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Approximating the standard condition number for cognitive radio spectrum sensing with finite number of sensors

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Abstract: In this study, the authors consider the standard condition number (SCN) detector for a cognitive radio with finite number of cooperative sensors. They derive an exact nested form of the distribution of the SCN for the central uncorrelated, non-central uncorrelated and central semi-correlated Wishart matrices under $\mathscr{H}_0$ and $\mathscr{H}_1$ hypotheses. Due to the complexity of these expressions, the authors approximate the distribution of the SCN by the generalised extreme value distribution using moment matching. They derive the exact form of the *p*th moment of the SCN for these cases. Consequently, the performance probabilities are approximated and a simple decision threshold formula is provided. In addition, a similar approximation for the detection probability is provided using non-central/central approximation. They show that the proposed analytical approximations provide high accuracy using Monte-Carlo simulations.

1 Introduction

In wireless communication systems, the concept of cognitive radio (CR) appeared as a new paradigm which aims to enhance the utilisation of the radio frequency spectrum. CR offers a great solution for spectrum scarcity as it transforms the current static spectrum allocation scheme into dynamic spectrum allocation. In CR networks, spectrum sensing (SS) is the task of obtaining awareness about the spectrum usage. Hence, SS plays a major role in the performance of the CR as well as the performance of the primary user (PU) networks that coexist. In this context, SS aims at reaching a high probability of accuracy in uncertain environments.

Several SS techniques have been developed in recent years [1]. Among others, matched filter, energy detector (ED) and cyclostationary feature detector have attracted a lot of attention due to their operational requirements and specifications [2, 3]. ED is among the most popular sensing techniques as it is non-coherent, blind towards the PU signal, the simplest for real-time implementation and performs well for high signal-to-noise ratio (SNR). However, it is limited by its dependency on noise uncertainty [4], one of the major drawbacks in SS. Recently, eigenvalue-based detector (EBD) has shown to overcome noise uncertainty challenges and to perform adequately even in low SNR conditions. Since then, there has been a significant amount of effort in the research community on this technique.

EBD is based on the eigenvalues of the received signal covariance matrix and includes the largest eigenvalue (LE) detector [5], the scaled LE detector [6, 7] and the standard condition number (SCN) detector [5, 8–13]. If the entries of the sample covariance matrix are considered Gaussian, then it is known as the Wishart matrix in random matrix theory (RMT) [14]. Recent results from RMT have been considered to study the properties of the eigenvalues of the covariance matrix.

If we denote the maximum and the minimum eigenvalues of the covariance matrix of the received signal by $\lambda_{\max}$ and $\lambda_{\min}$, respectively, then the SCN is defined as the ratio between $\lambda_{\max}$ and $\lambda_{\min}$ ($\lambda_{\max}/\lambda_{\min}$). The SCN detector compares this ratio to a threshold to decide whether the PU exists or not. The performance of a SS algorithm is usually characterised by two types of probabilities: the false-alarm probability (P_{fa}) and the detection probability (P_{d}). These latter depends on the decision threshold and is related to the statistics of the detector metric. Thus, the characteristics of the SCN distribution are of great importance in CR systems.

The SCN metric attracted a lot of attention in the recent years. This metric was studied asymptotically in [8] and the threshold was presented according to Marchenko–Pastur (MP) law [15]. If the size of the receiver matrix is $K \times N$, then MP law proves that the largest ($\lambda_{\max}$) [respectively the smallest ($\lambda_{\min}$)] eigenvalue of the receiver covariance matrix converges to a constant as $(K, N) \to \infty$ with $c = K/N$ is a constant ratio [16]. This constant depends on c and on the power of the signal and the noise. In practice, K could be the number of antennas or the number of cooperating nodes and N is the number of samples per antenna (or node) collected at the receiver.

In [5], the authors improved the accuracy of the asymptotic statistical distribution of the SCN by using the Tracy–Widom (TW) distribution to model the LE (numerator of the SCN) [17] while maintaining the MP representation of the smallest one. TW distribution is an asymptotic distribution of the LE of a central Wishart matrix as $(K, N) \to \infty$ [18]. This approximation of the SCN results in an approximated relation between the decision threshold and the P_{fa}.

This work was further extended in [9, 10] by using the Curtiss formula for the distribution of the ratio of two random variables ($\lambda_{\max}/\lambda_{\min}$) [19] where both the largest and the smallest eigenvalues converge to TW distributions when $(K, N) \to \infty$ as shown in [17, 20]. Moreover, by the exploitation of the normal and TW distributions and using the Curtiss formula, Penna *et al.* [11] provided a form for the probability of miss detection ($P_{\text{md}} = 1 - P_{\text{d}}$) for sufficiently large values of K and N. However, all these expressions include TW distribution and Curtiss formula are hard to evaluate numerically online. Moreover, the considered asymptotic assumptions are not practical since this assumes that the dimensions of the system are very large.

In this paper, we target the finite-size Wishart matrix case. In [21], the authors provided a general framework for the distribution of the SCN for finite Wishart matrix. Consequently, Zhang *et al.* [12] considered the dual Wishart case. Herein, a general nested sum form for the cumulative distribution function (CDF) and the probability density function (PDF) of the SCN are derived and we show that lookup tables (LUT) must be used to avoid the computational complexity. In this regard, we provide an alternative solution as we propose a new approximation for the SCN based on the generalised extreme value (GEV) distribution and its exact moments. For this purpose, we derive the exact form of the *p*th moment of the SCN to match the first three moments of the GEV distribution. This approximation is simple and accurate even though the computational complexity of the SCN moments still exists. However, as we need only the first three moments, the latter could be computed offline and saved. Moreover, we show that by using the non-central/central approximation along with the proposed approximation, the result is still accurate with a simpler moments form. As a result, the main contributions of this paper are summarised as follows:

- Derivation of the analytical form of the CDF and PDF for the SCN of the non-central uncorrelated Wishart (NUW) and central semi-correlated Wishart (CCW) matrices.
- Derivation of a nested form of the CDF and PDF for the SCN of the central uncorrelated Wishart (CUW), NUW and CCW matrices.
- Derivation of the analytical form of the *p*th moment for the SCN of CUW, NUW and CCW matrices.
- Approximating the SCN distribution using the GEV distribution for finite Wishart matrices.

The rest of this paper is organised as follows. In Section 2, the system model is introduced and discussed. In Section 3, we give the joint ordered eigenvalue distribution for the CUW, NUW and CCW matrices. In Section 4, we study the SCN distribution for the pre-mentioned Wishart matrices. Moments of the SCN are considered in Section 5 as we derive the exact form of the *p*th moment. Section 6 discusses the GEV approximation as well as the performance probabilities of the SCN detector. Computational complexity is also discussed. Simulation results and approximation accuracy are discussed in Section 7 and we conclude in Section 8.

Notations: vectors and matrices are indicated, respectively, by lower and upper case boldface. The symbol $|.|$ indicates the determinant of a matrix while $(\,.\,)^{\dagger}$ is the Hermitian symbol. $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. $E[.]$ stands for the expected value, and $\sim$ stands for distributed as.

2 System model

Let us consider a CR with K collaborative sensors aiming to detect the presence/absence of a single PU using N samples. Two hypotheses exist: $\mathscr{H}_0$ corresponds to the absence of PU; and $\mathscr{H}_1$ corresponds to the presence of the PU. The received vector, at instant n, is given by

$$\mathscr{H}_0:\ \boldsymbol{y}(n) = \boldsymbol{\eta}(n), \tag{1}$$

$$\mathscr{H}_1:\ \boldsymbol{y}(n) = \boldsymbol{h}(n)s(n) + \boldsymbol{\eta}(n), \tag{2}$$

where $\boldsymbol{y}(n)$ is the observed $K \times 1$ complex samples across all the sensors. $\boldsymbol{\eta}(n)$ is a $K \times 1$ complex circular white Gaussian noise vector. $\boldsymbol{h}(n)$ is a $K \times 1$ complex vector that represents the channels' coefficient between the PU and each of the CR sensor and $s(n)$ stands for the primary signal sample.

