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Economic production quantity with maintenance interruptions under random and correlated yields

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This paper considers an economic production quantity with imperfect items where the quality of items produced within the same production run is correlated. Production and scheduled maintenance policies for a correlated binomial production system are investigated. We study the impact of correlation on the system performance measures and draw insights in terms of the effect of correlation on the production and maintenance policies. We also illustrate that the popular and commonly used interrupted geometric production systems can be analysed by an equivalent correlated binomial production model.

Keywords: inventory management; production economics; random yield; probabilistic models; correlated binomial supply; manufacturing processes

1. Introduction

Production processes with random yield have been widely addressed in the literature due to their logistical implications on the performance of inventory systems. According to Chen and Yang (2014) and Gurnani, Akella, and Lehoczky (2000), the production processes in the semiconductor and electronics industries are highly uncertain where it is common to expect a yield of 50% in the LCD manufacturing industry. Random yields in the production of float glass, for example, are due to the random distribution of errors scattered on glass surfaces (Taskin and Ünal 2009). The impact of random yield in the manufacturing and remanufacturing industries is well established (Ojha, Sarker, and Biswas 2007; Inderfurth and Vogelgesang 2013), especially in the electronics goods industry (Huang and Song 2010; Chen and Xiao 2015). Applications on production systems with random yields are not restricted to manufacturing systems. Random yield is significant in agricultural production systems which includes the production of olive oil, orange juice, timber, see Kazaz (2004) or hybrid corn seed, see Jones et al. (2001), among other agricultural products. For example, the yield for olive oil production in Turkey can be as low as 30% or 40% (Kazaz 2004).

Capturing the behaviour of random yield by investigating the reliability of the production system becomes integral to the decision-making process. Yano and Lee (1995) present a review on random yield which categorises the literature on random yield into three main streams. One of the main streams is referred to as the interrupted geometric approach which assumes that a production system starts in an in-control state and can transition to an out-of-control state (Porteus 1986). The literature belonging to this stream, which includes extensions and generalisations of the interrupted geometric approach, model the distribution of the random yield by accounting for the reliability assumptions and maintenance policies of the production process.

The production and maintenance policies we consider in this paper are the length of the production runs and the savings associated with partitioning a production run into shorter sub-production runs where maintenance is performed between the sub-production runs. An investigation into the trade-off between long and short production runs is considered in Grosfeld-Nir and Gerchak (2004) where the authors present a review of the recent literature which addresses multiple lot sizing production with random yield. Kutzner and Kiesmüller (2013) point out that little research has been done on combining inventory management with production maintenance. The authors address the cost trade-off between performing frequent maintenance set-ups to achieve higher yields. A joint optimisation of the inventory and maintenance policy to reduce cost and improve system performance is also investigated.

Hu and Zong (2009) consider a production system which transitions from an in-control to an out-of-control state. The time spent in the in-control state follows a random distribution and a transition to the out-of-control state is not detected until the end of the run. The authors investigate the joint optimal run time and inspection policy. Xiang et al. (2014) investigate maintenance

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policies where the deterioration of the manufacturing system, and consequently the yield probability, is represented by a discrete-time Markov chain. Random yield has also been extensively addressed in the context of imperfect supply and a recent review is presented in Khan et al. (2011). We refer to Bendavid and Herer (2009), Giri and Dohi (2007) and Sarkar and Saren (2016) for applications of production systems transitioning from an in-control to an out-of-control state which relax the assumption that the in-control state only produces conforming items and the out-of-control state only produces non-conforming items. The work in Bendavid and Herer (2009) accounts for the penalty of accepting defective items and rejecting conforming items. Heuristics and a dynamic programming approach are presented to solve for the optimal policy. The authors in Guu and Zhang (2003) consider a multiple lot sizing problem with an interrupted geometric random yield. A dynamic programming approach is implemented for a finite-time horizon problem with a finite number of set-ups and holding costs. In this paper, we also consider the interrupted geometric production process where the probabilities of producing defective items in the in-control and out-of-control states are p_1 and p_2, respectively, $0 \leq p_1 < p_2 \leq 1$.

Inderfurth and Vogelgesang (2013) consider safety stocks in the context of random yield and present an approach for calculating dynamic safety stocks for different yield models. Random yield in the context of remanufacturing returned products in a decentralised closed loop supply chain is presented in Hosoda, Disney, and Gavirneni (2015). The work in Giri and Dohi (2007) considers inspection and preventive maintenance in the context of an in-control/out-of-control production system. Two policies are investigated where the first policy performs scheduled inspections and the system is reset only if an out-of-control state is identified. The second policy utilises preventive repair where the process is repaired even though it is operating in the in-control state. Sarkar and Saren (2016) consider an inspection policy where Type I and Type II errors are due to incorrectly identifying a non-conforming item as conforming or a conforming item as non-conforming. The trade-off between product inspection, where a warranty cost is assigned to a uninspected defective item, and process inspection is also investigated.

Most of the literature addressing maintenance policies in the context of production processes assume that the production process is interrupted during the duration of the maintenance process. Interrupting the production system is justified for a just-in-time (JIT) policy where a manager is allowed to halt production whenever quality problems arise (see Jaber 2006). A JIT environment justifies and motivates interrupting the production process to perform quality maintenance which transitions the process to the in-control state. The authors in Jaber and Guiffrida (2008) extend the model presented in Jaber and Guiffrida (2004) and assume that the system can be interrupted for maintenance in order to transition the production process to the in-control state.

Maintenance policies for an economic production quantity (EPQ) system is considered in Liao, Chen, and Sheu (2009) where the authors account for the possibility of perfect and imperfect maintenance. The production process can transition to the in-control state only in the case of perfect maintenance. Accordingly, the optimal run time is calculated. Liu and Cetinkaya (2011) integrate the computation of the manufacturer's production lot-size and the buyer's replenishment order quantity under random yield and quantify the impact of random yield on the system performance. Nasr, Maddah, and Salameh (2013), investigate an EOQ system with a correlated binomial supply. When solving for the optimal order size, the authors show that several models can modelled by as equivalent correlated binomial model by capturing the quality correlation of any two randomly selected items within an order. Nasr, Maddah, and Salameh (2013) illustrate that correlation leads to decreasing the order size and diversifying supplier reduces the impact of correlation on the cost.

