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Subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number

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ABSTRACT

Cohen et al. conjectured that, for each oriented cycle C, there exists an integer $f(C)$ such that any $f(C)$-chromatic strong digraph contains a subdivision of C as a subdigraph. In the same paper, Cohen et al. proved this conjecture for cycles with two blocks by showing that the chromatic number of strong digraphs that include no subdivision of a cycle with two blocks, with lengths of k_1 and k_2, is bounded from above by $O((k_1+k_2)^4)$. More recently, Kim et al. improved this upper bound to $O((k_1+k_2)^2)$ for the class of strong digraphs and to $O(k_1+k_2)$ for the class of Hamiltonian digraphs. In this paper, we confirm Cohen et al.'s conjecture for Hamiltonian digraphs. We demonstrate a stronger version by showing that every $3n$-chromatic Hamiltonian digraph contains a subdivision for every oriented cycle of order n.

1. Introduction

Throughout this paper, all graphs are assumed to be simple, that is, there are no loops and no multiple edges. A *digraph* is an oriented graph obtained by assigning an orientation to each edge of a given graph. Reciprocally, the graph obtained from a digraph D by ignoring its arc directions is called the *underlying graph* of D and is denoted by $G(D)$. Let x and y be two vertices of a digraph D, then the arc directed from x to y is denoted by (x, y). A *Hamiltonian path* (resp. *Hamiltonian cycle*) of a graph is a path (resp. cycle) that passes through all of its vertices. A *circuit* is an oriented cycle whose all arcs have the same direction. A *Hamiltonian digraph* is a digraph that contains a Hamiltonian circuit. The *length* of a path P (resp. cycle C), denoted by $l(P)$ (resp. $l(C)$), is its number of edges. Given a graph G, the *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. A *proper k-coloring* of G is a mapping $c : V(G) \to \{1, \dots, k\}$ such that no two adjacent vertices share the same color. A graph is *k-colorable* if it can be properly colored using k colors. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. A graph with chromatic number k is called a *k-chromatic graph*. The previous concepts are defined for digraphs by applying them simply to their underlying graphs.

One of the well-known problems in graph theory is to find an integer k such that every k-chromatic digraph contains a copy of each member of a certain family of digraphs. For example, Burr [4] proved that every $(n-1)^2$-chromatic digraph contains every oriented tree of order n, and he conjectured that this bound can be improved to $2n-2$. Of course, one way for a graph to have a large chromatic number is to contain a large complete subgraph, but this is not always the case. Indeed, Erdös showed in [9] that for any two positive integers g and k, there exists a graph G with girth larger than g and

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chromatic number larger than k. This means that if one of the elements of a given family of digraphs $\mathcal{D}$ has a bounded girth, then we cannot find an integer k in which every k-chromatic digraph contains a copy of each element of $\mathcal{D}$. This refocuses the problem to the study of two distinct classes of families of digraphs; those whose elements contain no oriented cycles and those whose elements have no upper bound on their girths.

In this context, concerning the oriented trees, Addario-Berry et al. [1] have improved Burr's bound by showing that every oriented tree of order n is contained in every $(\frac{n^2}{2} - \frac{n}{2} + 1)$-chromatic digraph, which is the best bound known reached so far. In contrast, considering some special oriented trees, better bounds on the chromatic number have been determined. An *antidirected tree* is an orientation of a tree in which every vertex has either in-degree zero or out-dergree zero. Burr showed in [4] that every antidirected tree of order n is contained in every $(8n-7)$-chromatic digraph. Then, in [1], Addario-Berry et al. decreased Burr's bound to $5n-9$, and in the same paper they proved every acyclic n-chromatic digraph contains every oriented trees of order n.

