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Artinian Gorenstein algebras with linear resolutions

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ABSTRACT

For each pair of positive integers n, d, we construct a complex $\widetilde{\mathbb{G}}'(n)$ of modules over the bi-graded polynomial ring $\widetilde{R} = \mathbb{Z}[x_1, \dots, x_d, \{t_M\}]$, where M roams over all monomials of degree $2n-2$ in $\{x_1, \dots, x_d\}$. The complex $\widetilde{\mathbb{G}}'(n)$ has the following universal property. Let P be the polynomial ring $\boldsymbol{k}[x_1, \dots, x_d]$, where $\boldsymbol{k}$ is a field, and let $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ be the set of homogeneous ideals I in P, which are generated by forms of degree n, and for which P/I is an Artinian Gorenstein algebra with a linear resolution. If I is an ideal from $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, then there exists a homomorphism $\widetilde{R} \to P$, so that $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(n)$ is a minimal homogeneous resolution of P/I by free P-modules. The construction of $\widetilde{\mathbb{G}}'(n)$ is equivariant and explicit. We give the differentials of $\widetilde{\mathbb{G}}'(n)$ as well as the modules. On the other hand, the homology of $\widetilde{\mathbb{G}}'(n)$ is unknown as are the properties of the modules that comprise $\widetilde{\mathbb{G}}'(n)$. Nonetheless, there is an ideal $\widetilde{I}$ of $\widetilde{R}$ and an element $\boldsymbol{\delta}$ of $\widetilde{R}$ so that $\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ is a Gorenstein ideal of $\widetilde{R}_{\boldsymbol{\delta}}$ and $\widetilde{\mathbb{G}}'(n)_{\boldsymbol{\delta}}$ is a resolution of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ by projective $\widetilde{R}_{\boldsymbol{\delta}}$-modules.

The complex $\widetilde{\mathbb{G}}'(n)$ is obtained from a less complicated complex $\widetilde{\mathbb{G}}(n)$ which is built directly, and in a polynomial manner, from the coefficients of a generic Macaulay inverse system Φ. Furthermore, $\widetilde{I}$ is the ideal of $\widetilde{R}$ determined

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Resolutions

by Φ. The modules of $\widetilde{\mathbb{G}}(n)$ are Schur and Weyl modules corresponding to hooks. The complex $\widetilde{\mathbb{G}}(n)$ is bi-homogeneous and every entry of every matrix in $\widetilde{\mathbb{G}}(n)$ is a monomial.
If $m_1, \dots, m_N$ is a list of the monomials in $x_1, \dots, x_d$ of degree $n-1$, then $\boldsymbol{\delta}$ is the determinant of the $N \times N$ matrix $(t_{m_i m_j})$.
The previously listed results exhibit a flat family of $\boldsymbol{k}$-algebras parameterized by $\mathbb{I}_n^{[d]}(\boldsymbol{k})$:

$$\boldsymbol{k}\big[\{t_M\}\big]_{\boldsymbol{\delta}} \to \left(\frac{\boldsymbol{k} \otimes_{\mathbb{Z}} \widetilde{R}}{\widetilde{I}}\right)_{\boldsymbol{\delta}}. \qquad (*)$$

Every algebra P/I, with $I \in \mathbb{I}_n^{[d]}(\boldsymbol{k})$, is a fiber of $(*)$. We simultaneously resolve all of these algebras P/I.
The natural action of $\mathrm{GL}_d(\boldsymbol{k})$ on P induces an action of $\mathrm{GL}_d(\boldsymbol{k})$ on $\mathbb{I}_n^{[d]}(\boldsymbol{k})$. We prove that if $d = 3$, $n \geq 3$, and the characteristic of $\boldsymbol{k}$ is zero, then $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ decomposes into at least four disjoint, non-empty orbits under this group action.

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0. Introduction

Fix a pair of positive integers d and n. We create a ring $\widetilde{R}$ and a complex $\widetilde{\mathbb{G}}'(n)$ of $\widetilde{R}$-modules with the following universal property. Let $P = \boldsymbol{k}[x_1, \dots, x_d]$ be a polynomial ring in d variables over the field $\boldsymbol{k}$ and let I be a grade d Gorenstein ideal in P which is generated by homogeneous forms of degree n. If the resolution of P/I by free P-modules is (Gorenstein) linear, then there exists a ring homomorphism $\widehat{\phi} : \widetilde{R} \to P$ such that $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(n)$ is a minimal homogeneous resolution of P/I by free P-modules. Our construction is coordinate free.

We briefly describe our construction, many more details will be given later. Let U be a free Abelian group of rank d. The ring $\widetilde{R}$ is equal to

$$\mathrm{Sym}_\bullet^{\mathbb{Z}}\big(U \oplus \mathrm{Sym}_{2n-2}^{\mathbb{Z}} U\big),$$

and the complex $\widetilde{\mathbb{G}}'(n)$ has the form

$$0 \to Y \to X_{d-1,n} \xrightarrow{\mathrm{Kos}^{\Psi}} X_{d-2,n} \xrightarrow{\mathrm{Kos}^{\Psi}} \dots \xrightarrow{\mathrm{Kos}^{\Psi}} X_{1,n} \xrightarrow{\mathrm{Kos}^{\Psi}} X_{0,n} \xrightarrow{\widehat{\Psi}} \widetilde{R},$$

where $X_{p,n}$ is a submodule of $\widetilde{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^p U \otimes_{\mathbb{Z}} \mathrm{Sym}_n^{\mathbb{Z}} U$ and Y is a free $\widetilde{R}$-module of rank one. The $\mathbb{Z}$-module homomorphism $\Psi : U \to \widetilde{R}$ is inclusion; $\widehat{\Psi} : \mathrm{Sym}_{\bullet}^{\mathbb{Z}} U \to \widetilde{R}$ is the $\mathbb{Z}$-algebra homomorphism induced by Ψ; and

$$\mathrm{Kos}^{\Psi} : \widetilde{R} \otimes_{\mathbb{Z}} \textstyle\bigwedge_{\mathbb{Z}}^p U \to \widetilde{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^{p-1} U$$

is the Koszul complex map induced by Ψ.