In this paper, we consider the realistic case where the quality of items produced in the same sub-production run is correlated. We assume that maintenance is performed between sub-production runs which reset the production process. This is similar to Kutzner and Kiesmüller (2013), among other works, where a maintenance interruption denotes a renewal interval since maintenance resets the production process. A consequence of this assumption is that the number of defective items across production runs is independent random variables but the quality of two randomly selected items within a production is correlated. A policy which performs maintenance more frequently will result in shorter sub-production runs which can reduce the impact of correlation on system performance.

To capture the quality correlation of items produced in the same sub-production run, we utilise the correlated binomial production process which assumes that the yield probability is identical for all items and the quality correlation between two randomly selected items is a constant value, ρ. We extend the correlated binomial model to include the interrupted geometric model where the probability of an item being defective becomes dependent on the in-control or out-of-control parameters of the production system. In such a case, the yield probability and quality correlation also become dependent on the duration between maintenance interruptions.

To the best of our knowledge, quality correlation has not been considered in the context of production systems. We evaluate the impact of correlation on the production policy as well as system performance and assess the trade-off between running smaller lots vs. larger lots. A sub-production run is defined as the interval between the scheduled maintenance epochs where the maintenance operation results in quality independence across lots produced in different sub-production cycles. A main motivation behind this work is to quantify the impact of correlation on the duration of the sub-production cycles (time between maintenance inspections) and the number of maintenance inspections to be performed in a production cycle.

Allowing for a production cycle to be split into multiple sub-production runs results in a system which is similar to the multiple lot sizing problem where the trade-off between smaller production runs vs. longer runs is evaluated. Note that in Guu and Zhang (2003) and Bendavid and Herer (2009), a MLPO can result in overstocking or understocking. To address understocking (shortages) which is a result of the variability in the number of defective items in a production cycle, our model adds a service constraint to account for the proportion production cycles where demand is not fully satisfied. Accordingly, the service constraint ensures that we produce enough items to satisfy the demand for a production cycle.

We begin by introducing the base model in Section 2 where the distribution of the number of good items produced in a sub-production run follows a Binomial distribution. The yield probability is assumed constant across all sub-production runs. In Section 3, we consider the realistic case where the quality of two randomly selected items produced within the same sub-production run is correlated. We refer to this model as the Correlated Binomial Production Model (CBPM). We extend the CBPM to the interrupted geometric model in Section 4 where the yield probability and quality correlation within a sub-production lot are dependent on the production run length. Numerical examples on the correlated binomial and interrupted geometric production models are presented in Section 5. Section 6 concludes the paper.

2. Base model: binomial production

The model considered is an EPQ model where scheduled maintenance is performed during fixed time intervals of length T. The maintenance time is assumed to be a fixed duration, t, during which production is discontinued and the production process is reset at the end of the maintenance process. A production cycle is composed of n ($n \geq 1$) consecutive sub-production cycles of length T where the sub-production runs are separated by the scheduled maintenance epochs. For clarity, the behaviour of the inventory over a cycle with $n = 3$ sub-production runs is presented in Figure 1. The model assumes an infinite time horizon where holding cost is per item per unit time, h. Every production run incurs a fixed set-up cost K, and every sub-production run incurs a fixed set-up cost, k, where $k < K$. The number of defective items during the ith sub-production run is denoted by the random variable X_i for $i = 1, \ldots, n$. We present a summary of the notation used throughout the paper.

Notation:

- α: Production rate (items/unit time)
- β: Demand rate (items/unit time)
- n: Number of sub-production runs per production cycle – Decision Variable
- T: Duration of a sub-production run (unit time) – Decision Variable
- X_n: Number of defective items during the nth sub-production run – Random Variable
- t: Fixed duration for maintenance (unit time)
- s: Selling price ($/item)
- s_s: Salvage price ($/item)
- c: Production cost ($/item)
- h: Holding cost ($/item/unit time)
- K: Ordering cost per cycle ($/production cycle)
- k: Ordering cost per sub-production run ($/sub-production run)
- p: Probability a randomly selected item is defective ($q = 1 - p$)
- ρ: Quality correlation between two randomly selected items within a production run

Our model assumes that defective items are shipped out at the end of a production run and not at the end of every sub-production cycle. So the removal of defective items occurs once per cycle instead of multiple times per cycles. The main rationale behind the assumption of removing the defective items once per cycle (as opposed to performing multiple removals per cycle), is that the percentage defective is usually small within a sub-production run. Consequently, the cost of carrying the defective items will be low in comparison to shipping out the items along with the set-up cost of initiating an inspection. It is also significant to note that although the set-up cost of setting up an inspection is not identified as a separate cost in the model, it is assumed to be a fixed cost per cycle which can be incorporated as part of the fixed set-up cost, K.

Defective items are removed from the inventory at the end of the production run and are salvaged at a price of s_s per item. Let y be the number of items produced over the time interval T, i.e. during a sub-production run,

$$y = T\alpha.$$

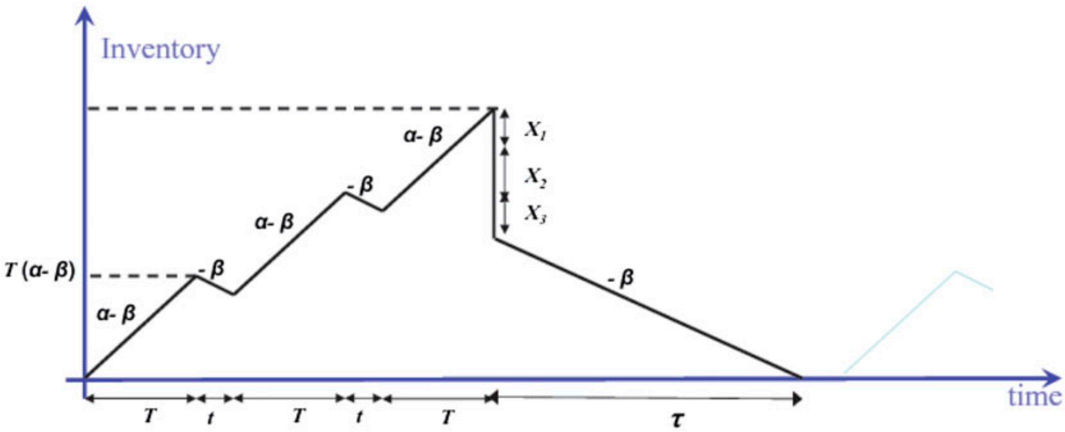

Figure 1. Inventory with multiple maintenance production periods, $n = 3$.

