



Hamiltonian formulation of ghost free mimetic massive gravity

O. Malaeb^a, C. Saghir^b

Physics Department, American University of Beirut, Beirut, Lebanon

Received: 29 May 2019 / Accepted: 17 June 2019 / Published online: 11 July 2019
© The Author(s) 2019

Abstract We present the hamiltonian formulation of the Ghost Free Mimetic Massive Gravity theory. The linearized theory is studied and the Hamiltonian equations of motion are analyzed. Poisson brackets are computed and closure is proved. To prove that this theory is ghost-free, the number of degrees of freedom is analyzed showing that we have only five degrees of freedom.

1 Introduction

The idea of Mimetic Gravity was first formulated by Chamseddine and Mukhanov [1] through the parametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ , dubbed mimetic field. The same idea was formulated in [2] without the need of auxiliary metric but through the imposition of an additional constraint on the mimetic field. In [3], it was shown that this theory can predict several cosmological solutions: inflation, bouncing universe... through the addition of an arbitrary potential $V(\phi)$ to the action. Dark energy could also be produced through the addition of an extra non dynamical scalar fields associated with new constraints [4]. In [5] and [6], the theory of mimetic gravity was extended by the addition of a new function $f(\chi)$, where $\chi = \square\phi$. Through this function, Chamseddine and Mukhanov were able to prove that both Big Bang and Black Hole singularities are resolved through a specific choice of $f(\chi)$ function. Mimetic gravity was extended to mimetic F(R) gravity which can describe inflation and late time acceleration era [7]. New versions of F(R) mimetic gravity was presented in [8] and proved to be ghost free. A comprehensive review of mimetic gravity theory and its extensions is presented in [9]. Moreover, the hamiltonian analysis of several mimetic gravity models is done in [10] with the elaboration of the presence of instabilities. Recently, Chamseddine and

Mukhanov [11] were able to generate mass of graviton field with a mass term differing from that of Fierz Pauli's one without the presence of an extra ghost field.

In this paper we are going to consider the canonical formulation of this theory. The ADM formalism of general relativity was studied by Arnowitt, Deser and Misner in 1959 [12]. The ADM formalism of mimetic theory was constructed in [13].

The aim of this paper is to analyze the hamiltonian equations of motion and to check that the theory doesn't lead to ghosts. Also, Poisson brackets will be computed.

2 Canonical form

It was proposed in [11] that to form a ghost free mimetic massive gravity, the mass term must necessarily be taken to be different from Fierz-Pauli type. The following action was considered

$$I = \int d^4x \sqrt{g} \left(-\frac{1}{2}R + \frac{m^2}{8} \left(\frac{1}{2}\bar{h}^2 - \bar{h}^{AB}\bar{h}_{AB} \right) + \lambda \left(g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 - 1 \right) \right), \quad (1)$$

where the last term is accounting for the mimetic origin of ϕ^0 and the mass term is distinguished from Fierz-Pauli by a relative coefficient of 1/2 instead of 1.

The induced metric perturbation \bar{h}^{AB} are given by

$$\bar{h}^{AB} = g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - \eta^{AB}, \quad (2)$$

where capital indices are raised and lowered by Minkowski metric $\eta^{AB} = (1, -1, -1, -1)$.

The purpose of this paper to construct and analyze the canonical formalism of this theory. The first step is to split the time component from space by rewriting the action in a 3 + 1 dimensional form. Considering small perturbations of