Regarding paths, the most celebrated one, known as Gallai-Hasse-Roy-Vitaver theorem [10,11,15,17], deals with directed paths and states that every n-chromatic digraph contains a directed path of length $n-1$. El Sahili [7] conjectured that every oriented path of order n exists in every n-chromatic digraph. Addario et al. [1] and El Sahili et al. [8], independently proved this conjecture for two blocks paths with $n \geq 4$. A *block* of a path P is a maximal directed sub-path of P. For paths with three blocks, El joubbeh [6] demonstrated that every oriented n-path (a path of order n) with three blocks is contained in every $(4n-4)$-chromatic digraph. For three-blocks paths that have length restrictions on their blocks, Mortada et al. [14] showed that each $(2n-5)$-chromatic digraph has an n-path with three blocks, the first and third of which are of length 1. Mortada et al. [13] also showed that each $(n+1)$-chromatic digraph has an n-path with three blocks, two consecutive of them are of length 1. Tarhini et al. [16] demonstrated the existence of each n-path in which the middle block is of length 1 in every $(2n+2)$-chromatic digraph, and more particularly in every n-chromatic digraph containing a Hamiltonian directed path.

On the other hand, Bondy [3] showed that each strong digraph has a circuit that is longer than its chromatic number. Note that the strong connectivity assumption is necessary because an acyclic k-chromatic digraph does not contain any circuit. Surprisingly, even if we are seeking non-directed oriented cycles, we cannot deal with general digraphs, due to the existence of acyclic digraphs with arbitrarily large chromatic number and no oriented cycles of two blocks (a *block* of an non-directed oriented cycle C is a maximal directed sub-path of C) as Gyárfás and Thomassen noticed (see [1]). Cohen et al. [5] extended this result to any number of blocks. More precisely, they proved for any positive integers b and c, there exists an acyclic digraph D with $\chi(D) \geq c$ in which all oriented cycles have more than b blocks. This highlights the necessity of restricting the problem of the existence of oriented cycles in chromatic digraphs to a particular class of digraphs, for example, Hamiltonian digraphs, which is the case that we have considered in this paper.

A *subdivision* of a digraph D is a digraph obtained from D by replacing each arc oriented from x to y with an x, y-directed path. Cohen et al. [5] conjectured in 2018 that Bondy's theorem can be extended to all oriented cycles:

Conjecture 1.1. *For any oriented cycle C, there is a constant $f(C)$ such that every strong $f(C)$-chromatic digraph contains a subdivision of C.*

In fact, Cohen et al. [5] proved this conjecture in their article for the case of two-blocks cycles. A cycle having two blocks of length k_1 and k_2 is known as a $C(k_1, k_2)$ cycle. More precisely, they showed that the chromatic number of strong digraphs with no subdivisions of two-blocks cycles $C(k_1, k_2)$ is bounded above by $(k_1+k_2-2)(k_1+k_2-3)(2k_2+2)(k_1+k_2+1)$. Addario et al. asked if the chromatic number of such digraphs can be bounded above by $O(k_1+k_2)$, which remains an open problem. Assuming that $k = \max\{k_1, k_2\}$, the best reached upper bound, found by Kim et al. [12], is $12k^2$. Kim et al. also showed in the same paper that every n-chromatic Hamiltonian digraph contains a subdivision of every two-blocks cycle of order n. In this article, we show that every $3n$-chromatic Hamiltonian digraph contains a subdivision of every oriented cycle of order n, proving Conjecture 1.1 for Hamiltonian digraphs.

2. The main results

Let G be a graph with a Hamiltonian path P and set $P = v_1 \dots v_t$. Al Mniny et al. [2] defined two edges $v_i v_j$ and $v_r v_l$ of G as *secant edges* with respect to P if $i < r < j < l$, and they proved that if G contains no secant edges then G is 3-colorable. The set of all positive integers that are not zero is denoted by $\mathbb{N}^*$. Let $k \in \mathbb{N}^*$. As an extension of the notion of secant edges, we define the notion of k-secant edges as follows: two edges $v_i v_j$ and $v_r v_l$ of G are *k-secant edges* with respect to P if they are secant and $j - r \geq k$.