After collecting N samples from each sensor, the received signal matrix $\boldsymbol{Y}$ is written as

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & \cdots & y_1(N) \\ \vdots & \ddots & \vdots \\ y_K(1) & \cdots & y_K(N) \end{pmatrix} \tag{3}$$

Suppose that $K \le N$ and define the sample covariance matrix as $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^{\dagger}$. Denote by $\lambda_1 \ge \cdots \ge \lambda_K > 0$ the eigenvalues of $\boldsymbol{W}$, then the SCN metric is given by

$$X = \frac{\lambda_1}{\lambda_K}. \tag{4}$$

Under $\mathscr{H}_0$, $\boldsymbol{W}$ is CUW matrix denoted by $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K)$ where K is the size of the matrix, N is the number of degrees of freedom, $\sigma_\eta^2 \boldsymbol{I}_K$ is the correlation matrix, σ_η^2 is the noise variance and $\mathscr{CW}$ stands for complex Wishart.

Under $\mathscr{H}_1$, we suppose a flat fading channel and we consider a single PU whose signal is drawn independently from Gaussian process for every sample. Consequently, $\boldsymbol{W}$ follows a NUW distribution denoted as $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with $\boldsymbol{\Omega}_K$ is a rank-1 non-centrality matrix [22].

3 Joint ordered eigenvalues distribution

The joint distribution of the ordered eigenvalues (JDOE) of the Wishart matrices for the three cases, (i) CUW, (ii) NUW with all but one of the eigenvalues of the non-centrality matrix are zeros and (iii) CCW with all but one of the eigenvalues of the correlation matrix are equal are considered.

3.1 CUW case

Under $\mathscr{H}_0$, $\boldsymbol{W}$ is a CUW matrix. Then, the JDOE of $\boldsymbol{W}$ is given by the following Lemma.

Lemma 1: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \boldsymbol{I}_K)$. Then, the JDOE of $\boldsymbol{W}$ is given by [23]

$$f(\boldsymbol{\lambda}) = \mathscr{K}_{\text{cu}} |\boldsymbol{V}(\boldsymbol{\lambda})|^2 \prod_{l=1}^{K} \lambda_l^{N-K} \mathrm{e}^{-\lambda_l}, \tag{5}$$

where $\boldsymbol{\lambda}$; is the vector of eigenvalues of $\boldsymbol{W}$, $\boldsymbol{V}(\boldsymbol{\lambda})$ is a Vandermonde matrix with $v_i(\lambda_j) = \lambda_j^{i-1}$ and $\mathscr{K}_{\text{cu}}$ is given by (6) with $\Gamma_n(m) = \prod_{i=0}^{n}(m-i)!$

$$\mathscr{K}_{\text{cu}} = [\Gamma_K(N)\Gamma_K(K)]^{-1} \tag{6}$$

3.2 NUW case

Under $\mathscr{H}_1$, $\boldsymbol{W}$ is a NUW matrix with rank-1 non-centrality matrix. Then, the JDOE of $\boldsymbol{W}$ is given by the following Lemma.

Lemma 2: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with non-centrality matrix $\boldsymbol{\Omega}_K$ of eigenvalues $\omega_1 > \omega_2 = \cdots = \omega_K = 0$. Then, the JDOE of $\boldsymbol{W}$ is given by

$$f(\boldsymbol{\lambda}) = \mathscr{K}_{\text{nu}} |\boldsymbol{U}(\boldsymbol{\lambda})| \times |\boldsymbol{F}(\boldsymbol{\lambda}, \omega_1)| \prod_{i=1}^{K} \lambda_i^{N-K} \cdot \mathrm{e}^{-\lambda_i} \tag{7}$$

where $\boldsymbol{U}(\boldsymbol{\lambda})$ and $\boldsymbol{F}(\boldsymbol{\lambda}, \omega_1)$ are $K \times K$ matrices of (i, j)th entries given, respectively, by $u_i(\lambda_j) = \lambda_j^{K-i}$ and (8) with ${}_0F_1(\,.\,,\,.\,)$ is the generalised hypergeometric function [24, Eq.(9.14.1)]. $\mathscr{K}_{\text{nu}}$ is defined in (9)

$$f_i(\lambda_j) = \begin{cases} {}_0F_1(N-K+1, \omega_1\lambda_j), & i = 1 \\ \dfrac{\lambda_j^{K-i} \cdot (N-K)!}{(N-i)!}, & i = 2, \ldots, K \end{cases} \tag{8}$$

$$\mathscr{K}_{\text{nu}} = \frac{\mathrm{e}^{-\omega_1} \cdot [(N-K)!]^{-K}}{\Gamma_{K-1}(K-1) \cdot \omega_1^{K-1}} \tag{9}$$

Proof: Particularise [25, Appendix-I] with $\boldsymbol{\Omega}_K$ is a rank-1 matrix. □

3.3 CCW case

First, let us recall the non-central/central approximation in Lemma 3.

Lemma 3 [26]: The complex NUW matrix $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ and the complex CCW matrix $\boldsymbol{W} \sim \mathscr{CW}_K(N, \hat{\boldsymbol{\Sigma}}_K)$ with effective correlation matrix $\hat{\boldsymbol{\Sigma}}_K = \sigma_\eta^2 I_K + \boldsymbol{\Omega}_K/N$ have the same first- and second-order moments differing by $\boldsymbol{\Omega}_K/N$.

Accordingly, $\boldsymbol{W}$ under $\mathscr{H}_1$ could be considered as a CCW matrix. Since $\boldsymbol{\Omega}_K$ is a rank-1 matrix then the effective correlation matrix has equal eigenvalues. Let $\boldsymbol{\sigma}; = [\sigma_1, \ldots, \sigma_K]^{\mathrm{T}}$ be the vector of ordered eigenvalues of $\hat{\boldsymbol{\Sigma}}_K$, then all but one of these eigenvalue are equal to σ_η^2 while $\sigma_1 = \sigma_\eta^2 + (\omega_1/N)$.

In this regard, the JDOE of this CCW matrix is given by the following Lemma:

Lemma 4: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \boldsymbol{\Sigma}_K)$ with correlation matrix $\boldsymbol{\Sigma}_K$ of eigenvalues $\sigma_1 > \sigma_2 = \cdots = \sigma_K$. Then, the JDOE of $\boldsymbol{W}$ is given by

$$f(\boldsymbol{\lambda}) = \mathscr{K}_{\mathrm{cc}} |\boldsymbol{V}(\boldsymbol{\lambda})| \, |\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\sigma})| \prod_{l=1}^{K} \lambda_l^{N-K}, \tag{10}$$

with (i, j)th entries of $\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\sigma})$ and $\mathscr{K}_{\mathrm{cc}}$ are given, respectively, by

$$e_i(\lambda_j) = \begin{cases} \mathrm{e}^{-(\lambda_j/\sigma_1)}, & i = 1 \\ (-\lambda_j)^{K-i} \mathrm{e}^{-(\lambda_j/\sigma_2)}, & 1 < i \le K \end{cases} \tag{11}$$

$$\mathscr{K}_{\mathrm{cc}} = \frac{\sigma_1^{K-N-1} \sigma_2^{(N-1)(1-K)}}{\Gamma_K(N) \Gamma_{K-1}(K-1)(\sigma_2 - \sigma_1)^{K-1}} \tag{12}$$

Proof: Particularise [27, Lemma-6] for $K - 1$ equal eigenvalues. □

4 SCN distribution

In this section, the exact SCN distribution is considered for the pre-mentioned three cases.