^a e-mail: oh09@aub.edu.lb

^b e-mail: cas25@mail.aub.edu

the fields around broken symmetry phase,

$$\phi^A = x^A + \chi^A, \tag{3}$$

the mass term and the mimetic one are then given by

$$\begin{aligned}
 S_\phi = \int d^4x \sqrt{g} \frac{m^2}{8} g^{00} & \left(-2 - 4\partial_0 \chi^0 \right. \\
 & \left. - 2\partial_0 \chi^0 \partial_0 \chi^0 - 2\partial_0 \chi^i \partial_0 \chi^k \eta_{ik} \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{0k} \left(-4\partial_k \chi^0 - 4\partial_k \chi^0 \partial_0 \chi^0 \right. \\
 & \left. - 4\eta_{mk} \partial_0 \chi^m - 4\partial_0 \chi^m \partial_k \chi^n \eta_{mn} \right) \\
 & + \sqrt{g} \frac{m^2}{8} + g^{ik} \left(-2\eta_{ik} - 4\eta_{mk} \partial_i \chi^m \right. \\
 & \left. - 2\partial_i \chi^0 \partial_k \chi^0 - 2\eta_{mn} \partial_i \chi^m \partial_k \chi^n \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{00} g^{00} \left(\frac{-1}{2} - 2\partial_0 \chi^0 \right. \\
 & \left. - 3\partial_0 \chi^0 \partial_0 \chi^0 - \eta_{rs} \partial_0 \chi^r \partial_0 \chi^s \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{00} g^{0k} \left(-2\partial_k \chi^0 - 6\partial_k \chi^0 \partial_0 \chi^0 - 2\eta_{rk} \partial_0 \chi^r \right. \\
 & \left. - 4\eta_{rk} \partial_0 \chi^r \partial_0 \chi^0 - 2\eta_{ir} \partial_0 \chi^i \partial_k \chi^r \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{00} g^{ik} \left(\eta_{ik} - \partial_i \chi^0 \partial_k \chi^0 + 2\partial_0 \chi^0 \eta_{ik} \right. \\
 & + 2\partial_i \chi^r \eta_{rk} + 4\partial_0 \chi^0 \partial_i \chi^r \eta_{rk} + \partial_i \chi^r \partial_k \chi^s \eta_{rs} \\
 & + \partial_0 \chi^0 \partial_0 \chi^0 \eta_{ik} + \eta_{mn} \eta_{ik} \partial_0 \chi^m \partial_0 \chi^n \\
 & \left. - 2\eta_{mi} \eta_{nk} \partial_0 \chi^m \partial_0 \chi^n - 4\eta_{rk} \partial_0 \chi^r \partial_i \chi^0 \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{0i} g^{0k} \left(-2\eta_{ik} - 2\partial_k \chi^0 \partial_i \chi^0 - 4\eta_{ik} \partial_0 \chi^0 \right. \\
 & \left. - 4\eta_{ri} \partial_k \chi^r - 2\eta_{mr} \eta_{ki} \partial_0 \chi^m \partial_0 \chi^r \right. \\
 & \left. - 2\eta_{ki} \partial_0 \chi^0 \partial_0 \chi^0 - 8\eta_{ri} \partial_0 \chi^0 \partial_k \chi^r - 2\eta_{rs} \partial_i \chi^s \partial_k \chi^r \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{0k} g^{lj} \left(2\partial_k \chi^0 \eta_{jl} - 4\eta_{kl} \partial_j \chi^0 + 2\eta_{mk} \eta_{lj} \partial_0 \chi^m \right. \\
 & \left. - 4\partial_0 \chi^m \eta_{ml} \eta_{kj} + 2\partial_k \chi^0 \partial_0 \chi^0 \eta_{lj} \right. \\
 & \left. - 4\eta_{kl} \partial_0 \chi^0 \partial_j \chi^0 + 4\partial_k \chi^0 \partial_j \chi^r \eta_{rl} - 4\eta_{rl} \partial_j \chi^0 \partial_k \chi^r \right. \\
 & \left. - 4\eta_{ks} \partial_j \chi^0 \partial_l \chi^s + 2\eta_{mn} \eta_{lj} \partial_0 \chi^m \partial_k \chi^n \right. \\
 & \left. - 4\partial_0 \chi^m \partial_l \chi^r \eta_{mr} \eta_{kj} - 4\partial_0 \chi^m \partial_k \chi^n \eta_{ml} \eta_{nj} \right. \\
 & \left. - 4\partial_0 \chi^m \partial_j \chi^s \eta_{ml} \eta_{ks} + 4\eta_{mk} \eta_{ls} \partial_0 \chi^m \partial_j \chi^s \right) \\
 & + \sqrt{g} \frac{m^2}{8} g^{ik} g^{lj} \left(\frac{1}{2} \eta_{ik} \eta_{lj} - \eta_{il} \eta_{kj} + 2\eta_{lj} \eta_{mk} \partial_i \chi^m \right. \\
 & \left. - 4\partial_i \chi^m \eta_{ml} \eta_{kj} + \partial_l \chi^0 \partial_j \chi^0 \eta_{ik} \right. \\
 & \left. - 2\eta_{kj} \partial_i \chi^0 \partial_l \chi^0 + \eta_{mn} \eta_{lj} \partial_i \chi^m \partial_k \chi^n \right. \\
 & \left. - 2\partial_i \chi^m \partial_l \chi^r \eta_{mr} \eta_{kj} - 2\partial_i \chi^m \partial_k \chi^n \eta_{ml} \eta_{nj} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + 2\eta_{mk} \eta_{rj} \partial_i \chi^m \partial_l \chi^r - 2\eta_{ml} \Big) \\
 & + \lambda \left(g^{00} \left(1 + 2\partial_0 \chi^0 + \partial_0 \chi^0 \partial_0 \chi^0 \right) \right. \\
 & \left. + 2g^{0k} \partial_k \chi^0 \left(1 + \partial_0 \chi^0 \right) + g^{ik} \partial_i \chi^0 \partial_k \chi^0 - 1 \right) \tag{4}
 \end{aligned}$$