Utilising the binomial production assumption for the base model, the first two moments of the number defective produced during the ith sub-production run,

$$\begin{aligned} &\mathrm{E}[X_i] = p\,y, \\ \text{and} \quad &\mathrm{E}[X_i^2] = y\,p\,(1-p) + (y\,p)^2, \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{1}$$

The duration of the production cycle is a random variable. Let C be the production cycle time,

$$C = n\,T + (n-1)\,t + \tau,$$

where τ is the time to complete the production cycle after halting production at the end of nth sub-production run as illustrated in Figure 1.

$$\tau = \left(n\,T\,(\alpha - \beta) - \sum_{i=1}^{n} X_i - (n-1)\,t\,\beta \right) \beta^{-1}. \tag{2}$$

The first two moments of τ are calculated as,

$$\begin{aligned} \mathrm{E}[\tau] &= \left(\omega - \sum_{i=1}^{n} \mathrm{E}[X_i] \right) \beta^{-1} = (\omega - n\,\mathrm{E}[X_i])\ \beta^{-1}, \\ \mathrm{E}[\tau^2] &= \left(\omega^2 - 2\,\omega \sum_{i=1}^{n} \mathrm{E}[X_i] + \mathrm{E}\Big[\Big(\sum_{i=1}^{n} X_i \Big)^2 \Big] \right) \beta^{-2} \\ &= \left(\omega^2 - 2\,\omega\,n\,\mathrm{E}[X_i] + n\,(n-1)\,\mathrm{E}[X_i]^2 + n\,\mathrm{E}[X_i^2] \right) \beta^{-2} \\ \text{and } \omega &= n\,T\,(\alpha - \beta) - (n-1)\,t\,\beta. \end{aligned} \tag{3}$$

The time epochs denoting the initiation of a production cycle are a renewal point process. Consequently, the renewal reward theorem is implemented to compute the long run profit of the inventory system. We also point out that the behaviour of the inventory model as described by Figure 1 was first presented in Salameh and Jaber (1997). The model of Salameh and Jaber (1997) does not account for defective items (deterministic model) and assumes that the maintenance sub-production run is given, i.e. T is a constant and not a decision variable. The authors solve for the optimal quantity to manufacture in a production run. It is worth noting that even for the relatively simple deterministic model, the authors present an algorithmic approach to solve for the global optima, i.e. it is not possible to express the optimal solution by a closed-form expression.

The performance of the inventory model considered is dependent on the probability of encountering a stock-out in a production cycle as a result of too many defective items being produced. A commonly used approach in inventory management is to define a service level, Ravindran and Warsing (2012), to address the probability of encountering a shortage in supply. The authors in Huang and Song (2010) implement a service constraint to address shortages in supply in the context of random yield in the semi-conductor industry. In this work, we implement a service level constraint where a service level denotes the probability of encountering shortages in supply within a production cycle. Next we detail our calculation of the service level constraint.

2.1 *Service level constraint*

The model considered utilises a service level constraint to ensure that the number of good items in a production cycle satisfies the demand with a probability of $1-\epsilon$. Let υ be the number in inventory before inspection, $\upsilon = n\,(T\,(\alpha-\beta) - t\,\beta)$. If $1-\epsilon$ is the service level probability, then,

$$\text{Prob}\Big(S_n > \upsilon\Big) < \epsilon \tag{4}$$

where $S_n = \sum_{i=1}^{n} X_i$. The service level constraint as represented in Equation (4) is approximated by,

$$\text{Prob}\Big(z > \frac{\upsilon - n\,\text{E}[X_i]}{\sigma_{X_i}\,\sqrt{n}}\Big) < \epsilon \tag{5}$$

where $\sigma_{X_i} = \sqrt{\text{E}[X_i^2] - \text{E}[X_i]^2}$ and z is the standard Normal distribution. The service level constraint, as expressed in Equation (5), is based on approximating the distribution of total yield in the production cycle, S_n, with a Normal distribution with mean $n\,\text{E}[X_i]$ and variance $n\,\sigma_{X_i}^2$.

Next we evaluate the profit and cost measures of the inventory system for a given maintenance policy. The sources of cost are the holding cost, the variable production cost, the production cycle set-up cost and the set-up costs incurred by the sub-production runs. The sources of revenue are the selling price of the conforming items as well as the salvage price of the non-conforming items.

2.2 *System performance measures*

Let I be the inventory carried over a production cycle. We can express I by $I = I_1 + I_2$ where I_1 is the amount carried during the n sub-production runs and I_2 is the amount carried after production is terminated over a duration of τ. The amount carried during the n sub-production runs.

$$\begin{aligned} I_1 &= \frac{1}{2}\,n\,T^2\,(\alpha-\beta) + (n-1)\Big(t\,T\,(\alpha-\beta) - \frac{1}{2}\,t^2\,\beta\Big) + \Big[\sum_{i=1}^{n-1}(n-i)\,T^2\,(\alpha-\beta)\Big] \\ &\quad + \Big[\sum_{i=2}^{n-1}(n-i)\,t\,T\,(\alpha-\beta)\Big] \\ &= \frac{1}{2}\,n\,T^2\,(\alpha-\beta) + (n-1)\Big(t\,T\,(\alpha-\beta) - \frac{1}{2}\,t^2\,\beta\Big) + \frac{1}{2}\,(n-1)\,n\,T^2\,(\alpha-\beta) \\ &\quad + \frac{1}{2}\,(n-2)\,(n-1)\,t\,T\,(\alpha-\beta) \end{aligned} \tag{6}$$

and the amount carried over the time interval of length τ,

$$I_2 = \frac{\tau^2\,\beta}{2}. \tag{7}$$

Since the quality of items is independent across sub-production runs, then the duration of a new cycle is independent of the duration of previous and future cycles. A new cycle with random duration C is initiated when the inventory is depleted. As a result, the time epochs representing the depletion of the inventory level denote a renewal point process. This enables us to utilise the renewal reward theorem to calculate the cost measures per unit time. The holding cost of the carried inventory per unit time (HC),

$$\text{HC} = h\,\frac{\text{E}[I_1+I_2]}{\text{E}[C]} = h\,\frac{\text{E}[I_1]+\text{E}[I_2]}{\text{E}[n\,T + (n-1)\,t + \tau]} = h\,\frac{\text{E}[I_1]+\text{E}[I_2]}{\text{E}[n\,T] + \text{E}[\tau] + (n-1)\,t} \tag{8}$$

