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Kasner branes with arbitrary signature

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ABSTRACT

We present static and time-dependent solutions for the theory of gravity with a dilaton field and an arbitrary rank antisymmetric tensor. The solutions constructed are valid for arbitrary space-time dimensions and signatures.

1. Introduction

Many years ago, Kasner [1] presented Euclidean four-dimensional vacuum solutions which depend only on one variable. Those solutions can be written in the form

$$ds^2 = x_1^{2a_1} dx_1^2 + x_1^{2a_2} dx_2^2 + x_1^{2a_3} dx_3^2 + x_1^{2a_4} dx_4^2, \tag{1.1}$$

where the constants appearing in the metric, the Kasner exponents, satisfy the following conditions

$$\begin{aligned} &a_2 + a_3 + a_4 = 1 + a_1, \\ &a_2^2 + a_3^2 + a_4^2 = (1 + a_1)^2 . \end{aligned} \tag{1.2}$$

The Kasner metric is actually valid for all space-time signatures [2]. If one writes

$$ds^2 = \epsilon_0 x_1^{2a_1} dx_1^2 + \epsilon_1 x_1^{2a_2} dx_2^2 + \epsilon_2 x_1^{2a_3} dx_3^2 + \epsilon_3 x_1^{2a_4} dx_4^2, \tag{1.3}$$

where ϵ_i takes the values ± 1, then the metric (1.3) can be shown to be a vacuum solution for four-dimensional gravity with all possible space-time signatures provided the Kasner conditions (1.2) are satisfied.

Dynamical time-dependent cosmological solutions can be obtained by simply setting $\epsilon_0 = -1$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, and after relabelling of the coordinates we obtain

$$ds^2 = -t^{2a_1} dt^2 + t^{2a_2} dx^2 + t^{2a_2} dy^2 + t^{2a_3} dz^2. \tag{1.4}$$

Static solutions can also be obtained and are given by

$$ds^2 = -x^{2a_2} dt^2 + x^{2a_1} dx^2 + x^{2a_3} dy^2 + x^{2a_4} dz^2. \tag{1.5}$$

After a redefinition of the coordinate t, the metric (1.4) can take the following form

$$ds^2 = -d\tau^2 + \tau^{2a} dx^2 + \tau^{2b} dy^2 + \tau^{2c} dz^2 \tag{1.6}$$

with

$$a + b + c = a^2 + b^2 + c^2 = 1. \tag{1.7}$$

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This metric is what is normally referred to in the literature as the Kasner metric. The Kasner metric is closely related to solutions found earlier by Weyl [3], Levi-Civita [4] and Wilson [5] and was subsequently rediscovered by many authors [6]. We note that the Kasner metric with two vanishing exponents corresponds to flat space-time, where $t = 0$ is only a coordinate singularity. Any other solution must have two of the exponents positive and the third negative. As such, the Kasner metric describes a homogenous universe which is expanding in two directions and contracting in the third. In terms of Bianchi classification of homogenous spaces, the Kasner metric corresponds to Bianchi type I. Generalised Kasner metrics played a central part in the study of the cosmological singularity, gravitational turbulence and chaos (see [7] and references therein).

The Kasner vacuum solution can be generalised to arbitrary space-time dimensions and signatures. In d dimensions, we have

$$ds^2 = \epsilon_0 d\tau^2 + \sum_{i=1}^{d-1} \epsilon_i \tau^{2a_i} dx_i^2 \tag{1.8}$$

with the conditions

$$\sum_{i=1}^{d-1} a_i = \sum_{i=1}^{d-1} a_i^2 = 1. \tag{1.9}$$

Let us now consider a d-dimensional gravity theory with an m-form F_m, with the action

$$S = \int d^d x \sqrt{|g|} \left(R - \frac{\epsilon}{2\,m!}\, F_m^2 \right), \tag{1.10}$$

where we allow for the possibility of a non-canonical sign of the kinetic term of the m-form, $\epsilon = \pm 1$. The equations of motion derived from the action (1.10) are given by

$$\begin{aligned} R_{\mu\nu} - \epsilon \left(\frac{1}{2(m-1)!} F_{\mu\alpha_2\cdots\alpha_m} F_\nu{}^{\alpha_2\cdots\alpha_m} - g_{\mu\nu} \frac{(m-1)}{2m!(d-2)} F_m^2 \right) &= 0, \\ \partial_\mu \left(\sqrt{|g|}\; F^{\mu\nu_2\cdots\nu_m} \right) &= 0. \end{aligned} \tag{1.11}$$