Inducing small metric perturbations

$$g^{\mu\nu} = h^{\mu\nu} + \eta^{\mu\nu}, \tag{5}$$

then to first order in perturbation, \bar{h}^{AB} becomes

$$\bar{h}_{\mu\nu} = -h_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu. \tag{6}$$

Upon restricting the action to second order in $h_{\mu\nu}$ and χ^A , noting that λ is of first order in perturbations, we get the simplified action

$$\begin{aligned}
 S_\phi = \int d^4x \sqrt{g} & \left(\partial_0 \chi^0 \left(\frac{m^2}{8} \left(-2h^{00} - 2\partial_0 \chi^0 \right) \right. \right. \\
 & \left. \left. + 4\partial_i \chi^i + 2h^{ik} \eta_{ik} \right) + 2\lambda \right) \\
 & + \partial_0 \chi^i \left(\frac{m^2}{8} \left(-4\partial_i \chi^0 - 2\partial_0 \chi_i - 4h^{0j} \eta_{ij} \right) \right) \\
 & + h^{0k} \left(\frac{m^2}{8} \left(-4\partial_k \chi^0 - 2h^{0i} \eta_{ik} \right) \right) \\
 & + h^{00} \left(\frac{m^2}{8} \left(\frac{-1}{2} h^{00} + 2\partial_i \chi^i + \eta_{ik} h^{ik} \right) + \lambda \right) \\
 & + h^{ik} \frac{m^2}{8} \left(2\eta_{ik} \partial^j \chi_j - 4\partial_k \chi_i \right) \\
 & + h^{ik} h^{lj} \frac{m^2}{8} \left(\frac{1}{2} \eta_{ik} \eta_{lj} - \eta_{il} \eta_{kj} \right) \\
 & + \frac{m^2}{8} \left(-2\partial_i \chi^0 \partial^i \chi^0 + 2\partial_i \chi^i \partial_k \chi^k \right. \\
 & \left. - 2\partial^k \chi^j \partial_j \chi_k - 2\partial^i \chi^j \partial_i \chi_j \right) \tag{7}
 \end{aligned}$$

The momenta conjugate to χ^0 and χ^i are given respectively

$$\begin{aligned}
 P = \frac{\partial L}{\partial \dot{\chi}^0} & = \frac{m^2}{8} \sqrt{g} \left(-2h^{00} + 4\partial_i \chi^i + 2h^{ik} \eta_{ik} \right) \\
 & + 2\sqrt{g} \lambda - 4\sqrt{g} \frac{m^2}{8} \dot{\chi}^0 \tag{8}
 \end{aligned}$$

$$\begin{aligned}
 P_i = \frac{\partial L}{\partial \dot{\chi}^i} & = \frac{m^2}{8} \sqrt{g} \left(-4\partial_i \chi^0 - 4h^{0j} \eta_{ij} \right) - \frac{m^2}{2} \sqrt{g} \partial_0 \chi_i \tag{9}
 \end{aligned}$$

These two equations can be inverted to write both $\dot{\chi}^0$ and $\dot{\chi}^i$ in terms of their corresponding momenta

$$\dot{\chi}^0 = \frac{-2}{\sqrt{g}m^2}P + \frac{1}{4} \left(-2h^{00} + 4\partial_i\chi^i + 2h^{ik}\eta_{ik} \right) + \frac{4\lambda}{m^2} \tag{10}$$

$$\dot{\chi}^i = \frac{-2P^i}{m^2\sqrt{g}} - \partial_j\chi^0\eta^{ij} - h^{0i} \tag{11}$$

In terms of momenta the Hamiltonian becomes

$$\begin{aligned} H_\phi &= P\dot{\chi}^0 + P_i\dot{\chi}^i - L_\phi \\ &= -\sqrt{g} \left(h^{ij}\frac{m^2}{8} \left(4\eta_{ij}\partial_k\chi^k - 4\partial_j\chi_i \right) \right. \\ &\quad + \frac{m^2}{8}h^{ij}h^{kl} \left(-\eta_{ik}\eta_{jl} + \frac{1}{2}\eta_{ij}\eta_{kl} \right) \\ &\quad + \frac{m^2}{8} \left(-2\partial_i\chi^j\partial^i\chi^j - 2\partial_i\chi^j\partial_j\chi^i + 4\partial_i\chi^i\partial_j\chi^j \right) \\ &\quad + \lambda\eta_{ij}h^{ij} + 2\lambda\partial_i\chi^i \left. - \frac{Ph^{00}}{2} + \eta_{ij}h^{ij}\frac{P}{2} \right. \\ &\quad + P\partial_i\chi^i - \frac{4\lambda^2\sqrt{g}}{m^2} \\ &\quad \left. + \frac{4P\lambda}{m^2} - \frac{P^2}{m^2\sqrt{g}} - \frac{P_iP^i}{m^2\sqrt{g}} - P_ih^{0i} - P^i\partial_i\chi^0 \right) \tag{12} \end{aligned}$$