The set-up cost during a production cycle accounts for the production set-up cost and the set-up cost for the n sub-production runs. Accordingly, the set-up and production cost per unit time (SC),

$$\text{SC} = \frac{(k\,n + K + c\,y)}{\text{E}[C]} = \frac{(k\,n + K + c\,y)}{\text{E}[n\,T] + \text{E}[\tau] + (n-1)\,t]}. \tag{9}$$

The profit per unit time as a result of the selling the conforming items and salvaging the defective items (SP),

$$\text{SP} = \frac{s\,\Big(n\,T - \sum_{i=1}^{n}\text{E}[X_i]\Big) + s_s\,\sum_{i=1}^{n}\text{E}[X_i]}{\text{E}[n\,T] + \text{E}[\tau] + (n-1)\,t} \tag{10}$$

The total profit per unit time which accounts for holding, production and set-up costs,

$$\begin{aligned}\Phi(n,T) &= \mathrm{SP} - \mathrm{SC} - \mathrm{HC}\\ &= \frac{s\left(n\,T - n\,\mathrm{E}[X_i]\right) + s_s\,n\,\mathrm{E}[X_i]}{\mathrm{E}[n\,T] + \mathrm{E}[\tau] + (n-1)\,t} - \frac{(k\,n + K + c\,y)}{\mathrm{E}[n\,T] + \mathrm{E}[\tau] + (n-1)\,t} - \\ &\quad h\,\frac{\mathrm{E}[I_1] + \mathrm{E}[I_2]}{\mathrm{E}[n\,T] + \mathrm{E}[\tau] + (n-1)\,t]}.\end{aligned} \tag{11}$$

Notice that the profit per unit time as expressed in Equation (11) is a function of the first two moments of the number of defective items in a sub-production lot of size y, $\mathrm{E}[X_i]$ and $\mathrm{E}[X_i^2]$. Next we investigate the yield distribution per sub-production cycle by considering a correlated binomial supply in Section 3 and an interrupted geometric production process in Section 4.

3. Correlated binomial production: yield probability independent of order size

In this section, we make the assumption that the yield probability, $q = 1 - p$, is not dependent on the size of the production lot. Re-initialising a sub-production run creates independence in quality across sub-production runs but the quality of the items produced within the same sub-production run is correlated. Such a production model is realistic in the case where the production process is dependent on the initialisation of the production run which can be prone to human or machine errors when setting the initial values (Yano and Lee 1995).

Let $Z_i = 0, 1$ denote the quality of the ith item within a production run for $i = 1, \ldots, n$, where $Z_i = 1$ if the item is of good quality and $Z_i = 0$ otherwise. If each production run produces y items, let the correlation between two randomly selected items within a production run be $\rho_{i,j} = \mathrm{Correlation}(Z_i, Z_j)$ for $i = 1, \ldots, y$, $j = 1, \ldots, y$ and $i \neq j$. The correlation between two randomly selected items within a production run is identical, i.e. $\rho = \rho_{i,j}$.

Let Y_j and X_j be the number of good items and bad items, respectively, during the jth production run for $j = 1, \cdots, n$, ($y = Y_i + X_i$). The first two moments of Y_i for $i = 1, \ldots, n$,

$$\begin{aligned} &\mathrm{E}[Y_i] = q\,y,\\ \text{and}\quad &\mathrm{E}[Y_i^2] = q\,(1-q)\,(1-\rho)\,y + \left(q\,(1-q)\,\rho + q^2\right)\,y^2. \end{aligned} \tag{12}$$

Proof of Equation (12) is based on expressing the variance of Y_i by the following equality,

$\mathrm{Var}(Y_i) = \sum_{i=1}^{y}\sum_{j=1}^{y}\mathrm{Cov}(Z_i\,Z_j) = \sum_{i=1}^{y}\mathrm{Var}(Z_i) + \sum_{i=1}^{y}\sum_{j=1, j\neq i}^{y}\rho_{ij}\mathrm{Var}(Z_i)$, and $\rho = \rho_{ij}$ for $i, j = 1, \ldots n$ and $i \neq j$.

It follows from Equation (12) and $y = Y_i + X_i$,

$$\begin{aligned} &\mathrm{E}[X_i] = (1-q)\,y,\\ \text{and}\quad &\mathrm{E}[X_i^2] = y^2 - 2\,q\,y^2 + q\,(1-q)\,(1-\rho)\,y + \left(q\,(1-q)\,\rho + q^2\right)\,y^2\\ &\qquad\;\; = q\,(1-q)\,(1-\rho)\,y + \left(q\,(1-q)\,\rho + (1-q)^2\right)\,y^2. \end{aligned} \tag{13}$$

Notice that for highly correlated production system, the variability in the number of defective items produced in a production cycle increases. This is illustrated by rearranging the right-hand side of $\mathrm{E}[X_i^2]$ in Equation (13),

$$\mathrm{E}[X_i^2] = y\,q\,(1-q)\,(y-1)\,\rho + (1-q)^2\,y^2. \tag{14}$$

For $y > 1$, the expression $y\,q\,(1-q)\,(y-1)$ is always positive, which illustrates that increasing ρ results in an increase in the variability of the number of defective items. Consequently, more items need to be produced to reduce the probability of having shortages and to account for the service level constraint in Equation (5). The model at this point assumes that the yield probability, q, is not dependent on the size of the production lot. This might not be the case if the production process degrades to an unreliable out-of-control state where the yield probability is significantly lower. We investigate such a case by considering the interrupted geometric production model in the next section.

4. Correlated binomial production: interrupted geometric

The production process begins in the in-control state at the beginning of production and can transition to the out-of-control state during the production of an item with probability $(1-\theta)$ (Porteus 1986). The out-of-control state is assumed to be an absorbing state, i.e. once the process reaches the out-of-control state it cannot return to the in-control state until the production process is restarted.