For the metric of (1.8), the nonvanishing components of the Ricci tensor are given by

$$\begin{aligned} R_{\tau\tau} &= \frac{1}{\tau^2} \sum_{i=1}^{d-1} \left(a_i - a_i^2 \right), \\ R_{x_i x_i} &= \epsilon_0 \epsilon_i \tau^{2a_i - 2} a_i \left(1 - \sum_{i=1}^{d-1} a_i \right). \end{aligned} \tag{1.12}$$

If we consider solutions with the metric (1.8) and with the m-form

$$F_m = P dx_1 \wedge dx_2 \wedge \ldots \wedge dx_m, \tag{1.13}$$

then the equations of motion are satisfied provided

$$\begin{aligned} P^2 &= \epsilon \epsilon_0 \epsilon_1 \ldots \epsilon_m \left(\frac{2\,(m-1)\,(d-2)}{m^2 (d-m-1)} \right), \\ a_1 &= a_2 = .. = a_m = \frac{1}{m}, \\ a_{m+1} &= \ldots = a_{d-1} = -\frac{m-1}{m(d-m-1)}. \end{aligned} \tag{1.14}$$

In four-dimensions with a Maxwell field, we simply get

$$\begin{aligned} P^2 &= \epsilon \epsilon_0 \epsilon_1 \epsilon_2 \\ a_1 &= a_2 = \frac{1}{2},\ a_3 = -\frac{1}{2}. \end{aligned} \tag{1.15}$$

For dynamical solutions with Lorentzian signature where $-\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, one must have $\epsilon = -1$, corresponding to the theory with the non-canonical sign of the Maxwell term in the action. The resulting solution obtained admits Killing spinors when the theory is viewed as the bosonic sector of $N = 2$ supergravity [8]. In four dimensions we can also have dyonic solutions, with the two-form given by

$$F = Q\, \tau^{a_3 - a_1 - a_2} d\tau \wedge dz + P dx \wedge dy \tag{1.16}$$

where P and Q are constant. Then for the metric with exponents given in (1.15), we obtain the condition

$$\epsilon \left(\epsilon_0 \epsilon_1 \epsilon_2 P^2 - Q^2 \epsilon_3 \right) = 1. \tag{1.17}$$

If we consider d-dimensional gravity with a dynamical scalar field

$$S=\int d^dx\sqrt{|g|}\left(R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right). \tag{1.18}$$

Then the metric (1.8) is a solution provided

$$\phi=d_1\log\tau+d_2, \tag{1.19}$$

$$\sum_{i=1}^{d-1}a_i=\sum_{i=1}^{d-1}a_i^2+\frac{1}{2}d_1^2=1 \tag{1.20}$$

where d_1 and d_2 are constants.

Static D-brane as well as time-dependent solutions (see for example [9] and references therein) in M and string theories are of importance for the study of duality symmetries, (A)dS/conformal field theory correspondence, stringy cosmological models and cosmological singularities. Our aim in this work is to find general Kasner-like brane solutions, which include static and time-dependent solutions, in gravitational theories with an arbitrary rank antisymmetric tensor field and a scalar dilaton. The general solutions can then be used to construct explicit solutions for all supergravity theories with non-trivial form fields and dilaton with various space-time signatures

2. Brane solutions

We start with d-dimensional gravity theory with an m-form F_m with the action given in (1.10). We consider the following generic metric solution

$$ds^2=e^{2U(\tau)}\left(\epsilon_0 d\tau^2+\sum_{i=1}^{p}\epsilon_i\tau^{2a_i}dx_i^2\right)+e^{2V(\tau)}\left(\sum_{j=p+1}^{d-1}\epsilon_j\tau^{2a_j}dx_j^2\right), \tag{2.1}$$

where ϵ_0, ϵ_i, ϵ_j take the values ±1 and a_i, a_j are all constants. Clearly we have $d=p+q+1$. As in the Kasner vacuum solutions with arbitrary space-time signature, dynamical cosmological solutions as well as static solutions can be obtained depending on whether τ is considered as a time or a spatial coordinate. Our metric solution (2.1) is, to a great extent, motivated by the results of [10]. There a correspondence between Melvin magnetic fluxtubes [11] and anisotropic cosmological solutions in four-dimensions was explored. Our static solutions can therefore be thought of as generalizations of Melvin fluxtubes to higher dimensions and with antisymmetric tensors and dilaton field switched on.