4.1 CUW case

Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K)$. Then, the general form of the CDF of the SCN of $\boldsymbol{W}$ is given by [21]

$$F^{\mathrm{cu}}(x) = \mathscr{K}_{\mathrm{cu}} \sum_{n=1}^{K} \int_0^\infty |\boldsymbol{\Upsilon}_n^{\mathrm{cu}}(x)| \, \mathrm{d}\lambda_K \tag{13}$$

with

$$\Upsilon_n^{\mathrm{cu}}(x)_{i,j} = \begin{cases} \gamma(N - K + i + j - 1, x\lambda_K) - \gamma(N - K + i + j - 1, \lambda_K), & i \ne n \\ \lambda_K^{N-K+i+j-2} \mathrm{e}^{-\lambda_K}, & i = n \end{cases} \tag{14}$$

where $\gamma(.,.)$ is the lower incomplete gamma function [24, Eq. (8.350.1)], $\mathscr{K}_{\mathrm{cu}}$ defined by (6).

Corollary 1: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K)$. Then, the exact general form of the CDF and PDF of the SCN of $\boldsymbol{W}$ are given, respectively, by (see (15))
and (see (16)) where $\mathbb{P}_1$ and $\mathbb{S}_1$ are, respectively, the set of all possible permutations and subsets of the vector $[1, \ldots, K - 1]$. $\mathrm{sgn}(\delta)$ is the permutation sign and $|\boldsymbol{s}|$ is the cardinality. $\sum_{l_1 \cdots l_{K-1}} = \sum_{l_1}^{L_1} \cdots \sum_{l_{K-1}}^{L_{K-1}}$ with $L_j = N + r_{\delta(j),n} + r_{j,m} - K - 2$. $\Sigma \boldsymbol{s}$, Σl_i and $\Pi l_i!$ are, respectively, the sum of the values of $l_i \in \boldsymbol{s}$, the sum of the values of $l_{1 \le i \le K-1}$ and the product of the factorial of the values of $l_{1 \le i \le K-1}$. Finally, $r_{i,j}$ is defined as

$$r_{i,j} = \begin{cases} i, & i < j \\ i + 1, & i \ge j \end{cases} \tag{17}$$

Proof: Refer to Appendix A. □

4.2 NUW case

The exact generic form of the SCN distribution of NUW matrix under $\mathscr{H}_1$ hypothesis is given by the following theorem.

Theorem 1: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with non-centrality matrix $\boldsymbol{\Omega}_K$ of eigenvalues $\omega_1 > \omega_2 = \cdots = \omega_K = 0$. Then, the CDF of the SCN of $\boldsymbol{W}$ is given by

$$F(x) = \mathscr{K}_{\mathrm{nu}} \sum_{n=1}^{K} \int_0^\infty |\boldsymbol{\Upsilon}_n^{\mathrm{nu}}(x)| \, \mathrm{d}\lambda_K \tag{18}$$

with

$$\Upsilon_n^{\mathrm{nu}}(x)_{i,j} = \begin{cases} \mathscr{I}_{i,j}^{\mathrm{nu}} = \displaystyle\int_{\lambda_K}^{x\lambda_K} f_j(u) u^{N-i} \mathrm{e}^{-u} \, \mathrm{d}u, & i \ne n \\ f_j(\lambda_K) \lambda_K^{N-i} \mathrm{e}^{-\lambda_K}, & i = n \end{cases} \tag{19}$$

where $f_j(.)$ is given in (8) and the integral in (19) has analytical solution given by (see (20))

Proof: Starting from the JDOE in (7), then the result in (18) follows the analytical derivation of (13) and considering the different cases of $f_i(\lambda_j)$. Finally, substitute the hypergeometric function and integrate using [24, Eq.(3.351.1)]. □

Corollary 2: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with non-centrality matrix $\boldsymbol{\Omega}_K$ of eigenvalues $\omega_1 > \omega_2 = \cdots = \omega_K = 0$. Then, the exact general form of the CDF and PDF of the SCN of $\boldsymbol{W}$ is given, respectively, by (see (21))
and (see (22)) with

$$\begin{aligned} F(x) = \mathscr{K}_{\mathrm{cu}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{\delta \in \mathbb{P}_1} \mathrm{sgn}(\delta) \prod_{i=1}^{K-1} (N + r_{\delta(i),n} + r_{i,m} - K - 2)! \\ \times \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|} \sum_{l_1 \cdots l_{K-1}} \frac{x^{\Sigma \boldsymbol{s}} \cdot (\Sigma l_i + N + n + m - K - 2)!}{\Pi l_i! \cdot (|\boldsymbol{s}|x - |\boldsymbol{s}| + K)^{\Sigma l_i + N + n + m - K - 1}} \end{aligned} \tag{15}$$

$$\begin{aligned} f(x) = \mathscr{K}_{\mathrm{cu}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{\delta \in \mathbb{P}_1} \mathrm{sgn}(\delta) \prod_{i=1}^{K-1} (N + r_{\delta(i),n} + r_{i,m} - K - 2)! \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|} \\ \times \sum_{l_1 \cdots l_{K-1}} \frac{x^{\Sigma \boldsymbol{s}} [\Sigma \boldsymbol{s} x^{-1} (|\boldsymbol{s}|x - |\boldsymbol{s}| + K) - |\boldsymbol{s}|(\Sigma l_i + N + n + m - K - 1)]}{\Pi l_i! \cdot [(\Sigma l_i + N + n + m - K - 2)!]^{-1} \cdot (|\boldsymbol{s}|x - |\boldsymbol{s}| + K)^{\Sigma l_i + N + n + m - K}} \end{aligned} \tag{16}$$

$$\mathscr{I}^{\text{nu}}_{i,j} = \begin{cases} \displaystyle\sum_{l=0}^{\infty} \frac{\omega_1^l}{(N-K+1)_l\, l!}[\gamma(N-i+l+1, x\lambda_K) - \gamma(N-i+l+1, \lambda_K)], & j = 1 \\ \dfrac{(N-K)!}{(N-j)!}[\gamma(N+K-i-j+1, x\lambda_K) - \gamma(N+K-i-j+1, \lambda_K)], & j = 2, \dots, K \end{cases} \tag{20}$$

$$F(x) = \mathscr{K}'_{\text{nu}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{h=0}^{\infty} \frac{\omega_1^h}{(N-K+h)!h!} \sum_{\delta \in \mathbb{P}_1} \text{sgn}(\delta) \prod_{i=1}^{K-1} (N - r_{\delta(i),n} + g_{r_{i,m},1}(h,K))! \\ \times \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|} \sum_{l_1 \cdots l_{K-1}} \frac{x^{\Sigma \boldsymbol{s}} \cdot (\Sigma l_i + N - n + g_{m,1}(h,K))!}{\Pi l_i! \cdot (|\boldsymbol{s}|x - |\boldsymbol{s}| + K)^{\Sigma l_i + N - n + g_{m,1}(h,K) + 1}} \tag{21}$$

