular” stream. In fact, for some fixed service rates and “regular” stream generation rate, the FCFS strategy performs better for a large interval of “high priority”- stream generation rate.

VII. APPENDIX

A. Proof of Theorem 1

Theorem. *The system described in Section III is stable, i.e. the average number of packets in the queue is finite, if and only if*

$$\mu_1 > \lambda_1 \left(1 + \frac{\lambda_2}{\mu_2} \right). \tag{27}$$

In this case the Markov chain shown in Fig. 3 has a stationary distribution $\Pi = [\pi_0, \pi_1, \dots, \pi_i, \dots, \pi'_1, \dots, \pi'_i, \dots]$, where π_i denotes the stationary probability of state q_i , $i \geq 0$, and π'_i denotes the stationary probability of state q'_i , $i > 0$. This stationary distribution is described by the following system of equations,

$$\pi_0 = \frac{\mu_2}{\mu_2 + \lambda_2} - \frac{\lambda_1}{\mu_1}, \tag{28}$$

$$\begin{bmatrix} \pi_i \\ \pi'_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 \end{bmatrix} \mathbf{H}^i \begin{bmatrix} \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\lambda_1 + \mu_2)} \\ \frac{\lambda_2}{\lambda_1 + \mu_2} \\ 1 \\ 0 \end{bmatrix} \pi_0, \quad i \geq 1 \tag{29}$$

where $\lambda = \lambda_1 + \lambda_2$, $\mathbf{H} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{I}_2 & \mathbf{0} \end{bmatrix}$,

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\mu_2 + \lambda_1)} & -\frac{\mu_2 \lambda_1}{\mu_1(\mu_2 + \lambda_1)} \\ \frac{\lambda_2}{\mu_2 + \lambda_1} & \frac{\lambda_1}{\mu_2 + \lambda_1} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -\frac{\lambda_1}{\mu_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

\mathbf{I}_2 is the 2×2 identity matrix and $\mathbf{0}$ is the 2×2 zero matrix.

Proof. Assume that

$$\mu_1 > \lambda_1 \left(1 + \frac{\lambda_2}{\mu_2} \right). \tag{30}$$

The detailed balance equations of the Markov chain given by Fig. 3 are given by:

$$\begin{cases} \lambda \pi_0 = \mu_1 \pi_1 + \mu_2 \pi'_1, \\ (\lambda_1 + \mu_2) \pi'_1 = \lambda_2 \pi_0, \\ \text{for } i \geq 1, \\ \pi_{i+1} = \left(1 + \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\mu_2 + \lambda_1)} \right) \pi_i - \frac{\mu_2 \lambda_1}{\mu_1(\mu_2 + \lambda_1)} \pi'_i \\ \quad - \frac{\lambda_1}{\mu_1} \pi_{i-1}, \\ \pi'_{i+1} = \frac{\lambda_2}{\mu_2 + \lambda_1} \pi_i + \frac{\lambda_1}{\mu_2 + \lambda_1} \pi'_i, \end{cases} \tag{31}$$

where $\lambda = \lambda_1 + \lambda_2$. For easier notation we denote

$$\begin{aligned} a_1 &= 1 + \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\mu_2 + \lambda_1)}, \\ a_2 &= \frac{\mu_2 \lambda_1}{\mu_1(\mu_2 + \lambda_1)}, \\ a_3 &= \frac{\lambda_1}{\mu_1}, \\ a_4 &= \frac{\lambda_2}{\mu_2 + \lambda_1}, \\ a_5 &= \frac{\lambda_1}{\mu_2 + \lambda_1}. \end{aligned}$$

Rewriting (31) in matrix form and using the above notation, we get

$$\begin{bmatrix} \pi_{i+1} \\ \pi'_{i+1} \\ \pi_i \\ \pi'_i \end{bmatrix} = \begin{bmatrix} a_1 & -a_2 & -a_3 & 0 \\ a_4 & a_5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_i \\ \pi'_i \\ \pi_{i-1} \\ \pi'_{i-1} \end{bmatrix}.$$

Let $\mathbf{A}_i = \begin{bmatrix} \pi_{i+1} \\ \pi'_{i+1} \\ \pi_i \\ \pi'_i \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} a_1 & -a_2 \\ a_4 & a_5 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -a_3 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{H} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{I}_2 & \mathbf{0} \end{bmatrix}$. Then

$$\mathbf{A}_i = \mathbf{H}\mathbf{A}_{i-1}.$$

Thus

$$\mathbf{A}_i = \mathbf{H}^i \mathbf{A}_0, \quad i \geq 0 \quad (32)$$

where $\mathbf{A}_0 = \begin{bmatrix} \pi_1 \\ \pi'_1 \\ \pi_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1(\lambda_1 + \mu_2)} \\ \frac{\lambda_2}{\lambda_1 + \mu_2} \\ 1 \\ 0 \end{bmatrix} \pi_0$, using the first

two equations of system (31).

(32) shows that in order to find the stability criterion of the system in (31) we first need to study the properties of \mathbf{H} . For that we compute its eigenvalues l_0, l_1, l_2, l_3 by solving the characteristic equation $|\mathbf{I}_4 - \mathbf{H}| = 0$. This leads to

$$|\mathbf{I}_4 - \mathbf{H}| = l(l-1)(l^2 - l(a_1 + a_5 - 1) + a_3 a_5). \quad (33)$$

\mathbf{H} has two obvious eigenvalues $l_0 = 0$ and $l_3 = 1$. To find the last two eigenvalues, let's find the root of the quadratic polynomial

$$p(l) = l^2 - l(a_1 + a_5 - 1) + a_3 a_5. \quad (34)$$

It can be shown through simple algebra that the discriminant of the above polynomial is strictly positive. Hence the remaining eigenvalues l_1 and l_2 are real and distinct. Let's assume that $l_1 < l_2$. This means that the matrix \mathbf{H} is diagonalizable and can be written as

$$\mathbf{H} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1},$$

where the columns of \mathbf{B} are the eigenvectors of \mathbf{H} and form a basis of \mathbb{R}^4 . We denote by $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the eigenvectors corresponding to l_0, l_1, l_2, l_3 .

So we can write \mathbf{A}_0 as

$$\mathbf{A}_0 = (\alpha_0 \mathbf{e}_0 + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3) \pi_0, \quad (35)$$

with $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Hence for $i > 0$,

$$\begin{aligned} \mathbf{A}_i &= \mathbf{H}^i \mathbf{A}_0 \\ &= (\alpha_0 \mathbf{H}^i \mathbf{e}_0 + \alpha_1 \mathbf{H}^i \mathbf{e}_1 + \alpha_2 \mathbf{H}^i \mathbf{e}_2 + \alpha_3 \mathbf{H}^i \mathbf{e}_3) \pi_0 \\ &= (\alpha_0 l_0^i \mathbf{e}_0 + \alpha_1 l_1^i \mathbf{e}_1 + \alpha_2 l_2^i \mathbf{e}_2 + \alpha_3 l_3^i \mathbf{e}_3) \pi_0 \\ &= (\alpha_1 l_1^i \mathbf{e}_1 + \alpha_2 l_2^i \mathbf{e}_2 + \alpha_3 \mathbf{e}_3) \pi_0, \end{aligned} \quad (36)$$

since $l_0 = 0$ and $l_3 = 1$. Equation (36) shows that three conditions need to be satisfied for the system to be stable and a steady-state distribution to exist:

- **Condition 1:** $|l_1| < 1$ and $|l_2| < 1$.
- **Condition 2:** $\alpha_3 = 0$.
- **Condition 3:** $\alpha_1 l_1^i \mathbf{e}_1 + \alpha_2 l_2^i \mathbf{e}_2$ has positive components for all $i > 0$.