If one chooses a basis $x_1, \dots, x_d$ for the free Abelian group U and a basis $\{t_M\}$, for the free Abelian group $\mathrm{Sym}_{2n-2}^{\mathbb{Z}}$, as M roams over all monomials in $x_1, \dots, x_d$ of degree $2n-2$, then one may think of $\widetilde{R}$ as the polynomial ring $\mathbb{Z}[x_1, \dots, x_d, \{t_M\}]$. There is a distinguished element $\boldsymbol{\delta} = \det(t_{m_i m_j})$ of $\widetilde{R}$, where $m_1, \dots, m_N$ is a list of the monomials in $x_1, \dots, x_d$ of degree $n-1$. The $\widetilde{R}_{\boldsymbol{\delta}}$-modules $(X_{p,r})_{\boldsymbol{\delta}}$ are projective and the complex $\widetilde{\mathbb{G}}'(n)_{\boldsymbol{\delta}}$ is a resolution of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$, where $\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ is a grade d Gorenstein ideal in $\widetilde{R}_{\boldsymbol{\delta}}$.

Let P be a standard graded polynomial ring $\boldsymbol{k}[x_1, \dots, x_d]$ over a field $\boldsymbol{k}$, and I be a homogeneous ideal of P so that the quotient ring P/I is Artinian and Gorenstein. Recall that P/I is said to have a (*Gorenstein*) *linear resolution over* P if the minimal homogeneous resolution of P/I by free P-modules has the form

$$\begin{aligned} 0 \to P(-2n-d+2) \to P(-n-d+2)^{\beta_{d-1}} \to \dots \to P(-n-j+1)^{\beta_j} \to \dots \\ \to P(-n-1)^{\beta_2} \to P(-n)^{\beta_1} \to P, \end{aligned}$$

for some positive integer n. (The Betti numbers β_j, with $1 \le j \le d-1$, may be computed using the Herzog–Kühl formulas [20].) The hypothesis that P/I is Gorenstein forces the entries in the first and last matrices in the minimal homogeneous resolution to have the same degree; all of the entries of the other matrices are linear forms. If $\boldsymbol{k}$ is a field and d and n are positive integers, then let $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ be the following set of ideals:

$$\mathbb{I}_n^{[d]}(\boldsymbol{k}) = \left\{ I \;\middle|\; \begin{array}{l} I \text{ is a homogeneous, grade } d \text{, Gorenstein ideal} \\ \text{of } P = \boldsymbol{k}[x_1, \dots, x_d] \text{ with a linear resolution,} \\ \text{and } I \text{ is generated by forms of degree } n \end{array} \right\}. \tag{0.1}$$

If I is in $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, then the socle degree of P/I is $2n-2$ and the Hilbert Function of P/I is

$$\dim_{\boldsymbol{k}}[P/I]_i = \dim_{\boldsymbol{k}}[P]_i = \dim_{\boldsymbol{k}}[P/I]_{2n-2-i} \quad \text{for } 0 \le i \le n-1. \tag{0.2}$$

Tony Iarrobino [22] initiated the use of the word “compressed” to describe the rings of (0.2). Among all Artinian Gorenstein $\boldsymbol{k}$-algebras with the specified embedding codimension and the specified socle degree, these have the largest total length. (The set of

Artinian Gorenstein algebras with a linear resolution is a proper subset of the set of compressed Artinian Gorenstein algebras.)

We consider the following questions and projects.

Project 0.3. Parameterize the elements of $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, and their resolutions, in a reasonable manner.

Project 0.4. The group $\mathrm{GL}_d\,\boldsymbol{k}$ acts on the set $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ (by way of the action of $\mathrm{GL}_d\,\boldsymbol{k}$ on the vector space with basis $x_1, \dots, x_d$). Decompose $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ into disjoint orbits under this group action.

Project 0.5. (This is a different way to say Project 0.4.) Classify all graded, Artinian, Gorenstein, $\boldsymbol{k}$-algebras with linear resolutions, of embedding codimension d and socle degree $2n-2$.

Question 0.6. If the complete answer to (0.4) and (0.5) is elusive, maybe one can answer: How many classes are there?

Question 0.7. If the complete answer to (0.6) is elusive, maybe one can answer: Is there more than one class?

Question 0.7 is already interesting when $d = 3$. According to Buchsbaum and Eisenbud [8], every grade three Gorenstein ideal is generated by the maximal order Pfaffians of an odd sized alternating matrix X. We consider such ideals when each entry of X is a linear form from $P = \boldsymbol{k}[x, y, z]$. Let $\boldsymbol{k}$ be a fixed field and n be a fixed positive integer. Consider

$$\mathbb{X}_n(\boldsymbol{k}) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a } (2n+1)\times(2n+1) \text{ alternating matrix of linear} \\ \text{forms from } P = \boldsymbol{k}[x,y,z] \text{ such that the ideal generated} \\ \text{by the maximal order Pfaffians of } X \text{ has grade } 3 \end{array} \right\}. \tag{0.8}$$

If X is in $\mathbb{X}_n(\boldsymbol{k})$, then the ideal generated by the maximal order Pfaffians of X is in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$. Buchsbaum and Eisenbud exhibited an element $H_n \in \mathbb{X}_n(\boldsymbol{k})$ for all n. The first two matrices H_n are

$$H_1 = \begin{bmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} 0 & x & 0 & 0 & z \\ -x & 0 & y & z & 0 \\ 0 & -y & 0 & x & 0 \\ 0 & -z & -x & 0 & y \\ -z & 0 & 0 & -y & 0 \end{bmatrix} \tag{0.9}$$

One follows the same pattern to build all of the rest of the H_n.