The hamiltonian is still a function of the Lagrange multiplier λ . However, it is independent of its time derivative; therefore, $p_\lambda = 0$. This is a primary constraint and will imply a secondary constraint by demanding its time consistency

$$0 = \dot{p}_\lambda = \{p_\lambda, H\} = \frac{\partial H}{\partial \lambda}. \tag{13}$$

This allows us to find λ and it turns out to be

$$\lambda = \frac{P}{2\sqrt{g}} - \frac{m^2}{8}\eta_{ij}h^{ij} - \frac{m^2}{4}\partial_i\chi^i. \tag{14}$$

Substituting back in the Hamiltonian, we end up with

$$\begin{aligned} H_\phi &= \int d^4x \left(-\frac{1}{2}Ph^{00} - \frac{1}{m^2\sqrt{g}}P_iP^i - P_ih^{0i} - P^i\partial_i\chi^0 \right. \\ &\quad - \frac{m^2}{8}h^{ij} \left(2\eta_{ij}\partial_k\chi^k - 4\partial_j\chi_i \right) \\ &\quad - \frac{m^2}{8}h^{ij}h^{kl} \left(-\eta_{ik}\eta_{jl} + \frac{1}{2}\eta_{ij}\eta_{kl} \right) \\ &\quad \left. - \frac{m^2}{8} \left(-2\partial_i\chi_j\partial^i\chi^j - 2\partial_i\chi^j\partial_j\chi^i + 2\partial_i\chi^i\partial_k\chi^k \right) \right) \tag{15} \end{aligned}$$

As mentioned above, this hamiltonian is for the mass term and the mimetic one. To form the complete hamiltonian, we still have to add the general relativity part. Considering the Einstein-Hilbert action in terms of the linearized metric $h^{\mu\nu}$ and keeping it up to second order, we have

$$\begin{aligned} S_g &= -\frac{1}{4} \int d^4x \left(\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\sigma}\partial_\nu h_\sigma^\nu \right. \\ &\quad \left. + \frac{1}{2}\partial_\sigma h^{\mu\nu}\partial^\sigma h_{\mu\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h \right) \tag{16} \end{aligned}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Writing this in a 3 + 1 dimensional form, it is expressed in terms of the three variables h^{00} , h^{0k} and h^{ik} . The conjugate momenta corresponding to these variables are [15]

$$\Pi = -\frac{1}{4}\partial_m h^{0m} \tag{17}$$

$$\Pi_l = \frac{1}{4}\partial_l h^{00} + \frac{1}{4}\partial_l\eta_{rs}h^{rs} - \frac{1}{2}\partial_i\eta_{lr}h^{ri} \tag{18}$$

$$\Pi_{mn} = \frac{1}{4}\partial_i h^{i0}\eta_{mn} - \frac{1}{4}\eta_{mn}\partial^0\eta_{rs}h^{rs} + \frac{1}{4}\eta_{mr}\eta_{ns}\partial^0h^{rs} \tag{19}$$

By inverting the equation of Π_{mn} we can write $\partial_0 h^{qt}$ as follow

$$\partial_0 h^{qt} = 4\eta^{mq}\eta^{nt}\Pi_{mn} - 2\eta^{qt}\eta^{ij}\Pi_{ij} + \frac{1}{2}\partial_i h^{i0}\eta^{qt} \tag{20}$$