The probability that an item is defective is dependent on the state of the production process where the in-control and out-of-control probabilities are p_1 and p_2, respectively, and $p_1 < p_2$. The number defective in a production run of size y is, $X = V_1 + V_2$, where V_1 and V_2 are the number defective when the process is in-control and out-of-control, respectively. The first two moments and the joint moment of V_1 and V_2 are,

$$\mathrm{E}[V_1] = \frac{p_1\,\theta\,(1-\theta^y)}{1-\theta}, \tag{15}$$

$$\mathrm{E}[V_2] = p_2\,y - \frac{p_2\,\theta\,(1-\theta^y)}{1-\theta}, \tag{16}$$

$$\mathrm{E}[V_1^2] = \theta\,p_1\left(\theta^{y+1}\,(2\,p_1\,y - 2\,p_1 + 1) - \theta^y\,(2\,p_1\,y + 1) + \theta\,(2\,p_1 - 1) + 1\right)(1-\theta)^{-2}, \tag{17}$$

$$\begin{aligned}\mathrm{E}[V_2^2] = p_2\Big(&\theta^{y+1}\,(-2\,p_2 - \theta + 1) + \theta^2\,(y^2\,p_2 + y\,p_2 + y + 1)\\ &+ \theta\,(-2\,p_2\,y^2 + 2\,p_2 - 2\,y - 1) + y^2\,p_2 - y\,p_2 + y)\Big)\,(1-\theta)^{-2},\end{aligned} \tag{18}$$

$$\mathrm{E}[V_1\,V_2] = \frac{p_1\,p_2\,\theta}{(1-\theta)^2}\left(\theta^{y+1}\,(1-y) + \theta^y\,(1+y) - \theta\,(1+y) + y - 1\right) \tag{19}$$

The proof for Equations (15)–(19) is presented in the Appendix 1. The first two moments of the number of defective items during the ith sub-production cycle are expressed as a function of the first two moments and the joint moment of V_1 and V_2,

$$\mathrm{E}[X_i] = \mathrm{E}[V_1] + \mathrm{E}[V_2], \tag{20}$$

$$\mathrm{E}[X_i^2] = \mathrm{E}[V_1^2] + 2\,\mathrm{E}[V_1\,V_2] + \mathrm{E}[V_2^2]. \tag{21}$$

Notice that the first two moments of X_i are a function of the production size y. Increasing the duration of the sub-production runs (increasing T) would result in a higher probability of obtaining defective items as the probability of transitioning to an out-of-control state increases. Let p be the probability that a randomly selected item in a sub-production run is of good quality and let $q = 1 - p$ be the probability of selecting an item of good quality. The yield probabilities are a function of the production size where

$$q = 1 - p = \mathrm{E}[Y_i]/y = 1 - \mathrm{E}[X_i]/y. \tag{22}$$

The correlation between the quality of two randomly selected items is also a function of the sub-production size y and can be calculated as (from Equation (12)),

$$\rho = \frac{\mathrm{E}[Y_i^2] - q^2\,y^2 - q\,(1-q)\,y}{q\,y\,(1-q)\,(y-1)}, \tag{23}$$

where $\mathrm{E}[Y_i^2] = \mathrm{E}[(y - X_i)^2] = y^2 - 2\,y\,\mathrm{E}[X_i] + \mathrm{E}[X_i^2]$. For the special case where $p_1 = 0$ and $p_2 = 1$, the model reduces to the in/out-of-control production presented in Porteus (1986) where the machine can transition to the out-of-control state when producing an item with probability θ. In such a case, all items produced when the machine is in the out-of-control state are defective and all items produces when the system is in-control are of good quality. For a production lot of size y, the yield probability and the quality correlation as expressed in Equations (22) and (23) reduce to,

$$q = 1 - p = \frac{\theta\,(1-\theta^y)}{(1-\theta)\,y}, \tag{24}$$

$$\rho = \frac{2\,y\,\theta^2\left(\theta^y\,(y-1) - \theta^{y-1}\,y + 1\right) - \theta^2\,(y-1)(1-\theta^y)^2}{\theta\,(1-\theta^y)(y-1)\left(y\,(1-\theta) - \theta\,(1-\theta^y)\right)}, \tag{25}$$

Equations (24) and (25) are consistent with the results of Porteus (1986) and Nasr, Maddah, and Salameh (2013)) for the special case where $p_1 = 0$ and $p_2 = 1$, which serves to validate Equations (22) and (23). It is also worth noting that Porteus (1986) utilises the approximation $(1-\theta)^y = 1 - (1-\theta)\,y - 0.5\,(1-\theta)^2\,y^2$ which is based on the Taylor series expansion of $(1-\theta)^y$ and is accurate for θ approaching 1. Applying the approximation on Equations (15) and (16), we obtain the following simplifying approximations on the first moment of V_1 and V_2,

$$\mathrm{E}[V_1] \approx p_1\,y\left(1 - \frac{y(1-\theta)}{2}\right), \quad \text{and} \quad \mathrm{E}[V_2] \approx \frac{p_2\,y^2\,(1-\theta)}{2}. \tag{26}$$

Table 1. Correlated binomial production.

Case	ρ	n_i	T_i	$n_i\,T_i$	$\Phi_i(n_i, T_i)$	y	$n\,y$	Δ_i (%)
1	0	1	7.375	7.375	46.72	295	295	–
2	0.1	2	3.825	7.65	45.62	153	306	−2.37
3	0.2	4	2.075	8.3	43.90	83	332	−6.04
4	0.3	6	1.975	11.85	41.17	79	474	−11.89
5	0.4	9	1.65	14.85	37.39	66	594	−19.99
6	0.5	11	1.7	18.7	32.94	68	748	−29.50
7	0.6	14	1.575	22.05	28.37	63	882	−39.27
8	0.7	16	1.6	25.6	23.77	64	1024	−49.13
9	0.8	19	1.525	28.975	19.02	61	1159	−59.29
10	0.9	21	1.55	32.55	14.21	62	1302	−69.60

If we make the simplifying assumption that defective items are identified and re-used immediately (as is the case in Porteus (1986)), then the approximations in Equations (26) can be utilised to simplify the cost function and obtain an approximate closed-form expression, similar to Porteus (1986).