Question 0.7, when $d = 3$ is: Can every element of $\mathbb{X}_n(\boldsymbol{k})$ be put in the form of H_n after row and column operations and linear change of variables?

The present paper contains a complete solution to Project 0.3, a complete answer to Question 0.7 when $d = 3$, and a partial answer to Question 0.6 when $d = 3$. We remain very interested in Projects 0.4 and 0.5.

To parameterize the elements of $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, we use Macaulay inverse systems. View the polynomial ring $P = \boldsymbol{k}[x_1, \dots, x_d]$ as the symmetric algebra $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$, where U is the $\boldsymbol{k}$-vector space with basis $x_1, \dots, x_d$. The divided power algebra $D_\bullet^{\boldsymbol{k}}(U^*)$ is a module over the polynomial ring $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$. (See Section 1.3.) Let I be a grade d Gorenstein ideal in $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$. One application of Macaulay's Theorem (see Theorem 1.4) is that the annihilator of I in $D_\bullet^{\boldsymbol{k}}(U^*)$ is a cyclic $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$-module, denoted $\operatorname{ann} I$, and called the Macaulay inverse system of I. If I is in $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, then the Macaulay inverse system of I is generated by an element ϕ of $D_{2n-2}^{\boldsymbol{k}}(U^*)$. In Corollary 1.13 we identify an open subset of the affine space $D_{2n-2}^{\boldsymbol{k}}(U^*)$ which parameterizes $\mathbb{I}_n^{[d]}(\boldsymbol{k})$.

As an intermediate step toward our solution of Project 0.3, we exhibit a complex $\widetilde{\mathbb{G}}(n)$ that depends only on the positive integer n. The complex $\widetilde{\mathbb{G}}(n)$ is built over the bi-graded polynomial ring

$$\widetilde{R} = \mathbb{Z}\left[x_1, \dots, x_d, \left\{ t_M \;\middle|\; \begin{array}{l} M \text{ is a monomial in } \{x_1, \dots, x_d\} \\ \text{of degree } 2n-2 \end{array} \right\}\right]; \tag{0.10}$$

where the variables x_i each have degree $(1, 0)$ and the variables t_M each have degree $(0, 1)$. Each element $\phi \in D_{2n-2}^{\boldsymbol{k}}(U^*)$ induces a $\mathbb{Z}[x_1, \dots, x_d]$-algebra homomorphism $\widehat{\phi}: \widetilde{R} \to \boldsymbol{k}[x_1, \dots, x_d] = P$, with $\widehat{\phi}(t_M) = \phi(M)$. In practice, $\phi = \sum \tau_M M^*$ in $D_{2n-2}^{\boldsymbol{k}}(U^*)$, where M varies over the monomials of $P = \operatorname{Sym}_\bullet^{\boldsymbol{k}}(U)$ of degree $2n-2$, and $\{M^*\}$ is the basis for $D_{2n-2}^{\boldsymbol{k}}(U^*)$ which is dual to $\{M\}$. The map $\widehat{\phi}: \widetilde{R} \to P$, which is induced by ϕ, sends the variable t_M to τ_M, which is an element of the field $\boldsymbol{k}$.

Theorem 0.11. *Fix n and $\widetilde{\mathbb{G}}(n)$ as described in (0.10). Let $m_1, \dots, m_N$ be a list of the monomials in $x_1, \dots, x_d$ of degree $n-1$.*

(a) *If $\boldsymbol{\delta}$ is the determinant of the $N \times N$ matrix $T = (t_{m_i m_j})$, then the localization $\widetilde{\mathbb{G}}(n)_{\boldsymbol{\delta}}$ is a resolution of a Gorenstein ring $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ by free $\widetilde{R}_{\boldsymbol{\delta}}$-modules.*
(b) *Let $\boldsymbol{k}$ be a field, P be the polynomial ring $\boldsymbol{k}[x_1, \dots, x_d]$, and U be the $\boldsymbol{k}$-vector space with basis $x_1, \dots, x_d$. If $I = \operatorname{ann}\phi$ is an element of $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ and P is an $\widetilde{R}$-algebra by way of $\widehat{\phi}$, then $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$ is a resolution of P/I by free P-modules.*
(c) *Some of the features of $\widetilde{\mathbb{G}}(n)$ are* **(1)** *it is bi-graded;* **(2)** *there is one $\widetilde{\mathbb{G}}(n)$ for each n; and* **(3)** *every entry of every matrix in $\widetilde{\mathbb{G}}(n)$ is a monomial.*

Parts (a) and (b) of Theorem 0.11 are established in Corollaries 4.16 and 4.22, respectively; the bi-graded Betti numbers of $\widetilde{\mathbb{G}}(n)$ are exhibited in Remark 4.3; assertion (c.3) is Theorem 5.2.

Feature (c.2) is interesting because, even when d is only 3, there are at least 4 disjoint families of minimal resolutions; see Theorem 0.14; but $\widetilde{\mathbb{G}}(n)$ specializes to all of these families. Feature (c.3) is interesting because, for example, the matrix which presents $I = \operatorname{ann}\phi$ is monomial and linear. Often the matrix that presents a module is more important than the generators of the module. One uses the presentation of M to compute $F(M)$ for any right exact functor F and any module. One uses the presentation to compute $\operatorname{Sym}_\bullet M$, and $\operatorname{Sym}_\bullet I$ is the first step toward studying the blow-up algebras – in particular the Rees algebra – associated to I. We have an uncomplicated presenting matrix!