Condition 1 and **Condition 2** ensure that

$$\lim_{i \rightarrow \infty} \pi_i = \lim_{i \rightarrow \infty} \pi'_i = 0$$

and thus the sum of all probabilities, $\pi_0 + \sum_{i=1}^{\infty} (\pi_i + \pi'_i)$, does not diverge. **Condition 3** makes sure that the components of \mathbf{A}_i are positive probabilities. We will show that (30) is sufficient for the above three conditions to hold.

Given that l_1 and l_2 are the roots of (34) then the following holds

$$\begin{aligned} l_1 l_2 &= a_3 a_5 \\ l_1 + l_2 &= a_1 + a_5 - 1. \end{aligned} \quad (37)$$

However, $l_1 l_2 = a_3 a_5 = \frac{\lambda_1^2}{\mu_1(\mu_2 + \lambda_1)} \geq 0$. This means that either both l_1 and l_2 are positive or they are both negative. Using (37) again, we notice that

$$l_1 + l_2 = a_1 + a_5 - 1 = \frac{\lambda_1 \mu_2 + \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 \mu_1}{\mu_1(\mu_2 + \lambda_1)} \geq 0.$$

This shows that both l_1 and l_2 are strictly positive (since 0 is not a root of $p(l)$). So to prove that **Condition 1** is satisfied we need to prove that $l_1 < l_2 < 1$. This is equivalent to show that (34) evaluated at 1 is strictly positive and that $l_1 l_2 < 1$ since $p(l)$ is a convex quadratic function in $l > 0$. Using simple algebra it can be shown that

$$p(1) = 1 - (a_1 + a_5 - 1) + a_3 a_5 = \frac{\mu_1 \mu_2 - \lambda_1(\mu_2 + \lambda_2)}{\mu_1(\mu_2 + \lambda_1)} > 0,$$

where the last inequality is due to (30). Moreover, (30) tells us that μ_1 should be strictly bigger than λ_1 . Thus we get that

$$l_1 l_2 = \frac{\lambda_1}{\mu_1} \frac{\lambda_1}{\mu_2 + \lambda_1} < 1.$$

This shows that $0 < l_1 < l_2 < 1$ and that **Condition 1** is satisfied.

To prove **Condition 2** we start by computing the eigenvectors of \mathbf{H} . For $l_0 = 0$, we solve the system given by $\mathbf{H}\mathbf{e}_0 = \mathbf{0}$. If $\mathbf{e}_0 = [u_1 \ u_2 \ u_3 \ 1]^T$ then

$$\begin{bmatrix} a_1 & -a_2 & -a_3 & 0 \\ a_4 & a_5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system leads to $\mathbf{e}_0 = [0 \ 0 \ 0 \ 1]^T$. Similarly, for $j = 1, 2, 3$, if $\mathbf{e}_j = [u_1 \ u_2 \ u_3 \ 1]^T$ then solving the system

$$\begin{bmatrix} a_1 & -a_2 & -a_3 & 0 \\ a_4 & a_5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 1 \end{bmatrix} = l_j \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 1 \end{bmatrix}$$

leads to $\mathbf{e}_j = [l_j(l_j - a_5) \ l_j a_4 \ l_j - a_5 \ a_4]^T$.

We know that $\mathbf{H} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$. If

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then

$$\mathbf{B} = \begin{bmatrix} 1 - a_5 & l_2(l_2 - a_5) & l_1(l_1 - a_5) & 0 \\ a_4 & l_2 a_4 & l_1 a_4 & 0 \\ 1 - a_5 & l_2 - a_5 & l_1 - a_5 & 0 \\ a_4 & a_4 & a_4 & 1 \end{bmatrix}.$$

Note that the determinant of \mathbf{B} , $|\mathbf{B}|$, is non-zero when we assume (30). Indeed,

$$|\mathbf{B}| = a_4 a_5 (l_2 - l_1) (-2 + a_5 - a_3 a_5 + a_1) < 0$$

since $l_2 > l_1$ and $-2 + a_5 - a_3 a_5 + a_1 = -p(1) < 0$ as shown before. In order to compute α_3 , we rewrite (35) as follows

$$\mathbf{A}_0 = [\mathbf{e}_3 \ \mathbf{e}_2 \ \mathbf{e}_1 \ \mathbf{e}_0] \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} \pi_0 = \mathbf{B} \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} \pi_0.$$

But we also know that

$$\mathbf{A}_0 = \begin{bmatrix} \frac{\lambda}{\mu_1} - \frac{\mu_2 \lambda_2}{\mu_1 (\lambda_1 + \mu_2)} \\ \frac{\lambda_2}{\lambda_1 + \mu_2} \\ 1 \\ 0 \end{bmatrix} \pi_0 = \begin{bmatrix} a_1 - 1 \\ a_4 \\ 1 \\ 0 \end{bmatrix} \pi_0.$$

Thus

$$\mathbf{B} \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} a_1 - 1 \\ a_4 \\ 1 \\ 0 \end{bmatrix}. \tag{38}$$

Solving the system in (38) with respect to $\alpha_3, \alpha_2, \alpha_1$ and α_0 we get that

$$\begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{l_2 - l_1} \\ \frac{-1}{l_2 - l_1} \\ 0 \end{bmatrix}.$$

Thus $\alpha_3 = 0$ and **Condition 2** is proved. Note that we didn't need any assumptions to prove this condition.

Given the above results, we can now rewrite the system in (36) as

$$\begin{cases} \mathbf{A}_i = (\alpha_2 l_2^i \mathbf{e}_2 + \alpha_1 l_1^i \mathbf{e}_1) \pi_0, & i > 0 \\ \mathbf{A}_0 = (\alpha_2 \mathbf{e}_2 + \alpha_1 \mathbf{e}_1) \pi_0 = \begin{bmatrix} a_1 - 1 \\ a_4 \\ 1 \\ 0 \end{bmatrix} \pi_0. \end{cases} \tag{39}$$

Using (39) we can prove **Condition 3**. In fact, for any $i > 0$,

$$\begin{aligned} \alpha_2 l_2^i \mathbf{e}_2 + \alpha_1 l_1^i \mathbf{e}_1 &\stackrel{(a)}{=} \alpha_2 (l_2^i \mathbf{e}_2 - l_1^i \mathbf{e}_1) \\ &\stackrel{(b)}{>} \alpha_2 l_1^i (\mathbf{e}_2 - \mathbf{e}_1) \\ &\stackrel{(c)}{=} l_1^i \begin{bmatrix} a_1 - 1 \\ a_4 \\ 1 \\ 0 \end{bmatrix} \\ &\stackrel{(d)}{>} \mathbf{0}, \end{aligned}$$

where $\mathbf{x} > \mathbf{y}$ for some vectors \mathbf{x} and \mathbf{y} means that the components of $\mathbf{x} - \mathbf{y}$ are strictly positive and

- (a) is because $\alpha_2 = -\alpha_1$,
- (b) is because $0 < l_1 < l_2$,
- (c) is obtained from the second equality in (39),
- (d) follows since $a_1 - 1 > 0$ and $a_4 > 0$.