Therefore, the total hamiltonian ($H_\phi + H_g$) is

$$\begin{aligned} H &= \Pi h^{\dot{0}0} + \Pi_i h^{\dot{0}i} + \Pi_{ij} h^{\dot{i}j} + P\dot{\chi}^0 + P_i\dot{\chi}^i - L \\ &= \int d^4x \left(+\frac{1}{2}\eta^{ij}\Pi_{ij}\partial_k h^{k0} \right. \\ &\quad + 2\Pi_{ij}\Pi_{kl}\eta^{ik}\eta^{jl} - \Pi_{ij}\Pi_{kl}\eta^{ij}\eta^{kl} \\ &\quad + \frac{1}{16}\partial_i h^{i0}\partial_j h^{j0} - \frac{1}{4}\partial_i h^{ij}\partial_j h^{kl}\eta_{kl} + \frac{1}{4}\partial_i h^{ij}\partial_l h^{kl}\eta_{jk} \\ &\quad - \frac{1}{4}\partial_i h^{0k}\partial^i h^{0l}\eta_{kl} - \frac{1}{8}\partial_i h^{km}\partial^i h^{ln}\eta_{kl}\eta_{mn} \\ &\quad + \frac{1}{8}\partial_i h^{kl}\partial^i h^{mn}\eta_{kl}\eta_{mn} \\ &\quad - \frac{1}{m^2\sqrt{g}}P_iP^i - P_ih^{0i} - P^i\partial_i\chi^0 \\ &\quad + h^{00} \left(\frac{1}{4}\partial_j\partial_i h^{ij} - \frac{1}{4}\partial_i\partial^i h^{kl}\eta_{kl} - \frac{1}{2}P \right) \\ &\quad - \frac{m^2}{8}h^{ij} \left(2\eta_{ij}\partial_k\chi^k - 4\partial_j\chi_i \right) \\ &\quad - \frac{m^2}{8}h^{ij}h^{kl} \left(-\eta_{ik}\eta_{jl} + \frac{1}{2}\eta_{ij}\eta_{kl} \right) \\ &\quad \left. - \frac{m^2}{8} \left(-2\partial_i\chi_j\partial^i\chi^j - 2\partial_i\chi^j\partial_j\chi^i + 2\partial_i\chi^i\partial_k\chi^k \right) \right) \tag{21} \end{aligned}$$

where h^{00} is appearing as a lagrange multiplier.

3 Equations of motion

Starting from the total hamiltonian constructed above the equations of motion are found by varying with respect to the variables χ^0, χ^i, P, P^i and h^{ij} .

The equations of motion of χ^0 and χ^i are respectively

$$\dot{P} = -\partial_i P^i \tag{22}$$

$$\dot{P}_i = -\frac{m^2}{4} \partial_i h^{kj} \eta_{kj} + \frac{m^2}{2} \partial_j h^{kj} \eta_{ki} + \frac{m^2}{2} \partial_j \partial^j \chi_i \tag{23}$$

The equations of motion of P and P_i are respectively

$$\dot{\chi}^0 = -\frac{h^{00}}{2} \tag{24}$$

$$\dot{\chi}_i + \frac{2}{m^2 \sqrt{g}} P_i + \partial_i \chi^0 + h^{0j} \eta_{ji} = 0 \tag{25}$$

Equation (24) is exactly the linearized mimetic constraint $\bar{h}^{00} = 0$

$$\begin{aligned} \bar{h}^{00} &= g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 - 1 \\ &= (h^{00} + 1) (1 + \partial_0 \chi^0) (1 + \partial_0 \chi^0) \\ &\quad - 2h^{0i} (1 + \partial_0 \chi^0) (\partial_i \chi^0) \\ &\quad + h^{ij} (\partial_i \chi^0) (\partial_j \chi^0) - 1 \\ &= h^{00} + 2\partial_0 \chi^0 = 0 \end{aligned} \tag{26}$$

Substituting for P_m in Eq. (25) using Eq. (9), and then replacing h by \bar{h} , we get

$$m^2 \left(\partial^\rho \bar{h}_{\rho k} - \frac{1}{2} \partial_k \bar{h} \right) = 0 \tag{27}$$

The linearized form of the equation $\dot{P} = -\partial_i P^i$ is

$$\begin{aligned} &\frac{m^2 \sqrt{g}}{8} \left(-2\dot{h}^{00} + 4\partial_0 \partial_i \chi^i + 2\partial_0 h^{ik} \eta_{ik} \right) \\ &\quad + 2\sqrt{g} \dot{\lambda} - 4\sqrt{g} \frac{m^2}{8} \ddot{\chi}^0 + \frac{m^2 \sqrt{g}}{8} \left(-4\partial_i \partial_j \chi^0 \eta^{ij} - 4\partial_i h^{0i} \right) \\ &\quad - \frac{m^2 \sqrt{g}}{2} \partial_i \partial_0 \chi^i = 0 \end{aligned} \tag{28}$$

which proves to be

$$\partial_0 \lambda - \frac{m^2}{4} \left(\partial^\rho \bar{h}_{\rho 0} - \frac{1}{2} \partial_0 \bar{h} \right) = 0 \tag{29}$$

The fourth Eq. (25) is simply returning the expression of P_i (Eq. 9).