Lemma 2.1. *Given a graph G contains a Hamiltonian path P and let $k \in \mathbb{N}^*$. If G contains no k-secant edges w.r.t. P, then G is $(4k-1)$-colorable.*

Proof. We use induction on $v(G)$. When $v(G) \leq 4k-1$, it is trivial. Now, suppose $v(G) \geq 4k$ and let $P = v_1 \dots v_t$ be Hamiltonian path of G. Let $v_i v_j \in E(G)$ such that $j - i \geq 2k$. If no such edge exists, we may easily divide the vertices of G into $2k$ stable sets, indicating that its chromatic number is smaller than $2k$, which is a better number than the one we seek.

Thus, there exists an integer $r \in \{i, \ldots, j\}$ such that $r - i \geq k$ and $j - r \geq k$, and since G has no k-secant edges w.r.t. P, then v_r has no neighbors outside of $\{v_i, \ldots, v_j\}$. Assume that $v_i v_j$ is selected so that $j - i$ is the lowest value under the constraint that $j - i \geq 2k$. We claim that $N_G(v_r) \subseteq \{v_{r-2k+1}, \ldots, v_{r-1}, v_{r+1}, \ldots, v_{r+2k-1}\}$. Indeed, $N_G(v_r) \subseteq \{v_i, \ldots, v_j\}$ and so for every $v_s \in N_G(v_r)$ we have $|r - s| < j - i$. Consequently, due to how $v_i v_j$ was selected, we have $|r - s| \leq 2k - 1$ and so $v_s \in \{v_{r-2k+1}, \ldots, v_{r+2k-1}\} \setminus \{v_r\}$. Hence, $d(v_r) \leq 4k - 2$.

Let G' be the graph obtained from G by deleting the vertex v_r, and adding the edge $v_{r-1}v_{r+1}$ in the case $v_{r-1}v_{r+1} \notin E(G)$. Set $P' = v_1 \ldots v_{r-1}v_{r+1} \ldots v_t$. Clearly, P' is a Hamiltonian path of G'. Furthermore, each k-secant edges w.r.t. P' in G' are also k-secant edges w.r.t. P in G, indicating that G' does not contain it. By induction, G' is $(4k-1)$-colorable and so there is a proper $(4k-1)$-coloring c' of G', and we can extend c' to a proper $(4k-1)$-coloring c of G owing to $d(v_r) \leq 4k - 2$. □

Given a digraph D containing a Hamiltonian directed path P and let $k \in \mathbb{N}^*$. Two arcs e and f in D are *k-secant arcs* w.r.t. P if their corresponding edges in $G(D)$ are k-secant edges w.r.t. $G(P)$. The following corollary derives from Lemma 2.1:

Corollary 2.1. *Let D be a digraph with a Hamiltonian directed path P and let $k \in \mathbb{N}^*$. If $\chi(D) \geq 4k$, then D contains k-secant arcs w.r.t. P.*

Presume that an non-directed oriented cycle C comprises of l blocks of successive lengths $k_1, k_2, \ldots, k_l$, then we write $C = C^+(k_1, \ldots, k_l)$ if the block of length k_1 is forward and $C = C^-(k_1, \ldots, k_l)$ otherwise. It is self-evident that l must be an even number according to the definition of a block. Note that in most cases $C^+(k_1, \ldots, k_l) \neq C^-(k_1, \ldots, k_l)$, for example $C^+(1, 1, 2, 2) \neq C^-(1, 1, 2, 2)$. A non-directed oriented cycle C is *antidirected* if all of its blocks are of length 1. Considering two vertices x and y in a circuit C, then $C_{[x,y]}$ is represented the unique directed sub-path of C that begins at x and ends at y. If $(x, y) \in E(C)$, then y is the *successor* of x on C. If a subdigraph H of a digraph D contains all arcs of D that connect two vertices in $V(H)$, it is termed *induced*. In this case, we write $H = D[V(H)]$.