Up till now we have shown that if $\mu_1 > \lambda_1 \left(1 + \frac{\lambda_2}{\mu_2}\right)$, the system described in Section II is stable and a steady-state distribution exists given by (39). The final point to prove in Theorem 1 is the expression of π_0 . For that we solve for π_0 the following equation

$$\pi_0 + \sum_{i=1}^{\infty} \pi_i + \pi'_i = \pi_0 + [0 \ 0 \ 1 \ 1] \sum_{i=1}^{\infty} \mathbf{A}_i = 1.$$

Using the first equation of (39) and replacing α_1 and α_2 by their expressions in function of l_1 and l_2 , using the fact that $l_1 + l_2$ and $l_1 l_2$ are given by (37) and finally replacing a_1, a_2, a_3, a_4 and a_5 by their expressions in function of $\lambda_1, \lambda_2, \mu_1, \mu_2$ we get

$$\pi_0 = \frac{\mu_2}{\mu_2 + \lambda_2} - \frac{\lambda_1}{\mu_1}.$$

□

B. Proof of Corollary 1

Corollary. *If we define $N(t)$ to be the number of stream \mathcal{U}_1 packets in the system at time t , then its moment generating function is $\phi_{N(t)}$*

$$\phi_{N(t)}(s) = \pi_0 \left(\frac{\mu_1 (\lambda_1 + \lambda_2 + \mu_2 - \lambda_1 e^s)}{\mu_1 \mu_2 + \mu_1 \lambda_1 - e^s (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 \mu_1 + \lambda_1 \mu_2) + \lambda_1^2 e^{2s}} \right), \tag{40}$$

where π_0 is given by (4). Particularly, the expected value of $N(t)$ is

$$\mathbb{E}(N(t)) = \frac{\lambda_1 (2\lambda_2 \mu_2 + \lambda_2 \mu_1 + \lambda_2^2 + \mu_2^2)}{(\mu_2 + \lambda_2) (\mu_1 \mu_2 - \lambda_1 (\mu_2 + \lambda_2))}. \tag{41}$$

Proof. At any point in time, there are exactly i stream \mathcal{U}_1 packets in the system if we are in state q_i or q'_{i+1} in the Markov chain given by Fig. 3. This means that the probability of having exactly i stream \mathcal{U}_1 packets in the system is $\pi_i + \pi'_{i+1}$. Hence, using the same quantities as in Section VII-A

$$\phi_{N(t)}(s) = \sum_{n=0}^{\infty} e^{sn} (\pi_n + \pi'_{n+1}) = \sum_{n=0}^{\infty} e^{sn} \left(\mathbf{A}_n^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

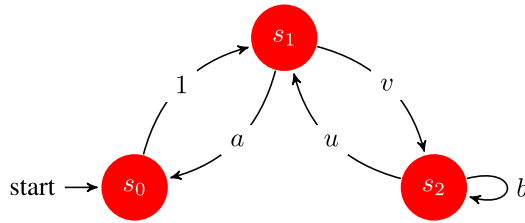


Fig. 9. Semi-Markov chain representing the “virtual” service time Y_j .

$$\begin{aligned}
&= \sum_{n=0}^{\infty} e^{sn} \alpha_2 \pi_0 \left((l_2^n \mathbf{e}_2 - l_1^n \mathbf{e}_1)^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \alpha_2 \pi_0 \left(\sum_{n=0}^{\infty} (e^s l_2)^n \mathbf{e}_2^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \sum_{n=0}^{\infty} (e^s l_1)^n \mathbf{e}_1^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \alpha_2 \pi_0 \left(\frac{1}{1-l_2 e^s} (l_2 a_4 + l_2 - a_5) - \frac{1}{1-l_1 e^s} (l_1 a_4 + l_1 - a_5) \right) \\
&= \alpha_2 \pi_0 (l_2 - l_1) \left(\frac{a_4 + 1 - a_5 e^s}{1 - (l_1 + l_2) e^s + l_1 l_2 e^{2s}} \right). \quad (42)
\end{aligned}$$

where the quantities used here are the one defined in the proof of Theorem 1. Thus,

$$\phi_{N(t)}(s) = \pi_0 \left(\frac{\mu_1 (\lambda_1 + \lambda_2 + \mu_2 - \lambda_1 e^s)}{\mu_1 \mu_2 + \mu_1 \lambda_1 - e^s (\lambda_1 \mu_2 + \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 \mu_1) + \lambda_1^2 e^{2s}} \right).$$

This last equality is obtained by using (37), $\alpha_2 = \frac{1}{l_2 - l_1}$ and replacing a_1, a_2, a_3, a_4, a_5 by their expressions in function of $\lambda_1, \lambda_2, \mu_1$ and μ_2 in (42). Finally,

$$\mathbb{E}(N(t)) = \left. \frac{d\phi_{N(t)}(s)}{ds} \right|_{s=0} = \frac{\lambda_1 (2\lambda_2 \mu_2 + \lambda_2 \mu_1 + \lambda_2^2 + \mu_2^2)}{(\mu_2 + \lambda_2)(\mu_1 \mu_2 - \lambda_1 (\mu_2 + \lambda_2))}. \quad \square$$

C. Proof of Lemma 1 and overview on the detour flow graph method

Lemma. Let Y_j be the “virtual” service time of packet j given that this packet does not find the system in state q'_1 , i.e. $\mathbb{P}(Y_j > t) = \mathbb{P}(Z_j > t | \overline{\Psi_j})$. Then, in steady state,

$$\phi_Y(s) = \mathbb{E}(e^{sY}) = \frac{\mu_1 (\mu_2 - s)}{s^2 - s(\mu_2 + \mu_1 + \lambda_2) + \mu_1 \mu_2}. \quad (43)$$

Similarly, let Y'_j be the “virtual” service time of packet j given that this packet finds the system in state q'_1 , i.e. $\mathbb{P}(Y'_j > t) = \mathbb{P}(Z_j > t | \Psi_j)$. Then, in steady state,

$$\phi_{Y'}(s) = \mathbb{E}(e^{sY'}) = \frac{\mu_1 \mu_2}{s^2 - s(\mu_2 + \mu_1 + \lambda_2) + \mu_1 \mu_2}. \quad (44)$$

Proof. We start by proving (43). For this, we use the detour flow graph method. Fig. 9 shows the semi-Markov chain relative to the “virtual” service time Y_j of the j^{th} packet of first stream \mathcal{U}_1 . Since the system is ergodic, it also applies to any packet at steady state. This chain is constituted of three states:

- s_0 : in this state, the system is idle from a stream- \mathcal{U}_1 point of view. This means that no stream- \mathcal{U}_1 packet is being served.
- s_1 : in this state, a stream- \mathcal{U}_1 packet is in service.
- s_2 : in this state, a stream- \mathcal{U}_2 packet is in service.