On the other hand, $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$ is not minimal. The final step in our solution of Project 0.3 is the complex $\widetilde{\mathbb{G}}'(n)$ which is described at the beginning of the Introduction and takes place in Section 6.

An alternate phrasing of Theorem 0.11 is also given in Corollary 4.22 where we exhibit a flat family of $\boldsymbol{k}$-algebras parameterized by $\mathbb{I}_n^{[d]}(\boldsymbol{k})$:

$$\boldsymbol{k}\big[\{t_M\}\big]_{\boldsymbol{\delta}} \to \left(\frac{\boldsymbol{k} \otimes_{\mathbb{Z}} \widetilde{R}}{\widetilde{I}}\right)_{\boldsymbol{\delta}}. \tag{4.26}$$

Every algebra $\boldsymbol{k}[x_1, \ldots, x_d]/I$, with $I \in \mathbb{I}_n^{[d]}(\boldsymbol{k})$, is a fiber of (4.26). We simultaneously resolve all of these algebras $\boldsymbol{k}[x_1, \ldots, x_d]/I$.

In order to provide some insight into the structure of the complex $\widetilde{\mathbb{G}}(n)$, we next describe the complex $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$, where the polynomial ring $P = \boldsymbol{k}[x_1, \ldots, x_d] = \operatorname{Sym}_\bullet^{\boldsymbol{k}} U$, the Macaulay inverse system $\phi \in D_{2n-2}^{\boldsymbol{k}}(U^*)$, and the $\mathbb{Z}[x_1, \ldots, x_d]$-algebra homomorphism $\widehat{\phi} : \widetilde{R} \to P$ have all been fixed. (In this discussion, U is the vector space over $\boldsymbol{k}$ with basis $x_1, \ldots, x_d$.) The complex $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$ is the mapping cone of

$$\begin{array}{ccccccccccccc}
 & & 0 & \to & L_{d-1,n} & \to & \cdots & \to & L_{1,n} & \to & L_{0,n} & \to & P \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & P \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^d U & \to & K_{d-1,n-2} & \to & \cdots & \to & K_{1,n-2} & \to & K_{0,n-2}, & &
\end{array} \tag{0.12}$$

where the top complex of (0.12) is a resolution of P/J^n by free P-modules for $J = (x_1, \ldots, x_d)$, and the bottom complex of (0.12) is a resolution of the canonical module of P/J^{n-1} by free P-modules. The P-modules $L_{i,n}$ and $K_{i,n-2}$ are contained in $P \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^i U \otimes_{\boldsymbol{k}} \operatorname{Sym}_n^{\boldsymbol{k}} U$ and $P \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^i U \otimes_{\boldsymbol{k}} D_{n-2}^{\boldsymbol{k}}(U^*)$, respectively; and the vertical map $L_{i,n} \to K_{i,n-2}$ is induced by the map $\boldsymbol{p}_n^{\phi} : \operatorname{Sym}_n^{\boldsymbol{k}} U \to D_{n-2}^{\boldsymbol{k}}(U^*)$ which sends the element u_n of $\operatorname{Sym}_n^{\boldsymbol{k}} U$ to $u_n(\phi)$ in $D_{n-2}^{\boldsymbol{k}}(U^*)$. The top complex of (0.12) is a well-known complex; it is called $L_n^1(\Psi)$ in [7, Cor. 3.2] (see also [32, Thm. 2.1]), where $\Psi : P \otimes_{\boldsymbol{k}} U \to P$ is the map given by multiplication in P. Of course, the bottom complex of (0.12) is isomorphic to $\operatorname{Hom}_P(L_{n-1}^1(\Psi), P)$. The properties of mapping cones automatically yield that $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$, which is the mapping cone of (0.12), is a resolution. The interesting step involves calculating the zeroth homology of this complex; see the proof of Theorem 4.7.

The bi-graded complex $\widetilde{\mathbb{G}}(n)$ is very similar to the graded complex $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)$. Indeed, we can use the double complex (0.12) to indicate the bi-degrees and the bi-homogeneous Betti number in $\widetilde{\mathbb{G}}(n)$. Recall the bi-degrees of the variables of $\widetilde{R}$ from (0.10): the x_i have bi-degree $(1, 0)$ and the t_M have bi-degree $(0, 1)$. The top complex of (0.12) only involves the x_i (even in the bi-homogeneous construction). The matrix on the far right has bi-homogeneous entries of degree $(n, 0)$; the other matrices in the top complex have bi-homogeneous entries of degree $(1, 0)$. The bottom complex of (0.12) also only involves the x_i (even in the bi-homogeneous construction). The matrix on the far left has bi-homogeneous entries of degree $(n - 1, 0)$; the other matrices in the bottom complex have bi-homogeneous entries of degree $(1, 0)$. The vertical maps in the bi-homogeneous construction have bi-homogeneous entries of degree $(0, 1)$. There are many ways to compute the Betti numbers in (0.12) (hence in $\widetilde{\mathbb{G}}(n)$). A formula is given in the Buchsbaum–Eisenbud paper [7]; the L's and K's are Schur modules and Weyl modules, respectively, and one can use combinatorial techniques to calculate the ranks; or one can use the Herzog–Kühl formula [20]. Our answer is given in Remark 4.3.