$$f(x) = \mathscr{K}'_{\text{nu}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{h=0}^{\infty} \frac{\omega_1^h}{(N-K+h)!h!} \sum_{\delta \in \mathbb{P}_1} \text{sgn}(\delta) \prod_{i=1}^{K-1} (N - r_{\delta(i),n} + g_{r_{i,m},1}(h,K))! \\ \times \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|} \sum_{l_1 \cdots l_{K-1}} \frac{x^{\Sigma \boldsymbol{s}}[\Sigma \boldsymbol{s} x^{-1}(|\boldsymbol{s}|x - |\boldsymbol{s}| + K) - |\boldsymbol{s}|(\Sigma l_i + N - n + g_{m,1}(h,K) + 1)]}{\Pi l_i! \cdot [(\Sigma l_i + N - n + g_{m,1}(h,K))!]^{-1} \cdot (|\boldsymbol{s}|x - |\boldsymbol{s}| + K)^{\Sigma l_i + N - n + g_{m,1}(h,K) + 2}} \tag{22}$$

$$\mathscr{K}'_{\text{nu}} = \left[\Gamma_{K-1}(K-1) \prod_{i=2}^{K} (N-i)!\,\omega_1^{K-1} \mathrm{e}^{\omega_1}\right]^{-1}, \tag{23}$$

and $\sum_{l_1 \cdots l_{K-1}} = \sum_{l_1}^{L_1} \cdots \sum_{l_{K-1}}^{L_{K-1}}$ with $L_j = N - r_{\delta(j),n} + g_{r_{j,m},1}(h,K)$ and we define $g_{i,j}(a,b)$ as

$$g_{i,j}(a,b) = \begin{cases} a, & i = j \\ b - i, & i \neq j \end{cases} \tag{24}$$

Proof: Refer to Appendix A. □

4.3 CCW case

The exact generic form of the SCN distribution of CCW matrix for the $\mathscr{H}_1$ hypothesis is given by the following theorem.

Theorem 2: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \boldsymbol{\Sigma}_K)$ with correlation matrix $\boldsymbol{\Sigma}_K$ of eigenvalues $\sigma_1 > \sigma_2 = \cdots = \sigma_K$. Then, the CDF of the SCN of $\boldsymbol{W}$ is given by

$$F(x) = \mathscr{K}_{\text{cc}} \sum_{n=1}^{K} \int_0^{\infty} |\boldsymbol{\Upsilon}_n^{\text{cc}}(x)|\, \mathrm{d}\lambda_K \tag{25}$$

with

$$\Upsilon_n^{\text{cc}}(x)_{i,j} = \begin{cases} \mathscr{I}^{\text{cc}}_{i,j} = \displaystyle\int_{\lambda_K}^{x\lambda_K} e_j(u) u^{N+i-K-1}\, \mathrm{d}u, & i \neq n \\ e_j(\lambda_K)\lambda_K^{N+i-K-1}, & i = n \end{cases} \tag{26}$$

where $e_j(.)$ is given in (11) and the integral in (26) has an analytical solution given by (see (27))

Proof: Starting from the JDOE in (10), then the result in (25) follows the analytical derivation of (13) and considering the different cases of $e_i(\lambda_j)$. Then, integrate using [24, Eq. (3.351.1)]. □

Corollary 3: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \boldsymbol{\Sigma}_K)$ with correlation matrix $\boldsymbol{\Sigma}_K$ of eigenvalues $\sigma_1 > \sigma_2 = \cdots = \sigma_K$. Then, the exact general form of the CDF and PDF of the SCN of $\boldsymbol{W}$ is given, respectively, by (see (28))
and (see (29)) with

$$\mathscr{K}'_{\text{cc}} = \frac{(-1)^{(K-1)(K-2)/2}\sigma_1^{K-N-1}\sigma_2^{(N-1)(1-K)}}{\Gamma_K(N)\Gamma_{K-1}(K-1)(\sigma_2 - \sigma_1)^{K-1}} \tag{30}$$

Proof: Refer to Appendix A. □

5 SCN moments

In this section, the exact expression of the pth moment of the SCN is considered for the three pre-mentioned cases.

5.1 CUW case

The exact expression of the pth moment of the SCN of the CUW matrix is given by the following theorem.

Theorem 3: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K)$. Then, the pth moment of the SCN of $\boldsymbol{W}$ is given by

$$\mathscr{I}^{\text{cc}}_{i,j} = \begin{cases} \left[\gamma\left(N+i-K, \dfrac{x\lambda_K}{\sigma_1}\right) - \gamma\left(N+i-K, \dfrac{\lambda_K}{\sigma_1}\right)\right] \cdot \sigma_1^{N+i-K}, & j = 1 \\ (-1)^{K-j}\left[\gamma\left(N+i-j, \dfrac{x\lambda_K}{\sigma_2}\right) - \gamma\left(N+i-j, \dfrac{\lambda_K}{\sigma_2}\right)\right]\sigma_2^{N+i-j}, & j = 2, \dots, K \end{cases} \tag{27}$$

$$F(x) = \mathscr{K}'_{\text{cc}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{\delta \in \mathbb{P}_1} \text{sgn}(\delta) \prod_{i=1}^{K-1} \frac{(N + r_{\delta(i),n} - g_{r_{i,m},1}(K,2i) - 1)!}{\sigma_{r_{i,m}}^{-(N + r_{\delta(i),n} - g_{r_{i,m},1}(K,2i))}} \\ \times \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|} \sum_{l_1 \cdots l_{K-1}} \frac{x^{\Sigma \boldsymbol{s}} \cdot (\Sigma l_i + N + n - g_{m,1}(K,2i) - 1)!}{\Pi l_i! \cdot \Pi \sigma_{r_{i,m}}^{l_i} \cdot \left(\Sigma_{\boldsymbol{s}}(x/\sigma_{s_i,m}) + \Sigma_{\bar{\boldsymbol{s}}}(1/\sigma_{\bar{s}_i,m}) + (1/\sigma_m)\right)^{\Sigma l_i + N + n - g_{m,1}(K,2i)}} \tag{28}$$

$$M(p) = \mathscr{K}_{\mathrm{cu}} \sum_{\delta \in \mathbb{P}_0} \mathrm{sgn}(\delta) \sum_{\alpha \in \mathbb{P}_0} \mathrm{sgn}(\alpha
) \sum_{l_1 \cdots l_{K-1}} \frac{\Pi_i (N - K - 2 + \delta(i) + \alpha(i) + l_{i-1} + C_{i,p})!}{\Pi_i l_i! \cdot \Pi_i i^{N - K - 1 + \delta(i) + \alpha(i) + l_{i-1} - l_i + C_{i,p}}} \tag{31}$$

with

$$C_{i,p} = \begin{cases} p, & i = 1 \\ 0, & 1 < i < K \\ -p, & i = K \end{cases} \tag{32}$$

and $\sum_{l_1 \cdots l_{K-1}} = \sum_{l_1 = 0}^{L_1} \cdots \sum_{l_{K-1} = 0}^{L_{K-1}}$ with $L_j = N - K - 2 + \delta(j) + \alpha(j) + l_{j-1} + C_{j,p}$, $\Pi_i(\,.\,)$ denotes the multiplication over $i = 1 \cdots K$ and $l_0 = l_K = 0$.

Proof: Refer to Appendix B. □

5.2 NUW case

The exact expression of the pth moment of the SCN of the NUW matrix under $\mathscr{H}_1$ hypothesis is given by the following theorem.