When the j^{th} packet reaches the head of the buffer, the system is in the idle state s_0 . Hence, with probability 1 it goes immediately to state s_1 where it starts serving the j^{th} packet. Due to the memoryless property of the interarrival time of the second stream $X^{(2)}$, two clocks start: a service clock $S^{(1)}$ and a clock $X^{(2)}$. The service clock ticks first with probability $a = \mathbb{P}(S^{(1)} < X^{(2)})$ and its value A has distribution $\mathbb{P}(A > t) = \mathbb{P}(S^{(1)} > t | S^{(1)} < X^{(2)})$. At this point, the stream \mathcal{U}_1 packet, currently being served, finishes service before any packet from the other stream is generated, and the system goes back to state s_0 . This ends the “virtual” service time Y_j . Clock $X^{(2)}$ ticks first with probability $v = 1 - a = \mathbb{P}(X^{(2)} < S^{(1)})$ and its value V has distribution $\mathbb{P}(V > t) = \mathbb{P}(X^{(2)} > t | X^{(2)} < S^{(1)})$. At this point, a new stream \mathcal{U}_2 update is generated and preempts the stream \mathcal{U}_1 packet currently in service. In this case, the system goes to state s_2 , where the preempted stream \mathcal{U}_1 update is placed back at the head of the buffer, and the system starts service of the stream \mathcal{U}_2 update.

When the system arrives in state s_2 , this means a new stream \mathcal{U}_2 packet was just generated and is starting its service. Thus, two clocks start: a service clock $S^{(2)}$ and a clock $X^{(2)}$. The service clock ticks first with probability $u = \mathbb{P}(S^{(2)} < X^{(2)})$ and its value U has distribution $\mathbb{P}(U > t) = \mathbb{P}(S^{(2)} > t | S^{(2)} < X^{(2)})$. At this point, the packet currently being served finishes service before any new stream \mathcal{U}_2 packet is generated, and the system goes back to state s_1 where the j^{th} packet of stream \mathcal{U}_1 starts its service again. However, clock $X^{(2)}$ ticks first with probability $b = 1 - u$, and its value B has distribution $\mathbb{P}(B > t) = \mathbb{P}(X^{(2)} > t | X^{(2)} < S^{(2)})$. At this point, a new stream \mathcal{U}_2 update is generated and preempts the one currently in service. In this case, the system stays in state s_2 .

From the above analysis, we see that the “virtual” service time is given by the sum of the values of the different clocks on the path starting and finishing at s_0 . For example, for the path $s_0 s_1 s_2 s_1 s_2 s_2 s_1 s_0$ in Fig. 9, the “virtual” service time $Y = V_1 + U_1 + V_2 + B_1 + U_2 + A_1$, where all the random variables in the sum are mutually independent. This value of Y is also valid for the path $s_0 s_1 s_2 s_2 s_1 s_2 s_1 s_0$. Hence, Y depends on the variables A_j, B_j, U_j, V_j and their number of occurrences and not on the path itself. Therefore, the probability that exactly (i_1, i_2, i_3, i_4) occurrences of (A, B, U, V) occur, which is equivalent to the probability that

$$Y = \sum_{k=1}^{i_1} A_k + \sum_{k=1}^{i_2} B_k + \sum_{k=1}^{i_3} U_k + \sum_{k=1}^{i_4} V_k$$

is given by $a^{i_1} b^{i_2} u^{i_3} v^{i_4} Q(i_1, i_2, i_3, i_4)$, where $Q(i_1, i_2, i_3, i_4)$ is the number of paths with this combination of occurrences. Taking into account the fact that the $\{A_k, B_k, U_k, V_k\}$ are mutually independent and denoting by $\{I_1, I_2, I_3, I_4\}$ the random variables associated with the number of occurrences

of $\{A, B, U, V\}$ respectively, the moment generating function of Y is,

$$\begin{aligned} \phi_Y(s) &= \mathbb{E} \left(\mathbb{E} \left(e^{sY} \mid (I_1, I_2, I_3, I_4) = (i_1, i_2, i_3, i_4) \right) \right) \\ &= \sum_{i_1, i_2, i_3, i_4} \left[a^{i_1} b^{i_2} u^{i_3} v^{i_4} Q(i_1, i_2, i_3, i_4) \right. \\ &\quad \left. \mathbb{E} \left(e^{s(\sum_{k=1}^{i_1} A_k + \sum_{k=1}^{i_2} B_k + \sum_{k=1}^{i_3} U_k + \sum_{k=1}^{i_4} V_k)} \right) \right] \\ &= \sum_{i_1, i_2, i_3, i_4} \left[a^{i_1} b^{i_2} u^{i_3} v^{i_4} Q(i_1, i_2, i_3, i_4) \right. \\ &\quad \left. \mathbb{E} \left(e^{sA} \right)^{i_1} \mathbb{E} \left(e^{sB} \right)^{i_2} \mathbb{E} \left(e^{sU} \right)^{i_3} \mathbb{E} \left(e^{sV} \right)^{i_4} \right]. \quad (45) \end{aligned}$$

In order to compute (45), we modify the state diagram in Fig. 9 and represent it as a *detour flow graph* (also called *signal flow graph* [19], [20]) as shown in Fig. 10a. For that, we first notice that the “virtual” service time Y is the interval of time spent by the system between two consecutive s_0 states. That’s why, in Fig. 10a, we split the s_0 state into two states: a starting state s_0 and an end state \bar{s}_0 . Hence, there is a one-to-one correspondence between the different paths from s_0 to \bar{s}_0 and the different combinations in which we can write Y in function of (A, B, U, V) . In order to capture the number of occurrences of the quantities (A, B, U, V) over a certain path, we associate with each label four “dummy” variables (D_1, D_2, D_3, D_4) and the exponents of (D_1, D_2, D_3, D_4) correspond to the number of occurrences of (A, B, U, V) respectively. For example, if between two states s_j and s_i (for any i, j), the edge has a label that contains the factor $D_1 D_4^2$, then it means that the system spent a time of $A + 2V$ when passing from s_j to s_i . Since we are interested in the distribution of Y , we multiply the labels of the edges in the detour flow graph by the probability of such label being visited. For example, given that the system is at state s_1 , it jumps to state s_2 with probability v and after spending a time V . Thus the label from s_1 to s_2 is vD_4 .