We denote the top complex of (0.12) by $\mathbb{L}(\Psi, n)$ and the bottom complex by $\mathbb{K}(\Psi, n-1)$, where, again, $\Psi : P \otimes_{\boldsymbol{k}} U \to P$ is multiplication in P. Our discussion of the complexes $\mathbb{L}$ and $\mathbb{K}$ is contained in Section 2. If one replaces $\mathbb{L}(\Psi, n)$ and $\mathbb{K}(\Psi, n - 1)$ in (0.12) by $\mathbb{L}(\Psi, n + \rho)$ and $\mathbb{K}(\Psi, n - 1 - \rho)$, respectively, for some positive integer ρ, then the mapping cone of the resulting double complex is a resolution $P/J^\rho I$. The ideal $J^\rho I$ is a "truncation" of I in the sense that $J^\rho I$ is equal to $I_{\geq n+\rho}$; in other words, $J^\rho I$ is generated by all elements in I of degree at least $n+\rho$. Our construction works for $0 \leq \rho \leq n-2$. On the other hand, $\rho \leq n-2$ is not a restrictive constraint because $I_{\geq r} = J^r$ for $2n-1 \leq r$, since the socle degree of P/I is $2n - 2$, and $\mathbb{L}(\Psi, r)$ is a resolution of P/J^r.

We now turn our attention to Questions 0.7 and 0.6, when $d = 3$; namely "How many orbits does $\mathbb{X}_n(\boldsymbol{k})$ have under the action of $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$?" If $n = 2$, then it is well-known (and not particularly hard to see) that $\mathbb{X}_n(\boldsymbol{k})$ has only one class; see, for example, Observation 7.1. Indeed, $\mathbb{I}_2$ consists of those ideals in $\boldsymbol{k}[x, y, z]$ which define homogeneous Gorenstein rings of minimal multiplicity. The $n = 2$ case causes one to consider multiplication tables, which is one of the key ideas in our work. The $n = 2$ case also reminds us of the resolutions of Kurt Behnke [1,2] and Eisenbud, Riemenschneider, and Schreyer [16].

Fix integers n and μ with $3 \leq n$ and $0 \leq \mu \leq 3$, let

$$\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k}) = \left\{ I \in \mathbb{I}^{[3]}_n(\boldsymbol{k}) \;\middle|\; \begin{array}{l} \exists \text{ linearly independent linear forms} \\ \ell_1, \dots, \ell_\mu \text{ in } P_1 \text{ with } \ell_1^n, \dots, \ell_\mu^n \text{ in } I \\ \text{and } \nexists\ \mu + 1 \text{ such forms} \end{array} \right\}. \tag{0.13}$$

It is clear that $\mathbb{I}^{[d]}_n(\boldsymbol{k})$ is the disjoint union of $\bigcup_{\mu=0}^{3} \mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ and each $\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ is closed under the action of $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$.

Theorem 0.14. *If $n \geq 3$ and the characteristic of $\boldsymbol{k}$ is zero, then $\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ is non-empty for $0 \leq \mu \leq 3$. In particular, if $n \geq 3$, then $\mathbb{X}_n(\boldsymbol{k})$ has at least four non-empty, disjoint orbits in the sense of Project 0.4.*

The proof of Theorem 0.14 is carried out in Section 7. To prove the result, we exhibit an element of $\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ for each μ. It turns out that the ideal BE_n, generated by the maximal order Pfaffians of the Buchsbaum–Eisenbud matrix H_n of (0.9), is in $\mathbb{I}^{[3]}_{n,2}(\boldsymbol{k})$;

$$J_{n,n-1} = \left(x^n, y^n, z^n\right) : (x + y + z)^{n-1}$$

is in $\mathbb{I}^{[3]}_{n,3}(\boldsymbol{k})$; and by modifying a homogeneous generator of the Macaulay inverse system for BE_n we produce ideals in $\mathbb{I}^{[3]}_{n,0}(\boldsymbol{k})$ and $\mathbb{I}^{[3]}_{n,1}(\boldsymbol{k})$. We do **not** claim that every ideal in $\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ may be converted into any other ideal in $\mathbb{I}^{[3]}_{n,\mu}(\boldsymbol{k})$ by using $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$. So, Projects 0.4 and 0.5 are far from resolved, even when $d = 3$.

There are at least two motivations for these projects. First of all, the ideals $J_{n,n-1}$ arise naturally in the study of the Weak Lefschetz Property (WLP) for monomial complete intersections in characteristic p. Let $\boldsymbol{k}$ be an infinite field and A be a standard graded Artinian $\boldsymbol{k}$-algebra. The ring A has the WLP if, for every general linear form ℓ, multiplication by ℓ from $[A]_i$ to $[A]_{i+1}$ is a map of maximal rank for all i. (That is, each of these maps is either injective or surjective.) Stanley [33] used the hard Lefschetz Theorem from Algebraic Geometry to show that, if $\boldsymbol{k}$ is the field of complex numbers, then every monomial complete intersection has the WLP. Alternate proofs of Stanley's Theorem, requiring only that $\boldsymbol{k}$ have characteristic zero, and using techniques from other branches of mathematics, may be found in [30] and [19]. The story is much different in positive characteristic. Let $A(\boldsymbol{k}, m, n) = \boldsymbol{k}[x_1, \dots, x_m]/(x_1^n, \dots, x_m^n)$, where $\boldsymbol{k}$ is an infinite field of positive characteristic p. If $m = 3$, then Brenner and Kaid [6] have identified all n, as a function of p, for which A has the WLP. For a given prime p, there are intervals of n for which $A(\boldsymbol{k}, 3, n)$ has the WLP and the position of these intervals is related to the Hilbert–Kunz multiplicity of the Fermat ring $\boldsymbol{k}[x, y, z]/(x^n + y^n + z^n)$ as studied by Han [18] and Monsky [29]. If $n = 4$, then it is shown in [24] that $A(\boldsymbol{k}, 4, n)$ rarely has the WLP and only for discrete values of n. Furthermore, the following statement holds.