Using the identity

$$\frac{\partial h^{ij}(x)}{\partial h^{kl}(y)} = \frac{1}{2} \left(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j \right) \delta(x, y), \tag{30}$$

the equation of h_{ij} proves to be

$$G_{ij}(-\bar{h}_{\rho\sigma}) = -\frac{m^2}{4} \left(\bar{h}_{ij} - \frac{1}{2} \eta_{ij} \bar{h} \right) \tag{31}$$

where

$$\begin{aligned} G_{\mu\nu}(h_{\rho\sigma}) &= -\frac{1}{2} \left(\partial^2 h_{\mu\nu} - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \partial_\mu \partial_\nu h \right) \\ &\quad + \frac{1}{2} \eta_{\mu\nu} \left(\partial^2 h - \partial^\sigma \partial^\rho h_{\rho\sigma} \right) \end{aligned} \tag{32}$$

is invariant under the coordinate transformation (6).

One of the equations of motion obtained is the constraint equation h^{00} and the other three Eqs. [(27), (29), (31)] are exactly what was found in [11]. There it was proven that all the remaining degrees of freedom are healthy and that mimetic massive gravity describes a massive graviton described by \bar{h}_{ij}^T and a mimetic matter described by λ .

4 Poisson bracket and degrees of freedom

Computing the poisson brackets is important in analyzing the number of physical degrees of freedom. Let us start by considering the primary first class constraints

$$\begin{aligned} T &= \Pi + \frac{1}{4} \partial_k h^{0k} \\ T_i &= \Pi_i - \frac{1}{4} \partial_i h^{00} - \frac{1}{4} \partial_i \eta_{rs} h^{rs} + \frac{1}{2} \partial_l \eta_{ir} h^{rl}. \end{aligned} \tag{33}$$

Furthermore, h^{00} appears as a lagrange multiplier. Therefore, we have one more first primary constraint

$$N = -\frac{1}{4} \partial_j \partial_i h^{ij} + \frac{1}{4} \partial_i \partial^i h^{kl} \eta_{kl} + \frac{P}{2} \tag{34}$$

The time change of these primary first class constraints will lead to a set of secondary first class constraints. To get the time change, we compute the poisson brackets of the primary constraints with the total hamiltonian. The time change of T is

$$\{T, H\} = +\frac{1}{4} \partial_j \partial_i h^{ij} - \frac{1}{4} \partial_i \partial^i h^{kl} \eta_{kl} - \frac{P}{2} = 0 \tag{35}$$

where $\frac{P}{2}$ is the linearized form of $\frac{m^2}{8}\sqrt{g}\bar{h} + \lambda\sqrt{g}$. Thus to first order, the above equation can be written as

$$G_{00}(-\bar{h}_{\rho\sigma}) = 2\lambda + \frac{m^2}{4}\bar{h} \tag{36}$$

Equation (36) is equivalent to

$$\Delta\bar{h} + \partial^i\partial^j\bar{h}_{ij} = 4\lambda + \frac{m^2}{8}\bar{h} \tag{37}$$

where $\Delta = -\partial^i\partial_i$.

Similarly, the time change of T_i is

$$\begin{aligned} \{T_k, H\} = & -\left(+\frac{1}{8}\partial_k\partial_j h^{0j} - \frac{1}{2}\partial_i\partial^i h^{0j}\eta_{kj} \right. \\ & \left. - P_k - \frac{1}{2}\partial_k(\eta^{ij}\Pi_{ij}) \right) \\ & + \frac{1}{4}\eta_{ij}\partial_k(\dot{h}^{ij}) - \frac{1}{2}\eta_{kj}\partial_i(\dot{h}^{ij}) = 0 \end{aligned} \tag{38}$$

Using the expression of Π_{ij} [Eq. (19)] and that of P_k [Eq. (9)], this equation will be given by

$$G_{0i}(-\bar{h}_{\rho\sigma}) = -\frac{m^2}{4}\bar{h}_{0i} \tag{39}$$

which is equivalent to

$$\Delta\bar{h}_{0i} + \partial_0\partial^k\bar{h}_{ki} + \partial_0\partial_i\left(\frac{4}{m^2}\lambda - \frac{1}{2}\bar{h}\right) = m^2\bar{h}_{0i}. \tag{40}$$

The above Eqs. (36, 37, 39, 40) are exactly those found in [11].

The time change of N is

$$\{N, H\} = 0 \tag{41}$$

Counting the degrees of freedom, we have ten independent fields $h_{\mu\nu}$ and four independent field X^A . This will give us a total of fourteen degrees of freedom. There are four primary first class constraints which lead to a set of four secondary class constraints. This will leave us with six degrees of freedom. There is one additional primary first class constraint; therefore, we end up having five degrees of freedom representing the massive graviton.