Using Mason’s gain formula [19], we know that the generating function $H_1(D_1, D_2, D_3, D_4)$ of the detour flow graph shown in Fig. 10a, can be written as

$$\begin{aligned} H_1(D_1, D_2, D_3, D_4) &= \sum_{i_1, i_2, i_3, i_4} \left[Q(i_1, i_2, i_3, i_4) a^{i_1} b^{i_2} u^{i_3} v^{i_4} D_1^{i_1} D_2^{i_2} D_3^{i_3} D_4^{i_4} \right], \quad (46) \end{aligned}$$

where $Q(i_1, i_2, i_3, i_4)$ is the number of paths with i_1 occurrences of A , i_2 occurrences of B , i_3 occurrences of U , i_4 occurrences of V . Comparing (45) and (46), we notice that

$$\phi_Y(s) = H_1 \left(\mathbb{E} \left(e^{sA} \right), \mathbb{E} \left(e^{sB} \right), \mathbb{E} \left(e^{sU} \right), \mathbb{E} \left(e^{sV} \right) \right).$$

Moreover, given a directed graph $G = (V, E)$ with algebraic label $L(e)$ on its edges, and a node $u \in V$ with no incoming edges, the transfer function $H(v)$ from u to a node v is the sum over of all paths from u to v with each path contributing the product of its edge labels to the sum (see [19]–[21]). The complete set of transfer functions $\{H(v) : v \in V\}$ can be computed easily by solving the linear equations:

$$\begin{cases} H(u) = 1 \\ H(w) = \sum_{w': (w', w) \in E} H(w') L((w', w)), \quad w \neq u. \end{cases}$$

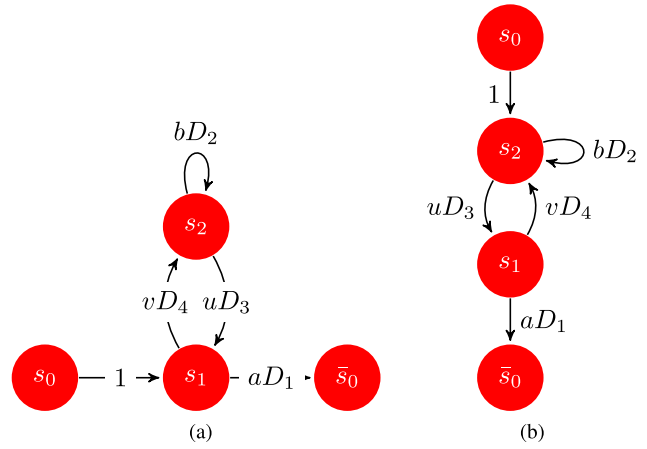


Fig. 10. Detour flow graphs for (a) Y and (b) Y' .

Solving the system of linear equations above for the detour flow graph of Fig. 10a, (46) becomes

$$H_1(D_1, D_2, D_3, D_4) = \frac{aD_1(1 - bD_2)}{1 - bD_2 - uD_3vD_4}. \quad (47)$$

From [8, Appendix A, Lemma 2], we know that A, B, U and V are exponentially distributed with $\mathbb{E} \left(e^{sB} \right) = \mathbb{E} \left(e^{sU} \right) = \frac{\lambda_2 + \mu_2}{\lambda_2 + \mu_2 - s}$ and $\mathbb{E} \left(e^{sA} \right) = \mathbb{E} \left(e^{sV} \right) = \frac{\lambda_2 + \mu_1}{\lambda_2 + \mu_1 - s}$. Simple computations show that $a = \frac{\mu_1}{\mu_1 + \lambda_2}$, $b = \frac{\lambda_2}{\mu_2 + \lambda_2}$, $u = \frac{\mu_2}{\mu_2 + \lambda_2}$, $v = \frac{\lambda_2}{\mu_1 + \lambda_2}$. Finally, replacing the above expressions into (47), we get our result.

To prove (44), we use the same method as before. But in this case, we notice that the j^{th} packet from stream \mathcal{U}_1 finds the system busy serving a packet from stream \mathcal{U}_2 . This translates in the detour flow graph shown in Fig. 10b. The generating function of this graph is

$$H_2(D_1, D_2, D_3, D_4) = \frac{aD_1uD_3}{1 - bD_2 - vD_4uD_3}. \quad (48)$$

For $(D_1, D_2, D_3, D_4) = (\mathbb{E} \left(e^{sA} \right), \mathbb{E} \left(e^{sB} \right), \mathbb{E} \left(e^{sU} \right), \mathbb{E} \left(e^{sV} \right))$ and replacing a, b, u and v by their values in (48), we obtain (44). \square

D. Proof of Lemma 2

Lemma. Assume an M/G/1 queue with interarrival time $X^{(1)}$ exponentially distributed with rate λ_1 and service time Y whose moment generating function is given by (9). The service time and the interarrival time are assumed to be independent. Then the distribution of the system time T is

$$f_T(t) = C_1 e^{-\alpha_1 t} (\mu_2 - \alpha_1) - C_1 e^{-\alpha_2 t} (\mu_2 - \alpha_2), \quad t \geq 0, \quad (49)$$

where $\alpha_1, \alpha_2 > 0$ are the roots of the quadratic expression

$$\begin{aligned} s^2 - s(\mu_1 + \mu_2 + \lambda_2 - \lambda_1) + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_1\lambda_2, \\ C_1 = \frac{(1 - \rho)\mu_1}{\alpha_2 - \alpha_1}, \end{aligned}$$

$$\text{and } \rho = \lambda_1 \mathbb{E} \left(Y \right) = \frac{\lambda_1(\mu_2 + \lambda_2)}{\mu_1\mu_2}.$$

Proof. From [22, p. 166], we know that the Laplace transform of the system time T is

$$\mathbb{E}(e^{-sT}) = \frac{(1-\rho)s\phi_Y(-s)}{s - \lambda_1(1 - \phi_Y(-s))}.$$

Replacing $\phi_Y(-s)$ by its expression in (43) we get

$$\begin{aligned} \mathbb{E}(e^{-sT}) &= \frac{(1-\rho)\mu_1(\mu_2+s)}{s^2+s(\mu_1+\mu_2+\lambda_2-\lambda_1)+\mu_1\mu_2-\lambda_1\mu_2-\lambda_1\lambda_2} \\ &= \frac{(1-\rho)\mu_1(\mu_2+s)}{(s-s_1)(s-s_2)} \\ &= s \frac{(1-\rho)\mu_1}{(s-s_1)(s-s_2)} + \frac{(1-\rho)\mu_1\mu_2}{(s-s_1)(s-s_2)}, \end{aligned} \quad (50)$$

where s_1 and s_2 are two real roots of the quadratic equation

$$s^2 + s(\mu_1 + \mu_2 + \lambda_2 - \lambda_1) + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_1\lambda_2.$$

Moreover, due to condition (3),

$$s_1 + s_2 = -\mu_1 - \mu_2 - \lambda_2 + \lambda_1 < 0$$

and

$$s_1s_2 = \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_1\lambda_2 > 0.$$