The non-vanishing components of the Ricci tensor for the metric (2.1) are given by

$$\begin{aligned}
R_{\tau\tau}&=-q\ddot{V}-p\ddot{U}-q\dot{V}\left(\dot{V}-\dot{U}\right)-\frac{1}{\tau}\left((s-l)\dot{U}+2l\dot{V}\right)-\frac{1}{\tau^2}\sum_{k=1}^{d-1}\left(a_k^2-a_k\right),\\
R_{x_ix_i}&=-\epsilon_0\epsilon_i\tau^{2a_i}\left[\ddot{U}-\frac{a_i}{\tau^2}+\left(\dot{U}+\frac{a_i}{\tau}\right)\left((p-1)\dot{U}+q\dot{V}+\frac{l+s}{\tau}\right)\right],\\
R_{x_jx_j}&=-\epsilon_0\epsilon_j e^{2V-2U}\tau^{2a_j}\left[\ddot{V}-\frac{a_j}{\tau^2}+\left(\dot{V}+\frac{a_j}{\tau}\right)\left(q\dot{V}+(p-1)\dot{U}+\frac{l+s}{\tau}\right)\right],
\end{aligned} \tag{2.2}$$

where we have defined

$$l=\sum_{j=p+1}^{d-1}a_j,\quad s=\sum_{i=1}^{p}a_i. \tag{2.3}$$

It can be easily seen that a major simplification significant occurs if in addition to keeping the Kasner conditions

$$\sum_{k=1}^{d-1}a_k=\sum_{k=1}^{d-1}a_k^2=1, \tag{2.4}$$

we also impose the relation

$$qV+(p-1)U=0. \tag{2.5}$$

The Ricci tensor non-vanishing components then take the much simpler form

$$\begin{aligned}
R_{\tau\tau}&=-\ddot{U}-\left[1-2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau}-\frac{(p-1)(d-2)}{q}\dot{U}^2,\\
R_{x_ix_i}&=-\epsilon_0\epsilon_i\tau^{2a_i}\left(\ddot{U}+\frac{\dot{U}}{\tau}\right),\\
R_{x_jx_j}&=\frac{(p-1)}{q}\epsilon_0\epsilon_j e^{2(\frac{2-d}{q})U}\tau^{2a_j}\left(\ddot{U}+\frac{\dot{U}}{\tau}\right).
\end{aligned} \tag{2.6}$$

We shall now consider solutions with a p-form given by

$$F_p = P dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p \tag{2.7}$$

with constant P. Then the Einstein equations of motion (1.11) reduce to the two equations

$$\ddot{U} + (1-2l)\frac{\dot{U}}{\tau} + (p-1)\dot{U}^2 = 0, \tag{2.8}$$

$$\ddot{U} + \frac{\dot{U}}{\tau} + \epsilon\epsilon_0\epsilon_1\ldots\epsilon_p \frac{q e^{2(1-p)U}}{2(d-2)} P^2 \tau^{-2s} = 0. \tag{2.9}$$

A solution for the equation (2.8) is given by

$$e^U = \left(c_1 + c_2\tau^{2l}\right)^{\frac{1}{p-1}} \tag{2.10}$$

which upon substitution in (2.9) gives the condition

$$\frac{8(d-2)l^2}{q(p-1)} c_2 c_1 + \epsilon\epsilon_0\epsilon_1\ldots\epsilon_p P^2 = 0. \tag{2.11}$$

One can also consider (the dual) solutions with a $q+1$-form given by

$$F_{q+1} = Q\, e^{2(1-p)U}\tau^{2l-1} d\tau \wedge dx_{p+1} \wedge \ldots \wedge dx_{d-1}. \tag{2.12}$$

In this case, we get the same form of solution with the various constants satisfying

$$\frac{8(d-2)l^2}{q(p-1)} \epsilon\epsilon_{p+1}\ldots\epsilon_{d-1} c_2 c_1 - Q^2 = 0. \tag{2.13}$$