5 Looking at the mimetic term

The Hamiltonian (21) appears to be independent of lambda. To investigate the energy density of the mimetic term, this hamiltonian must be expanded up to second order in scalar

perturbations [14]. For small perturbations, different fields are expanded as follows

$$\begin{aligned} \chi^0 &= \chi^0 \\ \chi^i &= \tilde{\chi}^i - \partial^i\pi \\ h^{00} &= -2\phi \\ h^{0i} &= 0 \\ h^{ij} &= 2\psi\eta^{ij} \end{aligned} \tag{42}$$

Due to the mimetic constraint $\bar{h}^{00} = 0$, we get $h^{00} = -2\dot{\chi}^0$. Substituting these perturbations in (21), the scalar part of the hamiltonian becomes

$$\begin{aligned} H_{scalar} = & \int d^4x \left(\psi \Delta \psi - \frac{3m^2}{4}\psi^2 - 2\dot{\chi}^0 \Delta \psi \right. \\ & \left. - \frac{m^2}{2}\psi \Delta \pi + \frac{m^2}{4} \Delta \pi \Delta \pi \right. \\ & \left. + \dot{\chi}^0 P - P^i \partial_i \chi^0 - 3\dot{\psi}^2 - \frac{1}{m^2} P_i P^i \right) \end{aligned} \tag{43}$$

Varying the above hamiltonian with respect to χ^0 where the momenta (8 and 9), up to first order in scalar perturbations, are respectively

$$\begin{aligned} P &= 2 \Delta \psi \\ P_i &= -\frac{m^2}{2} \partial_i \chi^0 + \frac{m^2}{2} \partial_i \dot{\pi}, \end{aligned} \tag{44}$$

we get

$$\dot{\psi} = \frac{-m^2}{4} (\chi^0 - \dot{\pi}). \tag{45}$$

This is exactly the scalar perturbation of Eq. (22)

$$\dot{P} = -\partial_i P^i. \tag{46}$$

In terms of ψ and π , the Hamiltonian (43) becomes

$$\begin{aligned} H_{scalar} = & \int d^4x \left(\psi \left(\Delta - \frac{3m^2}{4} \right) \psi + \frac{4}{m^2} \dot{\psi} \left(\Delta - \frac{3m^2}{4} \right) \dot{\psi} \right. \\ & \left. - \frac{m^2}{4} (2\psi \Delta \pi - \Delta \pi \Delta \pi) - 2\dot{\pi} \Delta \dot{\psi} \right) \end{aligned} \tag{47}$$

To get rid of the mixed terms, we need to diagonalize the above hamiltonian. Starting from the equation of motion of h_{00} (37) or from the equation of lambda (14), we get the equation of motion of ψ

$$\Delta\psi - \frac{3m^2}{4}\psi = \lambda + \frac{m^2}{4} \Delta \pi \tag{48}$$

which gives

$$\psi = \left(\Delta - \frac{3m^2}{4} \right)^{-1} \left(\lambda + \frac{m^2}{4} \Delta \pi \right) \quad (49)$$

Substituting ψ in Eq. (47), the Hamiltonian for the pure mimetic term becomes

$$H_\lambda = \int d^4x \left(\frac{-16\dot{\lambda}^2}{m^2(3m^2 - 4\Delta)} - \frac{4\lambda^2}{(3m^2 - 4\Delta)} \right). \quad (50)$$

This hamiltonian can be expressed in terms of momentum of λ which was found to be

$$p_\lambda = \frac{-32\dot{\lambda}}{m^2(3m^2 - 4\Delta)} \quad (51)$$

The hamiltonian becomes

$$H_\lambda = -\frac{p_\lambda^2 m^2 (3m^2 - 4\Delta)}{64} - \frac{4\lambda^2}{3m^2 - 4\Delta} \quad (52)$$

For plane wave modes of wave number \mathbf{k} , Δ is $-k^2$. For modes with $k \gg m$, the above energy density reduces to

$$-\frac{4}{m^2 k^2} \left(\dot{\lambda}^2 + \frac{m^2}{4} \lambda^2 \right). \quad (53)$$

This appears to be negative and singular as m^2 goes to zero. However, looking at the equation of motion that we get from H_λ

$$\ddot{\lambda} + \frac{m^2}{4} \lambda = 0,$$

we deduce that $\dot{\lambda} \propto m\lambda$. This fact avoid the singularity $m^2 \rightarrow 0$. For $m^2 \rightarrow 0$ the energy density of the λ term becomes

$$\epsilon_{mim} \simeq \lambda - \frac{\lambda^2}{k^2} \quad (54)$$

For $\lambda \ll k^2$, the energy density is positive and the linear term dominates. One should not worry about the other case where $\lambda > k^2$ since perturbation theory will be no more valid in this limit.