This proves that both roots s_1 and s_2 are negative. Let

$$G(s) = \frac{(1-\rho)\mu_1}{(s-s_1)(s-s_2)},$$

and $g(t)$ its inverse Laplace transform. Using the initial value theorem:

$$g(0^+) = \lim_{s \rightarrow \infty} sG(s) = 0. \quad (51)$$

Using (51) and the expression of $G(s)$, (50) can be written as

$$\mathbb{E}(e^{-sT}) = sG(s) - g(0^+) + \mu_2G(s). \quad (52)$$

Therefore, the probability density function of the system time $f_T(t)$ (which is the inverse Laplace transform of $\mathbb{E}(e^{-sT})$) is

$$f_T(t) = g'(t) + \mu_2g(t). \quad (53)$$

By partial fraction expansion,

$$G(s) = \frac{C_1}{s-s_1} - \frac{C_1}{s-s_2},$$

where $C_1 = \frac{(1-\rho)\mu_1}{s_1-s_2}$. Denoting $\alpha_1 = -s_1$ and $\alpha_2 = -s_2$, with $\alpha_1, \alpha_2 > 0$, we get

$$G(s) = \frac{C_1}{s+\alpha_1} - \frac{C_1}{s+\alpha_2}, \text{ and } C_1 = \frac{(1-\rho)\mu_1}{\alpha_2 - \alpha_1}.$$

Thus,

$$g(t) = C_1 e^{-\alpha_1 t} - C_1 e^{-\alpha_2 t},$$

and

$$f_T(t) = C_1 e^{-\alpha_1 t}(\mu_2 - \alpha_1) - C_1 e^{-\alpha_2 t}(\mu_2 - \alpha_2).$$

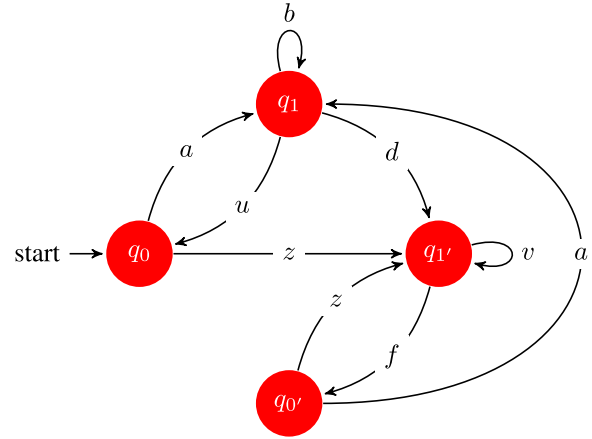


Fig. 11. Semi-Markov chain representing the M/G/1/1 interdeparture time for stream \mathcal{U}_1 .

E. Proof of Lemma 4

Lemma. The moment generating function of the interdeparture time of stream \mathcal{U}_1 , Y , is

$$\phi_Y(s) = \frac{\lambda_1 P_{\lambda-s}(\lambda_2 L_{\lambda_2-s} - s)}{\lambda_1 P_{\lambda-s}(\lambda_2 L_{\lambda_2-s} - s) - s(\lambda_2 - s)}. \quad (54)$$

Proof. We use the detour flow graph method. We define $\Lambda = \min(X^{(1)}, X^{(2)})$. As Λ is the minimum of independent exponential random variables, then it is also exponentially distributed with rates $\lambda = \lambda_1 + \lambda_2$. Fig. 11 shows the semi-Markov chain relative to the interdeparture time Y_j between the j^{th} and $j+1^{\text{th}}$ successfully received packet of stream \mathcal{U}_1 . This chain is composed of 4 states:

- q_0 is the idle state reached after the reception of a stream- \mathcal{U}_1 packet. This means the system is empty.
- q_1 is the state where a stream- \mathcal{U}_1 packet is being served.
- q_1' is the state where a stream- \mathcal{U}_2 packet is being served.
- q_0' is the idle state reached after the reception of a stream- \mathcal{U}_2 packet. At this point also the system is empty. The need to differentiate between states q_0 and q_0' will become clear shortly after.

When the j^{th} packet is delivered to the monitor, the system is in the idle state q_0 . Due to the memoryless property of the interarrival times of both streams, two clocks start: a clock $X^{(1)}$ and a clock $X^{(2)}$. Clock $X^{(1)}$ ticks first with probability $a = \mathbb{P}(X^{(1)} < X^{(2)})$, at which point a new packet from stream \mathcal{U}_1 will be generated first and the system goes to state q_1 . The value A of the clock when it ticks has distribution $\mathbb{P}(A > t) = \mathbb{P}(X^{(1)} > t | X^{(1)} < X^{(2)})$. Clock $X^{(2)}$ ticks first with probability $z = 1 - a = \mathbb{P}(X^{(2)} < X^{(1)})$, at which point a new packet from stream \mathcal{U}_2 is generated first and the system goes to state q_1' . The value Z of this second clock when it ticks has distribution $\mathbb{P}(Z > t) = \mathbb{P}(X^{(2)} > t | X^{(2)} < X^{(1)})$.

When the system arrives in state q_1 , this means a packet from stream \mathcal{U}_1 is starting its service. Thus, due to the

memoryless property of $X^{(2)}$, three clocks start: a service clock $S^{(1)}$, clock $X^{(1)}$ and clock $X^{(2)}$. The service clock ticks first with probability $u = \mathbb{P}(S^{(1)} < \Lambda)$ and its value U has distribution $\mathbb{P}(U > t) = \mathbb{P}(S^{(1)} > t | S^{(1)} < \Lambda)$. At this point, the stream \mathcal{U}_1 packet currently being served finishes service before any new packet is generated and the system goes back to state q_0 . This ends the interdeparture time Y_j . Clock $X^{(1)}$ ticks first with probability $b = \mathbb{P}(X^{(1)} < \min(S^{(1)}, X^{(2)}))$ and its value B has distribution $\mathbb{P}(B > t) = \mathbb{P}(X^{(1)} > t | X^{(1)} < \min(S^{(1)}, X^{(2)}))$. At this point, a new stream \mathcal{U}_1 update is generated before any other update from other streams and preempts the one currently in service. In this case the system stays in state q_1 . The third clock $X^{(2)}$ ticks first with probability $d = \mathbb{P}(X^{(2)} < \min(S^{(1)}, X^{(1)}))$ and its value D has distribution $\mathbb{P}(D > t) = \mathbb{P}(X^{(2)} > t | X^{(2)} < \min(S^{(1)}, X^{(1)}))$. At this point, a new update from stream \mathcal{U}_2 is generated, preempts the one currently in service and the system switches to state $q_{1'}$.

When the system arrives in state $q_{1'}$, this means a packet from stream \mathcal{U}_2 is starting its service. Thus, due to the memoryless property of $X^{(2)}$, two clocks are of interest: a service clock $S^{(2)}$ and clock $X^{(2)}$. What happens to stream \mathcal{U}_1 is irrelevant, as it has lower priority and any generated packet will be discarded. The service clock ticks first with probability $f = \mathbb{P}(S^{(2)} < X^{(2)})$ and its value F is distributed according to $\mathbb{P}(F > t) = \mathbb{P}(S^{(2)} > t | S^{(2)} < X^{(2)})$. At this point, the stream \mathcal{U}_2 packet currently being served finishes service before any new packet is generated and the system goes to state $q_{0'}$. Otherwise, clock $X^{(2)}$ ticks first with probability $v = 1 - f = \mathbb{P}(X^{(2)} < S^{(2)})$ and has value V distributed as $\mathbb{P}(V > t) = \mathbb{P}(X^{(2)} > t | X^{(2)} < S^{(2)})$. At this point, a new update from stream \mathcal{U}_2 is generated, preempts the one currently in service and the system stays in state $q_{1'}$.