Theorem 4: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with non-centrality matrix $\boldsymbol{\Omega}_K$ of eigenvalues $\omega_1 > \omega_2 = \cdots = \omega_K = 0$. Then, the pth moment, $M(p)$, of the SCN of $\boldsymbol{W}$ is given by (see (33)) with

$$G_{i,n}(h, K, \alpha) = \begin{cases} K - r_{\alpha(i),1}, & i < n \\ h, & i = n \\ K - r_{\alpha(i-1),1}, & i > n \end{cases} \tag{34}$$

and $\sum_{l_1 \cdots l_{K-1}} = \sum_{l_1 = 0}^{L_1} \cdots \sum_{l_{K-1} = 0}^{L_{K-1}}$ with $L_j = N - \delta(j) + G_{j,n}(h, K, \alpha) + l_{j-1} + C_{j,p}$.

Proof: Refer to Appendix B. □

5.3 CUW case

The exact expression of the pth moment of the SCN of the CCW matrix for $\mathscr{H}_1$ hypothesis is given by the following theorem.

Theorem 5: Let $\boldsymbol{W} \sim \mathscr{CW}_K(N, \boldsymbol{\Sigma}_K)$ with correlation matrix $\boldsymbol{\Sigma}_K$ of eigenvalues $\sigma_1 > \sigma_2 = \cdots = \sigma_K$. Then, the pth moment, $M(p)$, of the SCN of $\boldsymbol{W}$ is given by (see (35)) with

$$b_{i,n} = \begin{cases} 1, & i = n \\ 2, & i \neq n \end{cases} \tag{36}$$

and $\sum_{l_1 \cdots l_{K-1}} = \sum_{l_1 = 0}^{L_1} \cdots \sum_{l_{K-1} = 0}^{L_{K-1}}$ with $L_j = N + \delta(j) + G_{j,n}(0, K, \alpha) + l_{j-1} + C_{j,p} - K - 1$.

Proof: Refer to Appendix B. □

6 SCN approximation

This section provides an approximation for the SCN distribution based on the GEV distribution. Let X be a GEV distributed random variable, then the distribution of X is characterised by the location, scale and shape parameters denoted, respectively, by θ, β and ξ [28]. These parameters are given by the following lemma.

Lemma 5: Let X be a GEV random variable with mean, variance and skewness given by μ_X, σ_X^2 and $\mathscr{S}_X$, respectively. Then, the shape, scale and location of X are given, respectively, by

$$\xi = \begin{cases} a_1 \ln(b_1 \mathscr{S}_X^2 + c_1 \mathscr{S}_X + d_1), & \mathscr{S}_X < -0.63 \\ a_2 \mathscr{S}_X^2 + b_2 \mathscr{S}_X + c_2, & -0.63 \leq \mathscr{S}_X < 1.14 \\ 0, & \mathscr{S}_X = 1.14 \\ \dfrac{a_3 \mathscr{S}_X^2 + b_3 \mathscr{S}_X + c_3}{\mathscr{S}_X^2 + d_3 \mathscr{S}_X + e_3}, & \mathscr{S}_X > 1.14 \end{cases} \tag{37}$$

$$f(x) = \mathscr{K}'_{\mathrm{cc}} \sum_{n,m=1}^{K} (-1)^{n+m} \sum_{\delta \in \mathbb{P}_1} \mathrm{sgn}(\delta) \prod_{i=1}^{K-1} \frac{(N - r_{\delta(i),n} + g_{r_{i,m},1}(K, 2i) - 1)!}{\sigma_{r_{i,m}}^{-(N - r_{\delta(i),n} + g_{r_{i,m},1}(K,2i))}} \sum_{\boldsymbol{s} \in \mathbb{S}_1} (-1)^{|\boldsymbol{s}|}
\\ \times \sum_{l_1 \cdots l_{K-1}} \frac{\Pi l_i! \cdot \Pi \sigma_{r_{i,m}}^{l_i}}{(\Sigma l_i + N + n - g_{m,1}(K, 2i) - 1)!} \cdot
\\ \times \frac{x^{\Sigma \boldsymbol{s}} \left[\Sigma \boldsymbol{s} x^{-1} \left(\Sigma_{\boldsymbol{s}} (x/\sigma_{s_i,m}) + \Sigma_{\bar{\boldsymbol{s}}} (1/\sigma_{\bar{s}_i,m}) + (1/\sigma_m) \right) - (\Sigma l_i + N + n - g_{m,1}(K, 2i)) \Sigma_{\boldsymbol{s}} (1/\sigma_{s_i,m}) \right]}{\left(\Sigma_{\boldsymbol{s}} (x/\sigma_{s_i,m}) + \Sigma_{\bar{\boldsymbol{s}}} (1/\sigma_{\bar{s}_i,m}) + (1/\sigma_m) \right)^{\Sigma l_i + N + n - g_{m,1}(K,2i) + 1}} \tag{29}$$

$$M(p) = \mathscr{K}'_{\mathrm{nu}} \sum_{n=1}^{K} (-1)^{n+1} \sum_{\delta \in \mathbb{P}_0} \mathrm{sgn}(\delta) \sum_{\alpha \in \mathbb{P}_1} \mathrm{sgn}(\alpha) \sum_{h=0}^{\infty} \frac{\omega_1^h}{(N - K + h)!h!}
\\ \times \sum_{l_1 \cdots l_{K-1}} \frac{\Pi_i (N - \delta(i) + G_{i,n}(h, K, \alpha) + l_{i-1} + C_{i,p})!}{\Pi_i l_i! \cdot \Pi_i i^{N - \delta(i) + G_{i,n}(h,K,\alpha) + C_{i,p} + l_{i-1} - l_i + 1}} \tag{33}$$

$$M(p) = \mathscr{K}'_{\mathrm{cc}} \sum_{n=1}^{K} (-1)^{n+1} \sum_{\delta \in \mathbb{P}_0} \mathrm{sgn}(\delta) \sum_{\alpha \in \mathbb{P}_1} \mathrm{sgn}(\alpha)
\\ \times \sum_{l_1 \cdots l_{K-1}} \frac{\Pi_i (N + \delta(i) + G_{i,n}(0, K, \alpha) + l_{i-1} + C_{i,p} - K - 1)!}{\Pi_i l_i! \cdot \Pi_i \left[\sum_{j=1}^{i} 1/\sigma_{b_{j,n}} \right]^{N + \delta(i) + G_{i,n}(0,K,\alpha) + C_{i,p} + l_{i-1} - l_i - K}} \tag{35}$$

$$\beta = \begin{cases} \sqrt{\dfrac{\sigma_X^2 \xi^2}{g_2 - g_1^2}}, & \mathcal{S}_X \neq 1.14 \\ \dfrac{\sqrt{6\sigma_X^2}}{\pi}, & \mathcal{S}_X = 1.14 \end{cases} \quad (38)$$

$$\theta = \begin{cases} \mu_X - \dfrac{(g_1 - 1)\beta}{\xi}, & \mathcal{S}_X \neq 1.14 \\ \mu_X - \beta\gamma, & \mathcal{S}_X = 1.14 \end{cases} \quad (39)$$

where the constants in (37) are given in Table 1.

The mean, the variance and the skewness of X are provided by Table 2.

6.1 Distribution approximation

By considering the moments of the SCN, in Theorems 3–5, the mean, variance and skewness of the SCN are written as follows:

$$\mu_X = M(1) \quad (40)$$

$$\sigma_X^2 = M(2) - \mu_X^2 \quad (41)$$

$$\mathcal{S}_X = \frac{M(3) - 3M(2)\mu_X + 2\mu_X^3}{\sigma_X^3} \quad (42)$$

Accordingly, we give the following proposition.