Therefore, the main contribution to the energy density is linear in λ , and the second negative term just accounts for the negative contribution of gravitational self-interaction to the total energy density.

6 Conclusion

In this paper, we have constructed the Hamiltonian formulation of Ghost Free Mimetic Massive gravity. In their approach, small metric perturbations were induced small perturbations of the fields around broken symmetry phase were considered. Therefore, the Einstein–Hilbert action was expressed in terms of the linearized metric $h^{\mu\nu}$.

We wrote the total action in a 3 + 1 dimensional form up to second order in perturbations, and found the momenta of each field. This enabled us to write the total Hamiltonian. The equations of motion were all found and it was proved that they are the same equations obtained in [11]. At the end, the poisson brackets of the constraints were computed and degrees of freedom were counted proving that we have five degrees of freedom corresponding to a massive graviton without ghosts. IF the relative coefficient of the mass term is 1 instead of 1/2; similar to the Fierz Pauli's mass term; ghosts will appear again. Our future work will be to investigate the cosmological solutions of this mimetic massive gravity model and to study if the added mass term can generate inflation, spatially closed, open universe... Moreover, the possibility of extending this model to F(R) mimetic massive gravity can be investigated to check if it is ghost free; especially that F(R) mimetic gravity suffers from ghosts in general [16].

Acknowledgements We would like to thank Professor Ali Chamseddine for suggesting the problem and for his helpful discussions on the subject. We would like to thank also the American University of Beirut (Faculty of Science) for the support.

Data Availability Statement This manuscript has no associated data or the data will not be deposited [Authors' comments: All the necessary data are mentioned.]

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

References

1. A.H. Chamseddine, V. Mukhanov, Mimetic Dark Matter. JHEP **1311**, 135 (2013)
2. A. Golovnev, On the recently proposed Mimetic Dark Matter. Phys. Lett. B **728**, 39 (2014)
3. A.H. Chamseddine, V. Mukhanov, A. Vikman, Cosmology with Mimetic Matter. JCAP **1406**, 017 (2014). 19 pp
4. A.H. Chamseddine, V. Mukhanov, Inhomogeneous dark energy. JCAP **1602**(02), 040 (2016). 11 pp
5. A.H. Chamseddine, V. Mukhanov, Resolving cosmological singularities. JCAP **1703**(03), 009 (2017). 15 pp
6. A.H. Chamseddine, V. Mukhanov, Nonsingular black hole. Eur. Phys. J. C **77**(3), 183 (2017). 21 pp

7. S. Nojiri, S.D. Odintsov, Mimetic $F(R)$ gravity: inflation, dark energy and bounce. *Mod. Phys. Lett. A* **2940**, 1450211 (2014). <https://doi.org/10.1142/S0217732314502113>. [arXiv:1408.3561](https://arxiv.org/abs/1408.3561) [hep-th]
8. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Ghost-free $F(R)$ gravity with lagrange multiplier constraint. *Phys. Lett. B* (2017). <https://doi.org/10.1016/j.physletb.2017.10.045>
9. L. Sebastiani, S. Vagnozzi, R. Myrzakulov, Mimetic gravity: a review of recent developments and applications to cosmology and astrophysics. *Adv. High Energy Phys.* **2017**, 3156915 (2017). <https://doi.org/10.1155/2017/3156915>
10. A. Ganz, P. Karmakar, S. Matarrese, D. Sorokin, Hamiltonian analysis of mimetic scalar gravity revisited. *Phys. Rev. D* **99**, 064009 (2019)
11. A.H. Chamseddine, V. Mukhanov, Ghost free mimetic massive gravity. *JHEP* **1806**, 060 (2018). 3pp
12. R. Arnowitt, S. Deser, C.W. Misner, Republication of: the dynamics of general relativity. *Gen. Relat. Gravit.* **40**, 1997 (2008)
13. O. Malaeb, Hamiltonian formulation of mimetic gravity. *Phys. Rev. D* **91**, 103526 (2015). 15 PP
14. A.H. Chamseddine, V. Mukhanov, Mimetic massive gravity: beyond linear approximation. *JHEP* **1806**, 062 (2018). 25 pp
15. R. N. Ghalati, *Constraint Analysis of Linearized Gravity and a Generalization of the HTZ Approach*, [arXiv:hep-th/0703268v1](https://arxiv.org/abs/hep-th/0703268v1)
16. S. Nojiri, S. D. Odintsov, V. K. Oikonomou, *Ghost-Free $F(R)$ Gravity with Lagrange Multiplier Constraint*, [arXiv:1710.07838v1](https://arxiv.org/abs/1710.07838v1)