Finally, when the system arrives in state $q_{0'}$, this means the system is idle but no update from stream \mathcal{U}_1 has been delivered. Given that $X^{(1)}$ and $X^{(2)}$ are memoryless, the system in state $q_{0'}$ behaves exactly as if it were in state q_0 .

From the above analysis, we see that the interdeparture time is given by the sum of the values of the different clocks on the path starting and finishing at q_0 . For example, for the path $q_0 q_1 q_{1'} q_{0'} q_1 q_0$ in Fig. 11, the interdeparture time $Y = A_1 + D_1 + F_1 + Z_1 + F_2 + A_2 + U_1$, where all the random variables in the sum are mutually independent. This value of Y is also valid for the path $q_0 q_{1'} q_{0'} q_1 q_{1'} q_{0'} q_1 q_0$. Hence Y depends on the variables $A_j, B_j, D_j, F_j, U_j, V_j, Z_j$ and their number of occurrences and not on the path itself. Therefore, the probability that exactly $(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ occurrences of (A, B, D, F, U, V, Z) happen, which is equivalent to the probability that

$$Y = \sum_{k=1}^{i_1} A_k + \sum_{k=1}^{i_2} B_k + \sum_{k=1}^{i_3} D_k + \sum_{k=1}^{i_4} F_k + \sum_{k=1}^{i_5} U_k + \sum_{k=1}^{i_6} V_k + \sum_{k=1}^{i_7} Z_k$$

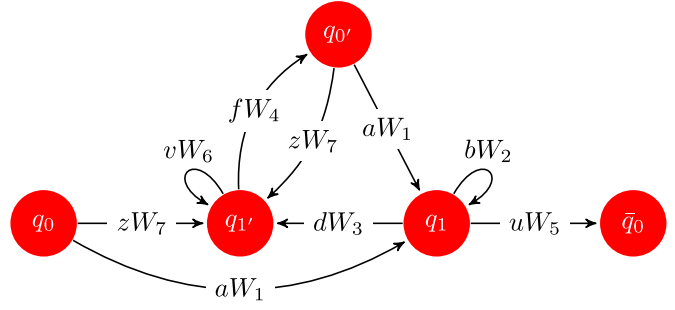


Fig. 12. Detour flow graph of the M/G/1/1 interdeparture time for stream \mathcal{U}_1 .

is given by $a^{i_1} b^{i_2} d^{i_3} f^{i_4} u^{i_5} v^{i_6} z^{i_7} Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$, where $Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ is the number of paths with this combination of occurrences. Taking into account the fact that the $\{A_k, B_k, D_k, F_k, U_k, V_k, Z_k\}$ are mutually independent, the moment generating function of Y is

$$\begin{aligned} \phi_Y(s) &= \mathbb{E}(\mathbb{E}(e^{sY} | (I_1, I_2, I_3, I_4, I_5, I_6, I_7) = (i_1, i_2, i_3, i_4, i_5, i_6, i_7))) \\ &= \sum_{\substack{i_1, i_2, i_3, \\ i_4, i_5, i_6, i_7}} [a^{i_1} b^{i_2} d^{i_3} f^{i_4} u^{i_5} v^{i_6} z^{i_7} Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7) \\ &\quad \mathbb{E}(e^{s(\sum_{k=1}^{i_1} A_k + \sum_{k=1}^{i_2} B_k + \sum_{k=1}^{i_3} D_k + \sum_{k=1}^{i_4} F_k + \sum_{k=1}^{i_5} U_k + \sum_{k=1}^{i_6} V_k + \sum_{k=1}^{i_7} Z_k)})] \\ &= \sum_{\substack{i_1, i_2, i_3, \\ i_4, i_5, i_6, i_7}} [a^{i_1} b^{i_2} d^{i_3} f^{i_4} u^{i_5} v^{i_6} z^{i_7} Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7) \\ &\quad \mathbb{E}(e^{sA})^{i_1} \mathbb{E}(e^{sB})^{i_2} \mathbb{E}(e^{sD})^{i_3} \mathbb{E}(e^{sF})^{i_4} \mathbb{E}(e^{sU})^{i_5} \mathbb{E}(e^{sV})^{i_6} \mathbb{E}(e^{sZ})^{i_7}], \end{aligned} \quad (55)$$

where $\{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$ are the random variables associated with the number of occurrences of $\{A, B, D, F, U, V, Z\}$, respectively.

In order to compute (55), we modify the state diagram in Fig. 11 and represent it as a *detour flow graph* (also called *signal flow graph* [19], [20]) as shown in Fig. 12. For that, we first notice that the interdeparture time Y is the interval of time spent by the system between two consecutive q_0 states. That's why, in Fig. 12, we split the q_0 state into two states: a starting state q_0 and an end state \bar{q}_0 . Hence, there is a one-to-one correspondence between the different paths from q_0 to \bar{q}_0 and the different combinations in which we can write Y in function of (A, B, D, F, U, V, Z) . In order to capture the number of occurrences of the quantities (A, B, D, F, U, V, Z) over a certain path, we associate with each label seven "dummy" variables $(W_1, W_2, W_3, W_4, W_5, W_6, W_7)$ and the exponents of $(W_1, W_2, W_3, W_4, W_5, W_6, W_7)$ correspond to the number of occurrences of (A, B, D, F, U, V, Z) respectively. For example, if between two states q_j and q_i (for any i, j), the edge has a label that contains the factor $W_1 W_3^2$, then it means that the system spent a time of $A + 2D$ when passing from q_j to q_i . Since we are interested in the distribution of Y , we multiply the labels of the edges in the detour flow graph by the probability of such label being visited. For example, given that the system is at state $q_{1'}$, it jumps to state $q_{0'}$ with probability f and after spending a time F . Thus the label from $q_{1'}$ to $q_{0'}$ is fW_4 .

Using Mason's gain formula [19], we know that the generating function $H(W_1, W_2, W_3, W_4, W_5, W_6, W_7)$ of the detour

flow graph shown in Fig. 12, can be written as

$$\begin{aligned}
& H(W_1, W_2, W_3, W_4, W_5, W_6, W_7) \\
&= \sum_{\substack{i_1, i_2, i_3, \\ i_4, i_5, i_6, i_7}} [Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7) a^{i_1} b^{i_2} d^{i_3} f^{i_4} u^{i_5} v^{i_6} z^{i_7} \\
&\quad W_1^{i_1} W_2^{i_2} W_3^{i_3} W_4^{i_4} W_5^{i_5} W_6^{i_6} W_7^{i_7}], \quad (56)
\end{aligned}$$

where $Q(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ is the number of paths with i_1 occurrences of A , i_2 occurrences of B , i_3 occurrences of D , i_4 occurrences of F , i_5 occurrences of U , i_6 occurrences of V , i_7 occurrences of Z . Comparing (55) and (56), we notice that

$$\phi_Y(s) = H(\mathbb{E}(e^{sA}), \mathbb{E}(e^{sB}), \mathbb{E}(e^{sD}), \mathbb{E}(e^{sF}), \mathbb{E}(e^{sU}), \mathbb{E}(e^{sV}), \mathbb{E}(e^{sZ})).$$