As examples we consider eleven-dimensional supergravity theories with (t,s) signatures with t being the number of the time directions and s the spatial ones. The relevant action [12,13] is

$$S = \int d^{11}x\sqrt{|g|}\left(R - \frac{\epsilon}{48}F_4^2\right) + \ldots \tag{2.14}$$

where ϵ takes the value 1 for the theories with space-time signatures $(1,10)$, $(5,6)$ and $(9,2)$, and $\epsilon = -1$ for the mirror theories with signatures $(10,1)$, $(6,5)$ and $(2,9)$. Using our general results, solutions for all space-time signatures in eleven dimensional supergravity theories can be constructed. For instance one can obtain the time-dependent solution for the standard $(1,10)$ theory ($\epsilon = 1$), given by

$$ds^2 = e^{2U}\left(-d\tau^2 + \tau^{2a_1}dx_1^2 + \tau^{2a_2}dx_2^2 + \tau^{2a_3}dx_3^2 + \tau^{2a_4}dx_4^2\right) + e^{-U}\left(\sum_{j=5}^{10}\tau^{2a_j}dx_j^2\right)$$
$$F_4 = P dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \tag{2.15}$$

with

$$e^U = \left(1 + c_2\tau^{2\sum_{j=5}^{10}a_j}\right)^{\frac{1}{3}}, \quad P^2 = 4\left(\sum_{j=5}^{10}a_j\right)^2 c_2. \tag{2.16}$$

Using the generic solution, setting $\epsilon_0 = 1$, $\epsilon_1 = -1$ and relabelling the coordinates $\tau = x$ and $x_1 = t$, then we can obtain static solutions given by

$$ds^2 = e^{2U}\left(-x^{2a_1}dt^2 + dx^2 + x^{2a_2}dx_2^2 + x^{2a_3}dx_3^2 + x^{2a_4}dx_4^2\right) + e^{-U}\left(\sum_{j=5}^{10}x^{2a_j}dx_j^2\right),$$
$$F_4 = P dt \wedge dx_2 \wedge dx_3 \wedge dx_4, \tag{2.17}$$

with

$$e^U = \left(1 + c_2 x^{2\sum_{j=5}^{10}a_j}\right)^{\frac{1}{3}}, \; P^2 = 4\left(\sum_{j=5}^{10}a_j\right)^2 c_2. \tag{2.18}$$

Similarly, we can also get the cosmological solutions

$$ds^2 = e^{2U}\left(-d\tau^2 + \sum_{i=1}^{7}\tau^{2a_i}dx_i^2\right) + e^{-4U}\left(\tau^{2a_8}dx_8^2 + \tau^{2a_9}dx_9^2 + \tau^{2a_{10}}dx_{10}^2\right),$$
$$F_4 = Q\, e^{-12U}\tau^{2(a_8+a_9+a_{10})-1}d\tau \wedge dx_8 \wedge dx_9 \wedge dx_{10}, \qquad Q^2 = 4l^2 c_2 (a_8+a_9+a_{10})^2$$
$$e^U = \left(1 + c_2\tau^{2(a_8+a_9+a_{10})}\right)^{\frac{1}{6}}. \tag{2.19}$$

For even-dimensional space-time one can consider "dyonic" solutions for $p=q+1=\frac{d}{2}$. The metric solutions for the $\frac{d}{2}$-form

$$F_{\frac{d}{2}}=Pdx_1\wedge dx_2\wedge\ldots\wedge dx_{\frac{d}{2}}+Q\,e^{-U(d-2)}\tau^{2l-1}d\tau\wedge dx_{\frac{d}{2}+1}\wedge\ldots\wedge dx_{d-1} \tag{2.20}$$

take the form

$$ds^2=e^{2U}\left(\epsilon_0 d\tau^2+\sum_{i=1}^{\frac{d}{2}}\epsilon_i\tau^{2a_i}dx_i^2\right)+e^{-2U}\left(\sum_{j=\frac{d}{2}+1}^{d-1}\epsilon_j\tau^{2a_j}dx_j^2\right). \tag{2.21}$$

The equations of motion in this case can be solved for

$$e^U=\left(c_1+c_2\tau^{2l}\right)^{\frac{2}{d-2}} \tag{2.22}$$

where

$$\epsilon\left(Q^2\epsilon_{p+1}\ldots\epsilon_{d-1}-P^2\epsilon_0\epsilon_1\ldots\epsilon_p\right)=\frac{32l^2}{d-2}c_1c_2. \tag{2.23}$$