Observation 0.15. *Let $\boldsymbol{k}$ be an infinite field and, for each pair of positive integers (a, b), let $J_{a,b}$ be the ideal $(x^a, y^a, z^a) : (x + y + z)^b$ of $\boldsymbol{k}[x, y, z]$. If $J_{n,n-1} \in \mathbb{I}^{[3]}_{n}(\boldsymbol{k})$ or $J_{n,n+1} \in \mathbb{I}^{[3]}_{n-1}(\boldsymbol{k})$, then $A(\boldsymbol{k}, 4, n)$ has the* WLP.

For a complete, up-to-date, history of the WLP see [28].

Also, there is much recent work concerning the equations that define the Rees algebra of ideals which are primary to the maximal ideal; see, for example, [9,13,14,21]. The driving force behind this work is the desire to understand the singularities of parameterized curves or surfaces; see [5,10–12,31] and especially [14]. One of the key steps in [14] is the decomposition of the space of balanced Hilbert–Burch matrices into disjoint orbits under the action of $\mathrm{GL}_3\,\boldsymbol{k} \times \mathrm{GL}_2\,\boldsymbol{k}$. The present paper includes a preliminary step toward obtaining a comparable decomposition of the space of syzygy matrices for the set of linearly presented grade three Gorenstein ideals.

1. Terminology, notation, and preliminary results

This section contains preliminary material. It is divided into four subsections: Miscellaneous information, Pfaffian conventions, Divided power structures, and Macaulay inverse systems.

1.1. Miscellaneous information

In this paper, $\boldsymbol{k}$ is always a field. Unless otherwise noted, the polynomial ring $\boldsymbol{k}[x_1,\dots,x_d]$ is assumed to be a standard graded $\boldsymbol{k}$-algebra; that is, each variable has degree one. For each graded module M we use $[M]_i$ to denote the *homogeneous component of M of degree i*.

If α is a real number then $\lfloor\alpha\rfloor$ is the *round down* of α; that is $\lfloor\alpha\rfloor$ is equal to the integer n with $n \le \alpha < n+1$. We use $\mathbb{Z}$ and $\mathbb{Q}$ to represent the ring of integers and the field of rational numbers, respectively.

Recall that if A, B, and C are R-modules, then the R-module homomorphism $F : A \otimes_R B \to C$ is a *perfect pairing* if the induced R-module homomorphisms $A \to \mathrm{Hom}_R(B,C)$ and $B \to \mathrm{Hom}_R(A,C)$, which are given by $a \mapsto F(a \otimes \_)$ and $b \mapsto F(\_ \otimes b)$, are isomorphisms.

If V is a free R-module, then

$$\Delta : \textstyle\bigwedge_R^\bullet V \to \bigwedge_R^\bullet V \otimes_R \bigwedge_R^\bullet V \tag{1.1}$$

is the usual co-multiplication map in the exterior algebra. In particular, the component $\Delta : \bigwedge_R^a V \to \bigwedge_R^1 V \otimes_R \bigwedge_R^{a-1} V$ of (1.1) sends

$$v_1 \wedge \cdots \wedge v_a \quad \text{to} \quad \sum_{i=1}^{a} (-1)^{i+1} v_i \otimes v_1 \wedge \dots \widehat{v_i} \cdots \wedge v_a,$$

for $v_1,\dots,v_a \in V$.

Let $\boldsymbol{k}$ be a field and $A = \bigoplus_{0\le i}[A]_i$ be a graded Artinian $\boldsymbol{k}$-algebra, with $[A]_0 = \boldsymbol{k}$ and maximal ideal $\mathfrak{m}_A = \bigoplus_{1\le i}[A]_i$. The *socle* of A is the $\boldsymbol{k}$-vector space

$$0 :_A \mathfrak{m}_A = \{a \in A \mid a\mathfrak{m}_A = 0\}.$$

The Artinian ring A is *Gorenstein* if the socle of A has dimension one. In this case, the degree of a generator of the socle of A is called the *socle degree* of A. If A is a graded Artinian Gorenstein $\boldsymbol{k}$-algebra with socle degree s, then the multiplication map

$$[A]_i \otimes_{\boldsymbol{k}} [A]_{s-i} \to [A]_s$$

is a perfect pairing for $0 \le i \le s$. The homogeneous ideal I of the polynomial ring $P = \boldsymbol{k}[x_1, \ldots, x_d]$ is a *grade d Gorenstein ideal* if P/I is an Artinian Gorenstein $\boldsymbol{k}$-algebra.

Let R be an arbitrary commutative Noetherian ring. The *grade* of a proper ideal I in R is the length of a maximal R-sequence contained in I. The ideal I is called *perfect* if the grade of I is equal to the projective dimension of R/I as an R-module. (The inequality $\operatorname{grade} I \le \operatorname{proj.\,dim.}_R R/I$ always holds.) An ideal I of grade g is a *Gorenstein ideal* if I is perfect and $\operatorname{Ext}^g_R(R/I, R)$ is a cyclic R/I-module.