5. Numerical examples

Consider an EPQ base case with the following parameters, $K = 100$ \$/production cycle, $k = 10$ \$/sub-production run, $h = 0.2$ \$/unit/unit time, $t = 0.2$ unit time, $\alpha = 40$ items/unit time, $\beta = 24$ items/unit time, $s = 5$ \$/item, $s_s = 1$ \$/item, and $c = 2$ \$/item. The service level constraint for the base case is implemented for $\epsilon = 0.01$. We investigate the optimal production and maintenance policies under different assumptions and present three examples which utilise the base case parameters. Example 1 considers the case where the quality correlation, ρ, and yield probability, $q = 1 - p$, are fixed and independent of the duration of the sub-production run, i.e. independent of the production lot size y. Example 2 assumes an interrupted geometric production with $p_1 = 0$ and $p_2 = 1$. Example 3 relaxes the assumption that the in-control and out-of-control states only produce conforming and non-conforming items, respectively.

Example 1 Consider the base case parameters with a fixed yield probability $q = 1 - p = 0.85$. Table 1 presents 10 cases where the quality correlation ρ is increased from 0 to 0.9 in increments of 0.1. When determining the production and maintenance policies, we utilise a line search heuristic. The line search is performed on the sub-production duration T while holding the number of maintenance interruptions, n, fixed. Then n is incrementally increased and a line search is performed on T.

Table 1 presents the solution of the line search heuristic for the 10 cases with different quality correlation, ρ. The solution of the line search heuristic for Case i, for $i = 1, \dots, 10$, is denoted by the number of sub-production runs, n_i, and the duration of the production run, T_i. The net profit is denoted by $\Phi_i(n_i, T_i)$ for $i = 1, \dots, 10$.

We refer to Case 1 as the base case since the quality correlation is set to 0, $\rho = 0$. Accordingly, we utilise Case 1 as a reference when calculating the reduction in profit incurred by an increase in quality correlation. Let Δ_i be the reduction in profit per unit time due to correlation. We calculate Δ_i as follows for $i = 2, \dots, 10$,

$$\Delta_i = \frac{\Phi_1(n_1, T_1) - \Phi_i(n_i, T_i)}{\Phi_1(n_1, T_1)} \times 100. \qquad (27)$$

The model introduces maintenance interruptions to reduce the impact of quality correlation on the system cost. Accordingly it should be obvious that in the case where $\rho = 0$, it is not optimal to interrupt a production run and perform maintenance. This behaviour is illustrated in Case 1 of Table 1 where $n_1 = 1$ and the duration of the uninterrupted production run is $n_1\,T_1 = T_1 = 7.375$ time units.

As the quality correlation increases, it becomes optimal to increase the number of sub-production runs to offset the impact of correlation. Another observation is that the optimal duration of the production run, $n_i\,T_i$, increases whereas the optimal duration of a sub-production run, T_i, decreases as ρ increases, Table 1. This observation is consistent with Figure 2 where an increase in quality correlation results in producing more items in a production run, $n\,y$, but the number produced in a sub-production run decreases, y.

Managerial implications related to the impact of highly correlated items is to increase the duration of the production run (consequently increases $n\,y$) and to further partition the production run into shorter sub-production runs. Although the

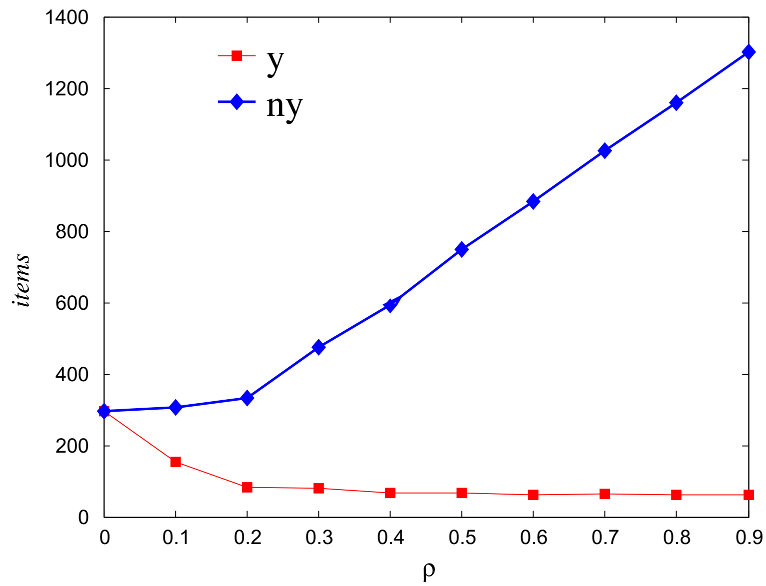

Figure 2. Production lot size per cycle – Example 1.

model presented attempts to reduce the impact of correlation on the system performance, the reduction in profit is still significant as shown in Table 1. This is illustrated by evaluating Δ_i, for $i = 1 \ldots, 10$, where higher values of ρ result in a significant decrease in profit where Δ_i exceeds 20% for $\rho > 0.4$. From a managerial perspective, the financial evaluation of the production system would be misleading if the effect of correlation, as quantified by Δ_i, is not factored in as part of the net profit calculation.

An important managerial insight regarding the impact of correlation on the production batch sizes is that higher correlation levels increase the variability in the number of defective items. This increase in variability can be seen from Equation (14) which illustrates that the second moment of the number defective increases as ρ increases. Consequently, the variability in the total number of good items produced, $S_n = \sum_{i=1}^{n} X_i$, increases significantly. This increase in variability has implications on the service level constraint which is accounted for by an increases in the number of items produced to achieve the desired probability ϵ. It is significant to point out from a managerial perspective that ignoring correlation by implementing the Case 1 policy ($\rho = 0$), which utilises $n = 1$ and $y = 295$, results in a violation of the service level constraint of $\epsilon = 0.01$, due to the increase in the variability of S_n.

Example 2 Next we consider the Interrupted Geometric as presented in Porteus (1986), where the process only produces good items when in the in-control state, $p_1 = 0$, and only produces defective items when in the out-of-control state, $p_2 = 1$. Recall that θ is the probability that the production process remains in the in-control state and $1 - \theta$ denotes the probability of transitioning to the out-of-control state. Table 2 varies $1 - \theta$ for the base case parameters from 0.001 to 0.009 in increments of 0.001. As can be seen in Table 2, the correlation remains within the range 0.657 to 0.639 for $1 - \theta \in [0.001, 0.009]$ whereas the yield probability increased from 0.044 to 0.117.