As an example, we consider solutions of the standard Einstein-Maxwell theory ($\epsilon=1$), we get the cosmological solutions [10]

$$\begin{aligned}ds^2&=\left(1+\frac{(Q^2+P^2)}{16a_3^2}\tau^{2a_3}\right)^2\left(-d\tau^2+\tau^{2a_1}dx_1^2+\tau^{2a_2}dx_2^2\right)+\left(1+\frac{(Q^2+P^2)}{16a_3^2}\tau^{2a_3}\right)^{-2}\tau^{2a_3}dx_3^2\\ F_2&=Pdx_1\wedge dx_2+Q\,e^{-2U}\tau^{2a_3-1}d\tau\wedge dx_3\end{aligned} \tag{2.24}$$

as well as the static solutions

$$\begin{aligned}ds^2&=\left(1+\frac{(Q^2+P^2)}{16a_3^2}x^{2a_3}\right)^2\left(-x^{2a_1}dt^2+dx^2+x^{2a_2}dx_2^2\right)+\left(1+\frac{(Q^2+P^2)}{16a_3^2}x^{2a_3}\right)^{-2}x^{2a_3}dx_3^2\\ F_2&=Pdt\wedge dx_2+Q\,e^{-2U}x^{2a_3-1}dx\wedge dx_3.\end{aligned} \tag{2.25}$$

We now turn to the construction of solutions associated with an m-form field, F_m, and a dilaton scalar, ϕ, coupled to the m-form field. We take the action (in the Einstein frame)

$$S=\int d^dx\sqrt{|g|}\left(R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{\epsilon}{2\,m!}e^{\beta\phi}\,F_m^2\right). \tag{2.26}$$

This represents the action of the bosonic fields of many supergravity theories. We have allowed for the possibility of non-canonical sign of the coupling term of the F_m form ($\epsilon=\pm1$) which will enable us to construct solutions for various supergravity theories constructed by Hull [13].

The equations of motion, derived from the variation of the action (2.26) are

$$\begin{aligned}R_{\mu\nu}&=\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{\epsilon e^{\beta\phi}}{2(m-1)!}\left[F_{\mu\alpha_2\cdots\alpha_m}F_\nu{}^{\alpha_2\cdots\alpha_m}-\frac{(m-1)}{m(d-2)}F_m^2\,g_{\mu\nu}\right],\\ &\partial_\mu\left(\sqrt{|g|}\,e^{\beta\phi}\,F^{\mu\nu_2\cdots\nu_m}\right)=0,\\ &\frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right)=\frac{\beta}{2\,m!}\epsilon e^{\beta\phi}F_m^2.\end{aligned} \tag{2.27}$$

Let us consider solution with the p-form given by

$$F_p=Pdx_1\wedge dx_2\wedge\ldots\wedge dx_p\ , \tag{2.28}$$

where P is a constant. We again take the metric (2.1) as our solution supplemented with the constraints (2.4) and (2.5). The Einstein equations of motion then give

$$\ddot U+\left[1-2\left(\frac{d-2}{q}\right)l\right]\frac{\dot U}{\tau}+\frac{(p-1)(d-2)}{q}\dot U^2=-\frac{\dot\phi^2}{2}+\frac{(p-1)}{2(d-2)}j, \tag{2.29}$$

$$\ddot U+\frac{\dot U}{\tau}=-\frac{q}{2\,(d-2)}j, \tag{2.30}$$

$$\frac{\dot\phi}{\tau}+\ddot\phi=\frac{\beta}{2}j, \tag{2.31}$$

where

$$j=\epsilon\epsilon_0\epsilon_1\ldots\epsilon_p e^{2(1-p)U+\beta\phi}P^2\tau^{-2s}. \tag{2.32}$$

The last two equations (2.30) and (2.31) imply that

$$\phi = -\frac{\beta(d-2)}{q}U. \tag{2.33}$$

Upon substituting (2.33) back in the equations (2.29) and (2.30) we obtain the following equations

$$\ddot{U} + (1-2l)\,\frac{\dot{U}}{\tau} + \mu\dot{U}^2 = 0, \tag{2.34}$$

$$\ddot{U} + \frac{\dot{U}}{\tau} + \epsilon\epsilon_0\epsilon_1...\epsilon_p \frac{qe^{-2\mu U}}{2\,(d-2)}P^2\tau^{-2s} = 0, \tag{2.35}$$

where we have defined $\mu = \left(p - 1 + \frac{\beta^2}{2q}(d-2)\right)$. The equation (2.34) admits the solution

$$e^U = \left(c_1 + c_2\tau^{2l}\right)^{\frac{1}{\mu}} \tag{2.36}$$

with the various constants satisfying the condition

$$\epsilon\epsilon_0\epsilon_1...\epsilon_p \frac{2(d-2)}{q}\frac{4l^2}{\mu}c_2c_1 + P^2 = 0. \tag{2.37}$$