Theorem 2.1. *Let D be a Hamiltonian digraph and let C be a non-directed oriented cycle with $2t$ blocks. Let $k_1, \ldots, k_{2t}$ be non-zero positive integers such that $C = C^+(k_1, \ldots, k_{2t})$. If $\chi(D) \geq (k_1 + 1) + 4k_2 + \cdots + (k_{2t-1} + 1) + 4k_{2t}$, then D contains a subdivision of C.*

Proof. Let C' be a Hamiltonian circuit of D, and build a list of induced sub-digraphs $(D_i)_{1 \leq i \leq 2t}$ of D that are validating 3 points:

- $\chi(D_i) = \begin{cases} k_i + 1 & \text{if } i \text{ is odd;} \\ 4k_i & \text{if } i \text{ is even;} \end{cases}$
- for $1 \leq i \leq 2t$, $C'[V(D_i)]$ is a Hamiltonian directed path of D_i;
- for $2 \leq i \leq 2t$, the start vertex of $C'[V(D_i)]$ is the successor of the end vertex of $C'[V(D_{i-1})]$.

To construct such a list, begin at any vertex in D, add its successor vertex on C', and repeat until you get an induced subdigraph of D with a chromatic number of $k_1 + 1$, then label it D_1. It is obvious that D_1 is formable since as whenever a new vertex is added, the chromatic number either keeps the same or goes up by one. Continue at the successor vertex of the end vertex of $C'[V(D_1)]$ and create the digraph D_2 in the same way as D_1, ending when its chromatic number becomes $4k_2$. Persist in this manner until all digraphs of this list have been created, keeping in mind that the chromatic number of D is adequate for this process to be completed.

After generating this list, we could use Corollary 2.1 to observe that whenever i is even, D_i contains two arcs e_i and f_i as k_i-secant arcs w.r.t. $C'[V(D_i)]$. Without loss of generality, suppose that the arc e_i comes before the arc f_i with respect to the direction of $C'[V(D_i)]$, and for each $e \in \{e_i, f_i\}$ we represent the two vertices of e as $x(e)$ and $y(e)$ so that $C'_{[x(e),y(e)]} \subseteq C'[V(D_i)]$. On the other hand, when i is odd, we have $l(C'[V(D_i)]) \geq k_i$ due to $\chi(D_i) = k_i + 1$. Thus, $e_2 \cup C'_{[x(f_2),y(e_2)]} \cup f_2 \cup C'_{[y(f_2),x(e_4)]} \cup e_4 \cup \cdots \cup e_{2t} \cup C'_{[x(f_{2t}),y(e_{2t})]} \cup f_{2t} \cup C'_{[y(f_{2t}),x(e_2)]}$ is a subdivision of $C^+(k_1, \ldots, k_{2t})$. Noting that no matter how any arc $e \in \{e_2, f_2, \ldots, e_{2t}, f_{2t}\}$ is oriented, neither of its two possibilities will disrupt the construction of the desired cycle. □

Theorem 2.2. *Every $3n$-chromatic Hamiltonian digraph contains a subdivision of every non-directed oriented cycle of order n.*

Proof. Let D be a $3n$-chromatic Hamiltonian digraph and let C be a non-directed oriented cycle of order n. Let $k_1, \ldots, k_{2t}$ be non-zero positive integers such that $C = C^+(k_1, \ldots, k_{2t})$. Because $k_1 + \cdots + k_{2t} = n$, either $k_1 + k_3 + \cdots + k_{2t-1} \leq \frac{n}{2}$ or $k_2 + k_4 + \cdots + k_{2t} \leq \frac{n}{2}$ must be true. Also, note that $C^+(k_1, \ldots, k_{2t}) = C^+(k_{2t}, k_{2t-1}, \ldots, k_1)$. Thus, with relabeling if necessary, we may assume that $k_2 + k_4 + \cdots + k_{2t} \leq \frac{n}{2}$. In terms of the number of blocks, the greatest value of $2t$ that may be reached is n, which occurs when the cycle is antidirected. Therefore, $\chi(D) = 3n = n + \frac{3n}{2} + \frac{n}{2} \geq (k_1 + \cdots + k_{2t}) + (3k_2 + 3k_4 + \cdots + 3k_{2t}) + t = (k_1 + 1) + 4k_2 + \cdots + (k_{2t-1} + 1) + 4k_{2t}$, and so by Theorem 2.1 D contains a subdivision of C. □

Declaration of competing interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data availability

No data was used for the research described in the article.

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