Proposition 1: Let X be the SCN of $\boldsymbol{W}$ and consider the following three cases:

- $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K)$.
- $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \boldsymbol{I}_K, \boldsymbol{\Omega}_K)$ with $\boldsymbol{\Omega}_K$ has only one non-zero eigenvalue ω_1.
- $\boldsymbol{W} \sim \mathcal{CW}_K(N, \boldsymbol{\Sigma}_K)$ with $\boldsymbol{\Sigma}_K$ has $K-1$ equal eigenvalues ($\sigma_1 > \sigma_2 = \cdots = \sigma_K$).

Then, the CDF and PDF of X can be tightly approximated, respectively, by

$$F(x; \theta, \beta, \xi) = \mathrm{e}^{-t(x)} \quad (43)$$

$$f(x; \theta, \beta, \xi) = \frac{1}{\beta} t(x)^{\xi + 1} \mathrm{e}^{-t(x)} \quad (44)$$

where $t(x)$ is defined in Table 2. ξ, β and θ are given by (37), (38) and (39) where the mean, variance and skewness are, respectively, given by (40), (41) and (42) with $M(p)$ is the pth moment of the SCN given, respectively for each case, by

i. Theorem 3.
ii. Theorem 4.
iii. Theorem 5.

6.2 Performance probabilities and decision threshold

Denoting by κ the decision threshold, then P_{d} and P_{fa} are, respectively, given by

$$P_{\mathrm{fa}} = P(X \geq \kappa / \mathcal{H}_0) \quad (45)$$

$$P_{\mathrm{d}} = P(X \geq \kappa / \mathcal{H}_1) \quad (46)$$

These performance probabilities depend on the threshold (κ) being used. However, if the expressions of the P_{fa} and P_{d} are known, then the threshold can be set according to a required error constraint. Consequently, P_{fa} and P_{d} are, due to Proposition 1, given by

$$P_{\{.\}} = \begin{cases} 1 - \mathrm{e}^{-(1 + ((\kappa - \theta)/\beta)\xi)^{-1\xi}}, & \xi \neq 0 \\ 1 - \mathrm{e}^{-\mathrm{e}^{-((\kappa - \theta)/\beta)}}, & \xi = 0 \end{cases} \quad (47)$$

where {.} refer to 'fa' or 'd'. GEV parameters are evaluated using Lemma 5 by considering the moments in Theorem 3 for P_{fa} and in Theorem 4 for P_{d}.

An alternative approximation could be provided for P_{d} using both Lemma 3 and case 3 of Proposition 1. We call this approximation the two-step approximation. In this approach, we first approximate the SCN of NUW matrix by the SCN of CCW matrix due to Lemma 3 and then we approximate the later using

Table 1 Constants of (37), Lemma 5

i	a_i	b_i	c_i	d_i	e_i
1	− 0.43544	2.3227	− 0.97563	1.3781	–
2	− 0.06393	0.3173	− 0.2771	–	–
3	0.333	− 0.09862	− 0.3195	0.9553	1.599

Table 2 Mean, variance, skewness and $t(x)$ for GEV

ξ	Mean (μ_X)	Variance (σ_X^2)	Skewness ($\mathcal{S}_X$)	$t(x)$
< 0	$\theta - (1 - g_1)\frac{\beta}{\xi}^{a}$	$(g_2 - g_1^2)\frac{\beta^2}{\xi^2}^{a}$	$-\frac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{32}}^{a}$	$\left(1 + \left(\frac{x - \theta}{\beta}\right)\xi\right)^{-1\xi}$
= 0	$\theta + \beta\gamma^{b}$	$\beta^2\frac{\pi^2}{6}$	$\frac{12\sqrt{6}\zeta(3)}{\pi^3}^{c}$	$\mathrm{e}^{-((x - \theta)/\beta)}$
$\left]0, \frac{1}{2}\right[$	$\theta - (1 - g_1)\frac{\beta}{\xi}^{a}$	$(g_2 - g_1^2)\frac{\beta^2}{\xi^2}^{a}$	$\frac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{32}}^{a}$	$\left(1 + \left(\frac{x - \theta}{\beta}\right)\xi\right)^{-1\xi}$
$\left[\frac{1}{2}, 1\right[$	$\theta - (1 - g_1)\frac{\beta}{\xi}^{a}$	∞	$\frac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{32}}^{a}$	$\left(1 + \left(\frac{x - \theta}{\beta}\right)\xi\right)^{-1\xi}$
≥ 1	∞	∞	$\frac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{32}}^{a}$	$\left(1 + \left(\frac{x - \theta}{\beta}\right)\xi\right)^{-1\xi}$

[a] $g_k = \Gamma(1 - k\xi)^{d}$.

[b] γ: Euler constant.

[c] $\zeta(.)$: Riemann zeta function.

[d] $\Gamma(.)$: Gamma function.

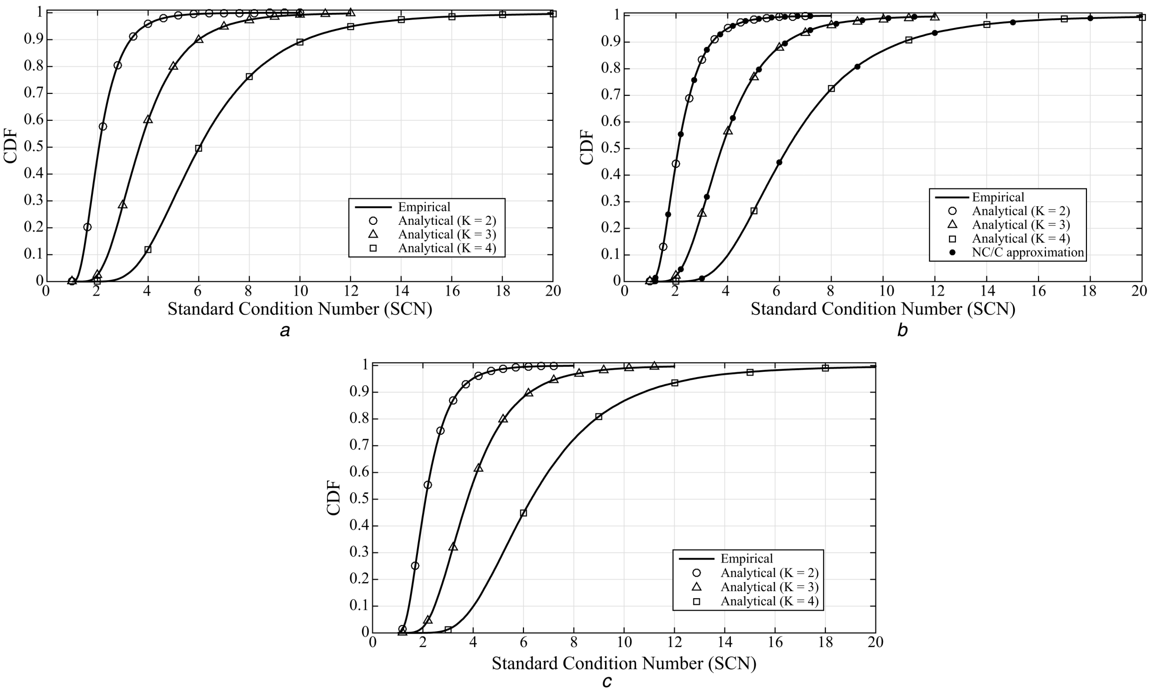


Fig. 1 *Empirical and analytical CDF of the SCN of Wishart matrices for different number of sensors K = {2, 3, 4}, N = 10 and SNR = − 10 dB*
(a) CUW case, ***(b)*** NUW case, ***(c)*** CCW case

Table 3 Empirical and analytical mean of the SCN of Wishart matrices

	N	10		20	
K		2	3	2	3
CUW case	mean-empirical	2.2527	4.002	1.7243	2.5021
	mean-analytical	2.2560	4.0041	1.7239	2.5020
NUW case	mean-empirical	2.3025	4.1495	1.7723	2.6267
	mean-analytical	2.3036	4.1442	1.7743	2.6271
CCW case	mean-empirical	2.3041	4.1489	1.7780	2.6311
	mean-analytical	2.3060	4.1529	1.7760	2.6308

GEV approximation. Then, P_{d} is still written as (47); however, the moments used in calculating μ, σ and ξ are given by Theorem 5.