If $\mathbb{D}$ is a double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{2,1} & \longrightarrow & D_{1,1} & \longrightarrow & D_{0,1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{2,0} & \longrightarrow & D_{1,0} & \longrightarrow & D_{0,0},
\end{array}$$

then $\mathbb{T}(\mathbb{D})$ is the *total complex* of $\mathbb{D}$. The modules of $\mathbb{T}(\mathbb{D})$ are

$$\cdots \to \mathbb{T}(\mathbb{D})_i \to \mathbb{T}(\mathbb{D})_{i-1} \to \cdots,$$

with $\mathbb{T}(\mathbb{D})_i = \bigoplus_{a+b=i} D_{a,b}$. The maps of $\mathbb{T}(\mathbb{D})$ are the maps of $\mathbb{D}$ with the signs adjusted.

If S is a statement, then we define

$$\chi(S) = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false.} \end{cases} \tag{1.2}$$

1.2. Pfaffian conventions

An *alternating* matrix is a square skew-symmetric matrix whose entries on the main diagonal are zero. Fix an alternating $n \times n$ matrix Z, with entries $(z_{i,j})$. The Pfaffian of Z, denoted $\operatorname{Pf}(Z)$, is a square root of a determinant of Z. We recall that $\operatorname{Pf}(Z) = 0$ if n is odd, and $\operatorname{Pf}(Z) = z_{1,2}$ if $n = 2$. If $a_1, \ldots, a_s$ are integers between 1 and n, then we let $\sigma(a_1, \ldots, a_s)$ be zero if there is some repetition among the indices $a_1, \ldots, a_s$, and if the indices $a_1, \ldots, a_s$ are distinct, then $\sigma(a_1, \ldots, a_s)$ is the sign of the permutation that puts the indices into ascending order. Let $Z_{a_1,\ldots,a_s}$ denote $\sigma(a_1, \ldots, a_s)$ times the Pfaffian of the matrix obtained by deleting rows and columns $a_1, \ldots, a_s$ from Z. We recall that $\operatorname{Pf}(Z)$ may be computed along any row or down any column; that is,

$$\operatorname{Pf}(Z) = \begin{cases} \sum_j (-1)^{i+j+1} z_{i,j} Z_{i,j}, & \text{with } i \text{ fixed} \\ \sum_i (-1)^{i+j+1} z_{i,j} Z_{i,j}, & \text{with } j \text{ fixed.} \end{cases}$$

1.3. Divided power structures

Let V be a free module of finite rank over the commutative Noetherian ring R. Form the polynomial ring

$$\operatorname{Sym}_\bullet^R(V)=\bigoplus_{0\le r}\operatorname{Sym}_r^R(V)$$

and the divided power R-algebra

$$D_\bullet^R(V^*)=\bigoplus_{0\le r}D_r^R(V^*),$$

where the functor * is equal to $\operatorname{Hom}_R(\_,R)$. Each graded component $\operatorname{Sym}_r^R(V)$ of $\operatorname{Sym}_\bullet^R(V)$ and $D_r^R(V^*)$ of $D_\bullet^R(V^*)$ is a free module of finite rank over R. The rules for a divided power algebra are recorded in [17, Section 7] or [15, Appendix 2]. (In practice these rules say that $w^{(a)}$ behaves like $w^a/(a!)$ would behave if $a!$ were a unit in R.) One makes $D_\bullet^R(V^*)$ become a $\operatorname{Sym}_\bullet^R V$-module by decreeing that each element v_1 of V acts like a divided power derivation on $D_\bullet^R(V^*)$. That is, $v_1(w_1^{(a)})=v_1(w_1)\cdot w_1^{(a-1)}$, v_1 of a product follows the product rule from calculus, and, once one knows how V acts on $D_\bullet^R(V^*)$, then one knows how all of $\operatorname{Sym}_\bullet^R V$ acts on $D_\bullet^R(V^*)$.

In a similar manner one makes $\operatorname{Sym}_\bullet^R V$ become a module over the divided power R-algebra $D_\bullet^R(V^*)$. If w_1 is in $D_1^R(V^*)$, then w_1 is a derivation on $\operatorname{Sym}_\bullet^R V$, and $w_1^{(a)}$ acts on $\operatorname{Sym}_\bullet^R V$ exactly like $\frac{1}{a!}w_1^a$ would act if $a!$ were a unit in R. The R-algebra $D_\bullet^R(V^*)$ is generated by elements of the form $w_1^{(a)}$, with $w_1\in V^*$; once one knows how these elements act on $\operatorname{Sym}_\bullet^R V$, then one knows how every element of $D_\bullet^R(V^*)$ acts on $\operatorname{Sym}_\bullet^R V$.

The above actions are defined in a coordinate free manner. It makes sense to see what happens when one picks bases. Let $x_1,\dots,x_d$ be a basis for V and $x_1^*,\dots,x_d^*$ be the corresponding dual basis for V^*. (We have $x_i^*(x_j)$ is equal to the Kronecker delta $\delta_{i,j}$.) The above rules show that

$$x_1^{a_1}\cdots x_d^{a_d}\bigl(x_1^{*(b_1)}\cdots x_d^{*(b_d)}\bigr)=x_1^{*(b_1-a_1)}\cdots x_d^{*(b_d-a_d)}\in D^R_{\sum b_i-\sum a_i}\bigl(V^*\bigr)$$

and

$$x_1^{*(b_1)}\cdots x_d^{*(b_d)}\bigl(x_1^{a_1}\cdots x_d^{a_d}\bigr)=\binom{a_1}{b_1}\cdots\binom{a_d}{b_d}x_1^{a_1-b_1}\cdots x_d^{a_d-b_d}\in \operatorname{Sym}^R_{\sum a_i-\sum b_i}(V),$$

where $x_i^r=0$ and $x_i^{*(r)}=0$ whenever $r<0$. Notice in particular, that if $\sum a_i=\sum b_i$, then

$$x_1^{a_1}\cdots x_d^{a_d}\bigl(x_1^{*(b_1)}\cdots x_d^{*(b_d)}\bigr)\quad\text{and}\quad x_1^{*(b_1)}\cdots x_d^{*(b_d)}\bigl(x_1^{a_1}\cdots x_d^{a_d}\bigr)$$

are both equal to

$$\begin{cases} 1 & \text{if } a_j = b_j \text{ for all } j \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\operatorname{Sym}_a^R V$ and $D_a^R(V^*)$ are **naturally** dual to one another.