Notice that the quality correlation ρ is a function of the sub-production lot size y, Equation (25). As θ is varied over the cases, ρ does not fluctuate significantly as a result of changes in the order size y.

As the probability of going out-of-control increases, shorter and more frequent sub-production runs become optimal. Table 2 illustrates that as the probability of transitioning to the out-of-control state increases, the optimal policy is to produce smaller lot sizes, y, but increase the number produced in per production cycle, $n\,y$.

Example 3 The next set of numerical cases presented in Table 3 consider the general interrupted geometric production where p_1 and p_2 are set to 0.05 and 0.5, respectively. In such a case, a production process can still produce good items even though the process is out-of-control. This reduces the impact of correlation within a lot size which ranges from 0.111 in Case 1 to 0.167 in Case 9 of Table 3. The range on ρ is significantly less than the quality correlation in Example 2 where $p_1 = 0$ and $p_2 = 1$. Figure 3 compares the behaviour of the optimal production policy for Examples 2 and 3. Although the sub-production size y is larger in Example 3, the total number of items produced over the entire cycle, $n\,y$ is larger

Table 2. Interupted geometric.

Case	$1-\theta$	p	ρ	n_i	T_i	n_i, T_i	$\Phi_i(n_i, T_i)$	y	$n\,y$
1	0.001	0.044	0.657	3	2.25	6.75	48.48	90	270
2	0.002	0.067	0.651	4	1.725	6.90	46.90	69	276
3	0.003	0.081	0.648	5	1.4	7.00	45.58	56	280
4	0.004	0.083	0.647	7	1.075	7.53	44.11	43	301
5	0.005	0.087	0.646	9	0.9	8.10	42.60	36	324
6	0.006	0.082	0.648	13	0.7	9.10	40.21	28	364
7	0.007	0.102	0.643	17	0.75	12.75	36.74	30	510
8	0.008	0.108	0.641	24	0.7	16.80	31.11	28	672
9	0.009	0.117	0.639	33	0.675	22.28	23.20	27	891

Table 3. Interupted geometric – general.

Case	$1-\theta$	p	ρ	n_i	T_i	n_i, T_i	$\Phi_i(n_i, T_i)$	y	$n\,y$
1	0.001	0.080	0.111	2	3.425	6.85	48.30	137	274
2	0.002	0.106	0.148	2	3.35	6.7	47.24	134	268
3	0.003	0.108	0.150	3	2.325	6.975	46.40	93	279
4	0.004	0.124	0.160	3	2.3	6.9	45.67	92	276
5	0.005	0.123	0.160	4	1.8	7.2	44.97	72	288
6	0.006	0.135	0.164	4	1.775	7.1	44.39	71	284
7	0.007	0.146	0.166	4	1.75	7	43.81	70	280
8	0.008	0.143	0.166	5	1.475	7.375	43.21	59	295
9	0.009	0.151	0.167	5	1.45	7.25	42.72	58	290

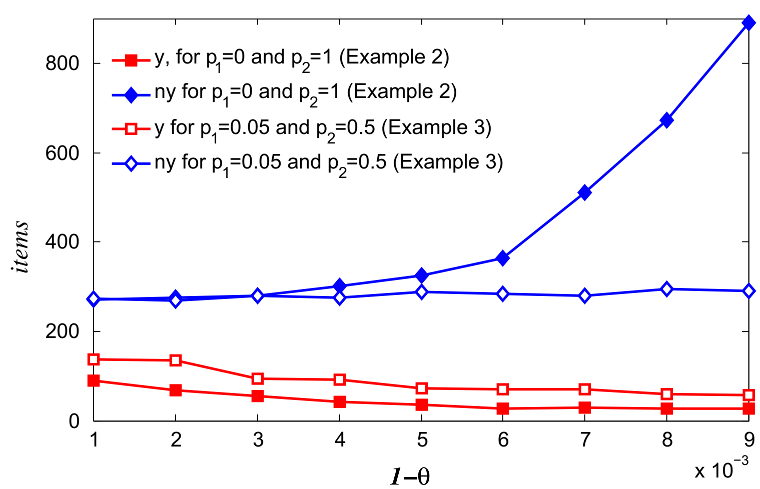

Figure 3. Comparison of production policies for the interrupted geometric cases – Examples 2 and 3.

in Example 2. The number of maintenance interruptions reaches a maximum of 5 in Example 3, compared to $n = 33$ in Example 2. The results of Examples 2 and 3 are consistent with the insights of Example 1 which state that highly correlated lots tend to decrease the sub-production lot size but increases the duration of the production cycle by increasing the number of sub-production runs.

We point out that the three examples considered utilise the net profit function per unit time as expressed in Equation (11). The distinguishing component when calculating the net profit function across the three examples are the first two moments of the number of good items in a sub-production lot, $\mathrm{E}[X_i]$ and $\mathrm{E}[X_i^2]$. The first two moments of the number of good items are calculated according to the modelling assumptions which describe the behaviour of the production process. Consequently, a manager investigating the performance of an EPQ system with a stochastic yield can utilise the approach presented in this paper to quantify the impact of the distribution of the number of good items, specifically the first two moments, on the performance measures of the system.

The majority of the literature on imperfect supply does not provide managerial insights or guidelines to quantify and reduce the impact of quality correlation on the performance of inventory systems. This work illustrates that it is not sufficient to simply evaluate the yield probability but a manager should also investigate the presence of correlation. We distinguish between two cases of imperfect supply. The first case assumes that the yield probability and correlation are independent of the production lot size. In such a case, a manager should consider implementing maintenance interruptions if significant correlation values are detected, $\rho > 0.3$. To reduce the impact of correlation, a production manager should increase the duration of the production cycle as well as schedule frequent (but shorter) maintenance interruptions. This work provides a manager with the tools, via a mathematical model, to determine the optimal maintenance policy and optimal production runs as a function of the supply correlation. In the second case where the production system can randomly deteriorate (interrupted geometric), the yield probability and correlation are dependent on the production lot size, y, the in-control probability, θ and the state dependent probabilities, $0 \leq p_1 < p_2$. A production system exhibiting high θ and where $p_1 \approx p_2$, the quality correlation becomes insignificant and it is optimal to conduct one production run without maintenance interruptions. For higher values of θ where the difference between p_1 and p_2 increases (e.g. $p_1 \to 0$ and $p_2 \to 1$), then it is profitable for a manager to conduct frequent maintenance interruptions with longer production durations where the optimal policy can be determined by examining the mathematical model presented. It is also significant for a manager to note that adjusting the production policy reduces but does not eliminate the impact of correlation on system performance. Consequently, the resulting increase in system cost should not be ignored; otherwise this leads to a misleading overestimation of the profit.