Accordingly, the threshold could be calculated. For example, for a constant false-alarm probability ($\hat{P}_{\mathrm{fa}}$), the threshold is given by

$$\kappa = \begin{cases} \theta + \dfrac{\beta}{\xi}(-1 + [-\ln(1 - \hat{P}_{\mathrm{fa}})]^{-\xi}), & \xi \neq 0 \\ -\theta - \beta \ln(-\ln(1 - \hat{P}_{\mathrm{fa}})), & \xi = 0 \end{cases} \tag{48}$$

6.3 Comments on the complexity

In practice, channel conditions are not stable and K and N may frequently change. Consequently, the implementation of the decision threshold must be dynamic and may rely on real-time computations rather than using LUTs.

The real-time computations are satisfied using the proposed threshold in (48), however, the complexity still exists in the computation of the GEV parameters due to the complexity of the exact SCN moments. This complexity could be avoided by an offline computation of these parameters.

As a comparison, the exact SCN distribution needs a one-dimensional (1D) LUT for every (K, N) value under $\mathscr{H}_0$ hypothesis whereas only three values are needed in the proposed GEV approximation (i.e. parameters). Further step is to approximate the SCN central moments using a simple formulation and thus no need for the offline computation. In [29], we have studied this approach asymptotically, however, an approximation for the SCN moments for practical (K, N) values is a target in our future work.

7 Simulation and discussion

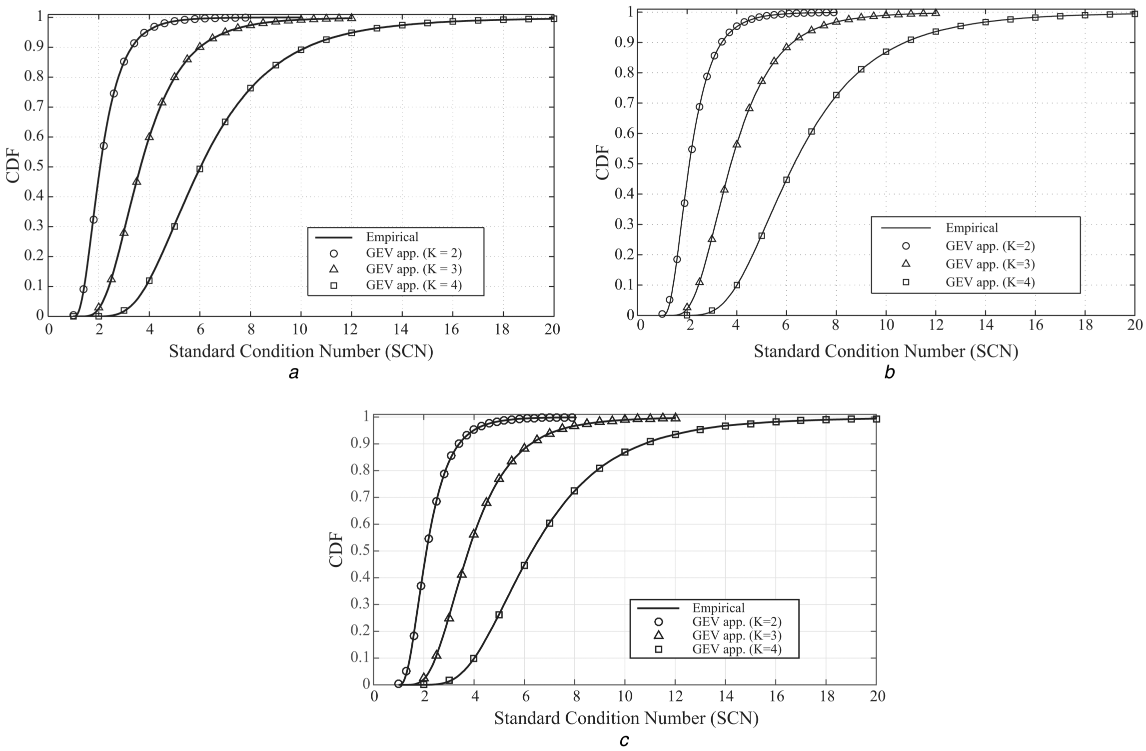
In this section, we discuss the analytical results through Monte-Carlo simulations, as presented in Sections 4–6. The simulation results are obtained by generating 10^5 random realisations of $\boldsymbol{Y}$.

Fig. 1 validates the analytical form of the distribution of the SCN of the three considered Wishart cases derived in Section 4. Results show a perfect match between the empirical SCN distribution and the corresponding analytical form in each case.

Table 3 validates the analytical expressions of the pth moment of the SCN of the three considered Wishart cases derived in Section 5 by considering the first moment. The SNR value is set − 10 dB and the correlation matrix is considered as given by Lemma 3 to show the accuracy of non-central/central approximation. Table 3 shows a perfect accuracy in the analytical representation in each case. Moreover, from the table and Figs. 1*b* and *c*, results also show a high accuracy in the non-central/central approximation presented in Lemma 3.

Fig. 2 shows the accuracy of the GEV approximation of the SCN of the three considered Wishart cases as proposed in Section 6. The results show high accuracy in the GEV approximation in all cases for different number of sensors.

Table 4 shows the rejection rate of the Kolmogorov–Smirnov test (KS-test) of the GEV approximation for different values of K and N with SNR = − 10 dB. KS-test is a technique for testing the goodness of fit of a samples from some unknown distribution. It is a non-parametric test used to compare continuous 1D distributions by comparing the distance between the empirical distribution function of the samples and the CDF of the assumed distribution. The rejection rate is calculated over 5×10^4 realisations of SCN sets. Each set is composed of 10^2 SCN samples. For each



a

b

c

Fig. 2 *Empirical CDF of the SCN of Wishart matrices and its corresponding GEV approximation*
(a) CUW case, ***(b)*** NUW case, ***(c)*** CCW case

Table 4 Rejection rate of the KS-test of the GEV approximation

$(K \times N)$	(2×10)	(3×10)	(3×20)	(4×50)
CUW case	1.178	1.058	1.02	0.992
NUW case	1.062	1.042	1.034	0.9040
CCW case	1.264	1.06	1.004	1.002

realisation, the KS-test tries to reject the null hypothesis with significance level of 0.01. The null hypothesis states that the SCN sample follows the GEV distribution as stated in Proposition 1. Indeed, the results of the KS test are small enough for the tested data. This shows the accuracy of the approximation.

The performance of the CR system is considered in Fig. 3. Figs. 3*a* and *b* show the empirical P_{fa} and P_{d} of the SCN and its corresponding GEV approximation and the two-step approximation given by Section 6.2. Fig. 3*c* shows the empirical results of the P_{d} as a function of SNR and its corresponding GEV approximation and two-step approximation. Fig. 3*d* shows the empirical receiver operating characteristics (ROC) of the SCN detector and its corresponding approximation using GEV for P_{fa} and the GEV or the two-step approximation for P_{d}. Results show that the proposed approximations are perfect and match the empirical results with high accuracy. In addition, results show that the two-step approximation could be used as an alternative for the GEV approximation of P_{d} since both have approximately same accuracy, especially as N increases, and since the moments of the SCN of CCW matrix are simpler than those of NUW matrix.