We can also construct the dual solutions with

$$F_{q+1} = e^{-\beta\phi}\,Q\,e^{2(1-p)U}\tau^{2l-1}d\tau \wedge dx_{p+1} \wedge ... \wedge dx_{d-1}, \tag{2.38}$$

$$\phi = \beta\left(\frac{d-2}{q}\right)U, \tag{2.39}$$

and where

$$Q^2 = \epsilon\epsilon_{p+1}...\epsilon_{d-1}\frac{8(d-2)}{q}\frac{l^2c_1c_2}{\mu}. \tag{2.40}$$

Our analysis can be used to find non-trivial solutions for all the theories of type IIA, type IIA*, type IIB, type IIB* and type IIB′ supergravity theories with various space-time signatures [13]. In all those theories, the bosonic part consists of a dilaton and a 3-form coming from the NS-NS sector. All type IIA and IIA* theories also have a 2-form and a 4-form coming from the RR sector of the theory. The type IIB, IIB* and IIB′ have 1-form, 3-form and 5-form coming from the RR sector. A list of these theories with corresponding space-time signatures and signs for gauge kinetic terms can be found in [13]. For a given non-trivial RR field, the bosonic action in these theories takes the form

$$S = \int d^{10}x\sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2\,m!}\,e^{\frac{1}{2}(5-m)\phi}F_m^2\right). \tag{2.41}$$

For a p-form given by

$$F_p = P dx_1 \wedge dx_2 \wedge ... \wedge dx_p. \tag{2.42}$$

A generic solution is given by

$$ds^2 = \left(c_1 + c_2\tau^{2l}\right)^{\frac{(9-p)}{8}}\left(\epsilon_0 d\tau^2 + \sum_{i=1}^{p}\epsilon_i\tau^{2a_i}dx_i^2\right) + \left(c_1 + c_2\tau^{2l}\right)^{\frac{(1-p)}{8}}\left(\sum_{j=p+1}^{9}\epsilon_j\tau^{2a_j}dx_j^2\right) \tag{2.43}$$

with

$$\phi = -\frac{4\,(5-p)}{(9-p)}U, \tag{2.44}$$

and with the various constants satisfying the condition

$$4\epsilon\epsilon_0\epsilon_1...\epsilon_p l^2c_2c_1 + P^2 = 0.$$

The dual solution is obtained with the $q+1$-form and scalar field given by

$$F_{q+1} = Q\,e^{-\frac{32}{q}U}\tau^{2l-1}d\tau \wedge dx_{p+1} \wedge ... \wedge dx_{d-1}, \tag{2.45}$$

$$\phi = 4\left(\frac{4}{q} - 1\right)U, \tag{2.46}$$

$$Q^2 = 4\epsilon\epsilon_{p+1}...\epsilon_{d-1}l^2c_1c_2. \tag{2.47}$$

The self-dual $p = 5$ case in various type IIB theories is treated separately. In this case, we have $p = q + 1 = 5$ and no dilaton coupling to F_5. Solutions can be obtained using the general "dyonic" solutions constructed above. For example, we take for the five-form

$$F_5 = P\left(dx_1 \wedge dx_2 \wedge .. \wedge dx_5 + \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 e^{-8U}\tau^{2l-1}d\tau \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9\right) \tag{2.48}$$

and the metric can take the form

$$ds^2 = e^{2U}\left(\epsilon_0 d\tau^2 + \sum_{i=1}^{5}\epsilon_i \tau^{2a_i}dx_i^2\right) + e^{-2U}\left(\sum_{j=6}^{9}\epsilon_j \tau^{2a_j}dx_j^2\right) \tag{2.49}$$

where

$$e^{2U} = \left(1 + c_2\tau^{2(a_6+a_7+a_8+a_9)}\right)^{\frac{1}{2}} \tag{2.50}$$

$$\epsilon P^2\left(\epsilon_6\epsilon_7\epsilon_8\epsilon_9 - \epsilon_0\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\right) = 4l^2 c_2. \tag{2.51}$$

In this paper, we have constructed time-dependent and static solutions generalising Kasner solutions to include matter fields. The solutions constructed are for arbitrary space-time dimensions and signatures. The formalism can be applied to find non-trivial cosmological and static solutions in all known supergravity theories with form fields and a dilaton in various space-time signatures. It is of importance to investigate the relevance of our time-dependent solutions to string cosmology and the study of cosmological singularities [7]. We should also investigate the generalisation of our results to find solutions in theories with many scalars and Maxwell fields, such as $N = 2$ supergravity theories in four and five dimensions. Work along these lines is in progress and we hope to report on it in the near future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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