6. Conclusion

Although an abundance of literature exists on modelling random yield in production systems, very few models consider correlation between imperfect items. We investigate scheduled maintenance policies as an approach to reduce the impact of correlation and imperfect production on the system performance measures. The EPQ system presented in this work centres around expressing the characteristics of the random yield of a production system, specifically, the first two moments of the number of conforming items in a production lot. This is achieved by expressing the performance measures and eventually the net profit per unit time as a function of the first two moments of non-conforming items in a sub-production lot. The first two moments in turn are expressed in accordance to the behaviour of the production system, where we consider the scenario which accounts for ρ and q being independent of the production lot size. We also consider the case where ρ and q are dependent on the production lot size, as in the case of the interrupted geometric production model. Accordingly, expressing the net profit per unit time as a function of the first two moments enables a production manager to quantify the impact of variability on the random yield which can significantly increase as illustrated by the correlated binomial model.

One of the main findings of this work is highlighted in the numerical results section of this paper which illustrates that highly correlated lot sizes result in longer production runs with more frequent and shorter sub-production runs. Also, not accounting for the effect of quality correlation on the system performance can lead to misleading analysis of system performance and a significant over-estimation of the profit. A manager can further utilise the mathematical model to calculate the optimal number of maintenance inspections to quantify and reduce the impact of correlation on the profit.

The production models considered in this paper assume that the quality of products within the same lot is correlated but independent of the quality of other lots. As mentioned in Section 3, such a model is realistic in the case where the quality correlation is dependent on the initial setting of the production run. Yano and Lee (1995) give examples of situating a die properly, or selecting the temperature for the substance being heated often vary by input and sometimes require trial and error. A limitation of this work is that in many real-world production cases quality correlation exists across batches. Future work can relax the assumption of independence in quality across batches and account for a correlation factor between lots produces with different initialisation set-ups. Another significant assumption that can be relaxed is that defective items can either be of good quality or defective. Future work can focus on quantifying different levels of quality and not restrict the analysis to either good or defective. Extensions of this work can also include the scenario where backordering is accounted for by a shortage cost per item per unit time as an alternative to utilising a service level constraint.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1.

Assume that the production process goes out-of-control while producing the hth item.

$$
\begin{aligned}
\mathrm{E}[V_1] &= \sum_{i=1}^{y} p_1\,(i-1) \times \mathrm{Prob}(h=i) \;+\; p_1\,y \times \mathrm{Prob}(h>y) \\
&= \left(\sum_{i=1}^{y} p_1\,(i-1)\,(1-\theta)\,\theta^{i-1}\right) \;+\; p_1\,y\,\theta^{y} \\
&= p_1\,(1-\theta)\left(\sum_{i=1}^{y}(i-1)\,\theta^{i-1}\right) \;+\; p_1\,y\,\theta^{y}.
\end{aligned} \tag{A1}
$$

Expanding the summation $\sum_{i=1}^{y}(i-1)\,\theta^{i-1}$ and rearranging the terms results in Equation (15).

$$
\begin{aligned}
\mathrm{E}[V_1^2] &= \left(\sum_{i=1}^{y}(i-1)\,p_1\,(1-p_1+(i-1)\,p_1) \times \mathrm{Prob}(h=i)\right) \;+\; y\,p_1\,(1-p_1+y\,p_1) \times \mathrm{Prob}(h>y) \\
&= \left(\sum_{i=1}^{y}(i-1)\,p_1\,(1-p_1+(i-1)\,p_1)\,(1-\theta)\,\theta^{i-1}\right) \;+\; y\,p_1\,(1-p_1+y\,p_1)\,\theta^{y} \\
&= (1-\theta)\,p_1\left(\sum_{i=1}^{y}(i-1)\,(1-p_1+(i-1)\,p_1)\,\theta^{i-1}\right) \;+\; y\,p_1\,(1-p_1+y\,p_1)\,\theta^{y}.
\end{aligned} \tag{A2}
$$

Expanding the summation $\sum_{i=1}^{y}(i-1)\,\theta^{i-1}$ and rearranging the terms results in Equation (17).

$$
\mathrm{E}[V_2] = \sum_{i=1}^{y} p_2\,(y-i+1) \times \mathrm{Prob}(h=i) \;=\; p_2\,(1-\theta)\sum_{i=1}^{y}(y-i+1)\,\theta^{i-1}. \tag{A3}
$$

Expanding the summation $\sum_{i=1}^{y}(y-i+1)\,\theta^{i-1}$ and rearranging the terms results in Equation (16).

$$
\begin{aligned}
\mathrm{E}[V_2^2] &= \sum_{i=1}^{y} p_2\,(y-i+1)\,(1-p_2+(y-i+1)\,p_2) \times \mathrm{Prob}(h=i) \\
&= \sum_{i=1}^{y} p_2\,(y-i+1)\,(1-p_2+(y-i+1)\,p_2)\,(1-\theta)\,\theta^{i-1} \\
&= p_2\,(1-\theta)\sum_{i=1}^{y}(y-i+1)\,(1-p_2+(y-i+1)\,p_2)\,\theta^{i-1}
\end{aligned} \tag{A4}
$$

Expanding the summation $\sum_{i=1}^{y}(y-i+1)\,(1-p_2+(y-i+1)\,p_2)\,\theta^{i-1}$ and rearranging the terms results in Equation (18).

$$
\begin{aligned}
\mathrm{E}[V_1\,V_2] &= \sum_{i=1}^{y} p_1\,(i-1)\,p_2\,(y-i+1)\,(1-\theta)\,\theta^{i-1} \\
&= p_1\,p_2\,(1-\theta)\sum_{i=1}^{y}(i-1)\,(y-i+1)\,\theta^{i-1}
\end{aligned} \tag{A5}
$$

Expanding the summation $\sum_{i=1}^{y}(i-1)\,(y-i+1)\,\theta^{i-1}$ and rearranging the terms results in Equation (19).