Fig. 4 compares the SCN detector and the ED when noise uncertainty is considered. This figure shows the ROC of both detectors for $K = 3$, $N = 500$, SNR = -10 dB and noise uncertainty of 0.1 dB. The decision threshold for the SCN detector is calculated using the proposed equation in (48). Results show that the SCN detector outperforms the ED as noise uncertainty is considered. It is indeed due to the blind detection nature of the SCN.

8 Conclusion

In this paper, we have considered the SCN of complex Wishart matrices in a CR system with finite number of cooperative sensors and finite number of samples. We derived the exact distribution of the SCN of the CUW, CCW and NUW matrices. In addition, the pth moment of the SCN is considered as we derived the exact form for the pre-mentioned Wishart matrices. To overcome the computational complexity of the exact SCN distribution, we provided an approximation for this distribution using the GEV distribution based on moment matching criteria. Accordingly, an approximation for the performance probabilities of the SCN detector and its decision threshold are given through simple expressions. Moreover, and using the non-central/central approximation we provide a two-step approximation for the detection probability of the SCN detector which provides analogous accuracy with simpler moment complexity. The analytical results are validated through extensive Monte-Carlo simulations with different system parameters. Results have shown that the approximation perfectly fits the empirical results.

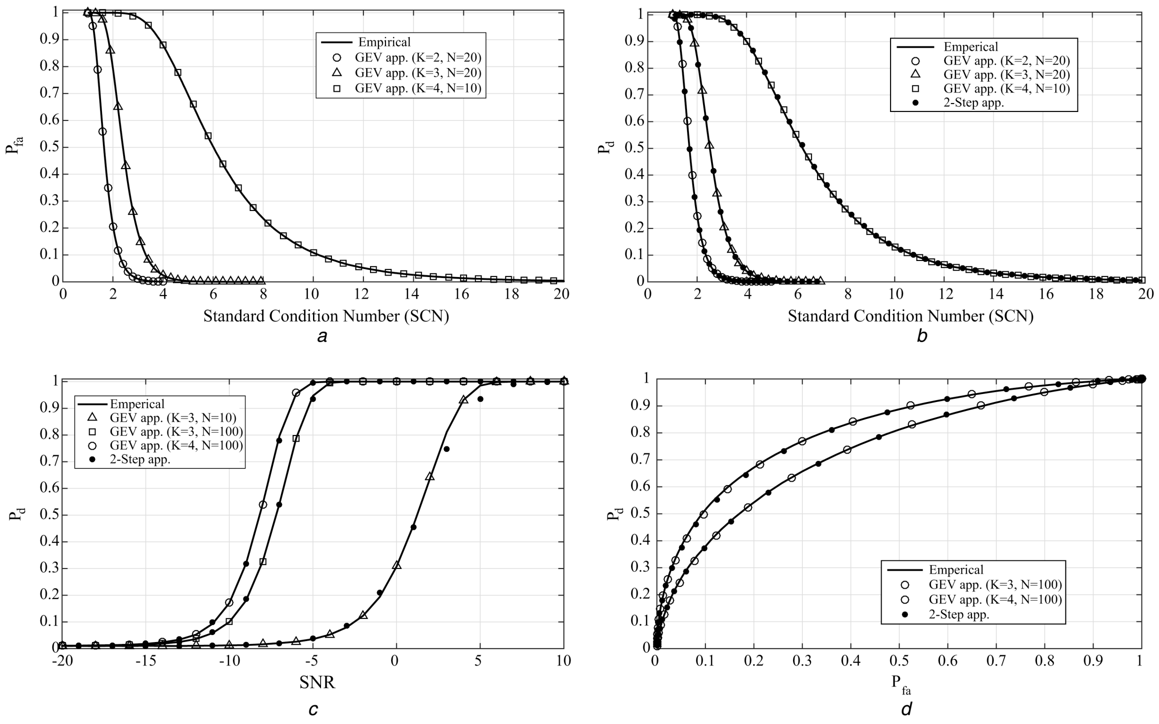


Fig. 3 *Empirical performance probabilities and ROC of the SCN detector and its corresponding GEV and two-step approximations for different K, N and SNR = − 10 dB*

(a) Probability of false-alarm (P_{fa}), ***(b)*** Probability of detection (P_{d}), ***(c)*** P_{d} versus SNR. dROC at −10 dB

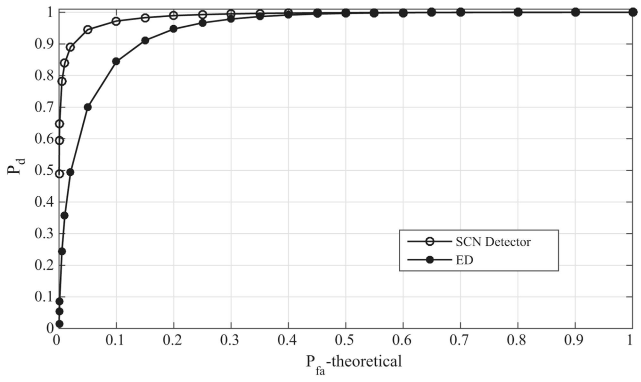


Fig. 4 *ROC of the SCN detector versus ROC of the ED for K = 3, N = 500, SNR = − 10 dB and 0.1 dB noise uncertainty*

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10 Appendix

10.1 Appendix A: Proof of Equations (15), (16), (21), (22), (28), (29).

Considering (13), (18) and (25) and applying Laplace expansion followed by Leibniz formula of the determinant [30] we get (see (49)) where {.} refers to one of these cases (cu, nu or cc).

Knowing that $\gamma(n, \lambda) = (n-1)!\left(1 - \mathrm{e}^{-\lambda}\sum_{l=0}^{n-1}\lambda^{l}/l!\right)$ and ${}_0F_1(n, \lambda) = \sum_{l=0}^{\infty}\lambda^{l}/(n)_l l!$, then (15), (21) and (28) are derived as follows:

- substitute (14), (19) or (26) in (49) for (15), (21) and (28), respectively.
- algebraic manipulation and integrate.

Equations (16), (22) and (29) are, respectively, the derivatives of (15), (21) and (28).

10.2 Appendix B: Proof of Theorems 3–5.

The *p*th moment of the SCN is given by

$$M(p) = \int_{0}^{\infty}\int_{\lambda_K}^{\infty}\cdots\int_{\lambda_2}^{\infty}\left(\frac{\lambda_1}{\lambda_K}\right)^{p}\cdot f(\boldsymbol{\lambda})\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_K \tag{50}$$

then the result is derived as follows:

- substitute the JDOE for the CUW, NUW and CCW cases.
- apply Laplace expansion.
- apply Leibniz formula.
- algebraic manipulation to transform the product into sum then collect common terms.
- integrate using [24, Eq. (3.351.2)] and [24, Eq. (3.351.3)] and watch the recurrence.

$$F(x) = \mathscr{K}_{\{.\}}\sum_{n,m=1}^{K}(-1)^{n+m}\sum_{\delta\in\mathbb{P}_1}\mathrm{sgn}(\delta)\int_{0}^{\infty}\Upsilon_{n,m}^{\{.\}}(\lambda_K)\prod_{i=1}^{K-1}\int_{\lambda_K}^{x\lambda_K}\Upsilon_{r_{\delta_i,n},r_{i,m}}^{\{.\}}(u)\,\mathrm{d}u\,\mathrm{d}\lambda_K \tag{49}$$