Moreover, given a directed graph $G = (V, E)$ with algebraic label $L(e)$ on its edges, and a node $u \in V$ with no incoming edges, the transfer function $H(v)$ from u to a node v is the sum over of all paths from u to v with each path contributing the product of its edge labels to the sum (see [19]–[21]). The complete set of transfer functions $\{H(v) : v \in V\}$ can be computed easily by solving the linear equations:

$$\begin{cases} H(u) = 1 \\ H(w) = \sum_{w':(w',w) \in E} H(w')L((w',w)), \quad w \neq u. \end{cases}$$

Solving the system of linear equations above yields the transfer function as

$$\begin{aligned}
& H(W_1, W_2, W_3, W_4, W_5, W_6, W_7) \\
&= \frac{uW_5aW_1(1-vW_6)}{(1-zW_7fW_4-vW_6)(1-bW_2)-dW_3aW_1fW_4}. \quad (57)
\end{aligned}$$

Using [10, Lemma1] and Lemma 3, we know that $\mathbb{E}(e^{sB}) = \mathbb{E}(e^{sD}) = \frac{\lambda(1-P_{\lambda-s})}{(\lambda-s)(1-P_{\lambda})}$, $\mathbb{E}(e^{sA}) = \mathbb{E}(e^{sZ}) = \frac{\lambda}{\lambda-s}$, $\mathbb{E}(e^{sF}) = \frac{L_{\lambda_2-s}}{L_{\lambda_2}}$ and $\mathbb{E}(e^{sV}) = \frac{\lambda_2(1-L_{\lambda_2-s})}{(\lambda_2-s)(1-L_{\lambda_2})}$. Moreover, we can notice that U has the same distribution as the system time T so $\mathbb{E}(e^{sU}) = \frac{P_{\lambda-s}}{P_{\lambda}}$. Simple computations show that $a = \frac{\lambda_1}{\lambda}$, $b = \frac{\lambda_1}{\lambda}(1-P_{\lambda})$, $d = \frac{\lambda_2}{\lambda}(1-P_{\lambda})$, $f = L_{\lambda_2}$, $u = P_{\lambda}$, $v = 1-L_{\lambda_2}$, $z = \frac{\lambda-\lambda_1}{\lambda}$. Finally, replacing the above expressions into (57), we get our result. \square

F. Proof of Lemma 5

Lemma. Consider stream \mathcal{U}_1 . For any $j \geq 1$, the random variables T_j and Y_j relative to the j^{th} successful packet are independent. Moreover the process $(Y_j)_{j \geq 1}$ is i.i.d, with its distribution given by Lemma 4, and the process $R(\tau) = \sup\{n \in \mathbb{N}; D_n \leq \tau\}$ is a renewal process.

Proof. Let $L_j = \min(X_j^{(1)}, X^{(2)})$. Since the interarrival times for both streams are exponential and independent, L_j is also exponential with rate $\lambda = \lambda_1 + \lambda_2$. Except L_j , all other variables are relative to stream \mathcal{U}_1 . The j^{th} successful packet leaves the queue empty hence $Y_j = \hat{X}_j + Z_j$. $\hat{X}_j = L_j - T_j$ is the remaining time between the departure of the stream- \mathcal{U}_1 j^{th} successful packet, and the generation time of the next packet to be transmitted (it can belong to stream \mathcal{U}_1 or stream \mathcal{U}_2).

Z_j is the time for a new stream- \mathcal{U}_1 packet to be successfully delivered. Z_j does not overlap with T_j and thus is independent from it. As for \hat{X}_j , we also obtain that it is independent of T_j . To prove this, notice that for a successfully received packet j the joint distribution $f_{L_j, T_j}(x, t)$ can be written as

$$\begin{aligned}
f_{L_j, T_j}(x, t) &= f_{L, T|L \geq T}(x, t|x \geq t) \\
&= \begin{cases} 0 & \text{if } x < t \\ \frac{f_{L, S}(x, t)}{\mathbb{P}(S < L)} & \text{if } x > t \end{cases}, \quad (58)
\end{aligned}$$

where $L = \min(X^{(1)}, X^{(2)})$ and S is the generic service time. These two variables are independent. Now, using a change of variable we obtain

$$\begin{aligned}
f_{\hat{X}_j, T_j}(\hat{x}, t) &= f_{L_j - T_j, T_j}(\hat{x}, t) = f_{L_j, T_j}(\hat{x} + t, t) \\
&= \begin{cases} 0 & \text{if } \hat{x} < 0 \\ \frac{f_{L, S}(\hat{x} + t, t)}{\mathbb{P}(S < L)} & \text{if } \hat{x} > 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } \hat{x} < 0 \\ \frac{\lambda e^{-\lambda(\hat{x} + t)} f_S(t)}{\mathbb{P}(S < L)} & \text{if } \hat{x} > 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } \hat{x} < 0 \\ (\lambda e^{-\lambda \hat{x}}) \frac{e^{-\lambda t} f_S(t)}{\mathbb{P}(S < L)} & \text{if } \hat{x} > 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } \hat{x} < 0 \\ h(\hat{x})g(t) & \text{if } \hat{x} > 0 \end{cases}. \quad (59)
\end{aligned}$$

Moreover, \hat{X}_i is exponential with rate λ since

$$\begin{aligned}
\mathbb{P}(\hat{X}_j > t) &= \mathbb{P}(L_j > t + S_j | L_j > S_j) \\
&= \frac{\mathbb{P}(L_j > t + S_j)}{\mathbb{P}(L_j > S_j)} \\
&= \frac{1}{\mathbb{P}(L_j > S_j)} \left(\int_0^\infty e^{-\lambda(t+s)} f_{S_j}(s) ds \right) \\
&= (1 + \lambda \theta)^k \left(\frac{e^{-\lambda t}}{(1 + \lambda \theta)^k} \right) \\
&= e^{-\lambda t}. \quad (60)
\end{aligned}$$

(59) and (60) show that \hat{X}_j and T_j are indeed independent. Given that \hat{X}_j and Z_j are both independent from T_j , then Y_j and T_j are also independent.

Furthermore, since $Y_{j-1} = \hat{X}_{j-1} + Z_{j-1}$, \hat{X}_j is independent from T_j and the interarrival process is i.i.d and independent from the i.i.d service process, then \hat{X}_j and Z_j are independent of Y_{j-1} . This implies that for any $j \geq 1$, Y_{j-1} and Y_j are independent. Moreover, it is clear that the Z_j 's have the same distribution. Since the \hat{X}_j 's are exponential with rate λ then the $(Y_j)_{j \geq 1}$ is an i.i.d process. Given that Y_j is the interval of time between the receptions of two consecutive successful stream- \mathcal{U}_1 packets, then the number of successfully received packets in the interval $[0, \tau]$, $R(\tau)$, is a renewal process. \square

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