We make use of the dual of the natural evaluation map

$$\operatorname{ev}_j : D_j^R\big(V^*\big) \otimes_R \operatorname{Sym}_j^R V \to R.$$

Indeed, $\operatorname{ev}_j^*(1)$ is a well-defined coordinate-free element of $\operatorname{Sym}_j^R(V) \otimes_R D_j^R(V^*)$. If $\{A_i\}$ and $\{A_i^*\}$ are a pair of dual bases for the free modules $\operatorname{Sym}_j^R V$ and $D_j^R(V^*)$, then

$$\operatorname{ev}_j^*(1) = \sum_i A_i \otimes A_i^* \in \operatorname{Sym}_j^R(V) \otimes_R D_j^R\big(V^*\big). \tag{1.3}$$

1.4. Macaulay inverse systems

Theorem 1.4. *(See Macaulay, [26].) Let U be a vector space of dimension d over the field $\boldsymbol{k}$. Then there exists a one-to-one correspondence between the set of non-zero homogeneous grade d Gorenstein ideals of $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$ and the set of non-zero homogeneous cyclic submodules of $D_\bullet^{\boldsymbol{k}}(U^*)$:*

$$\left\{ \begin{matrix} \textit{homogeneous grade } d \textit{ Gorenstein} \\ \textit{ideals of } \operatorname{Sym}_\bullet^{\boldsymbol{k}} U \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{non-zero homogeneous cyclic} \\ \textit{submodules of } D_\bullet^{\boldsymbol{k}}(U^*) \end{matrix} \right\}.$$

If I is an ideal from the set on the left, then the corresponding submodule of $D_\bullet^{\boldsymbol{k}}(U^*)$ is

$$\operatorname{ann} I = \big\{ w \in D_\bullet^{\boldsymbol{k}}\big(U^*\big) \bigm| rw = 0 \textit{ in } D_\bullet^{\boldsymbol{k}}\big(U^*\big), \textit{ for all } r \textit{ in } I \big\},$$

and if M is a homogeneous cyclic submodule from the set on the right, then the corresponding ideal is

$$\operatorname{ann}(M) = \big\{ r \in \operatorname{Sym}_\bullet^{\boldsymbol{k}} U \bigm| rM = 0 \textit{ in } D_\bullet^{\boldsymbol{k}}\big(U^*\big) \big\}.$$

Furthermore, the socle degree of $\frac{\operatorname{Sym}_\bullet^{\boldsymbol{k}} U}{I}$ is equal to the degree of a homogeneous generator of $\operatorname{ann} I$.

Definition. In the language of Theorem 1.4, the homogeneous cyclic submodule $\operatorname{ann}(I)$ of $D_\bullet^{\boldsymbol{k}}(U^*)$ is the *Macaulay inverse system* of the grade d Gorenstein ideal I of $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$.

The outline of a Proof of Theorem 1.4. Let P denote $\operatorname{Sym}_\bullet^{\boldsymbol{k}} U$ and * denote $\boldsymbol{k}$-dual. Let I be an ideal from the set on the left and let s be the socle degree of P/I. Fix an isomorphism $[P/I]_s \xrightarrow{\simeq} \boldsymbol{k}$. Define $w \in D_s^{\boldsymbol{k}}(U^*) = \operatorname{Hom}_{\boldsymbol{k}}(\operatorname{Sym}_s^{\boldsymbol{k}}(U), \boldsymbol{k})$ to be the homomorphism $w : \operatorname{Sym}_s^{\boldsymbol{k}}(U) \to \boldsymbol{k}$ which is the composition

$$\operatorname{Sym}_s^{\boldsymbol{k}}(U) = P_s \to [P/I]_s \xrightarrow{\simeq} \boldsymbol{k}.$$

The definition of w shows that $[P]_s \cap \operatorname{ann} w = [I]_s$. The ring P/I is Gorenstein so multiplication $[P/I]_i \otimes_{\boldsymbol{k}} [P/I]_{s-i} \to [P/I]_s$ is a perfect pairing for all i. It follows that $\operatorname{ann}(w) = I$.

Now let w be a non-zero element of $D_s^{\boldsymbol{k}}(U^*)$ and let $I = \operatorname{ann}(w)$. The homomorphism $w : \operatorname{Sym}_s^{\boldsymbol{k}} U \to \boldsymbol{k}$ is non-zero; so the vector space $[P/I]_s$ has dimension 1 and this vector space is contained in the socle of P/I. On the other hand, if r is a homogeneous element of P with $r \notin I$ and $\deg r < s$, then the hypothesis $rw \neq 0$ guarantees that there is a homogeneous element r' of P with $r'r \in P_s$ and $r'rw \neq 0$. In particular, r is not in the socle of P/I. Thus, the socle of P/I has dimension one and P/I is Gorenstein. □

Let I be a fixed homogeneous grade d Gorenstein ideal in

$$P = \boldsymbol{k}[x_1, \dots, x_d] = \operatorname{Sym}_\bullet^{\boldsymbol{k}} U,$$

where U is the d-dimensional vector space $\bigoplus_{i=1}^d \boldsymbol{k} x_i$, and let $\phi \in D_\bullet^{\boldsymbol{k}}(U^*)$ be a homogeneous generator for the Macaulay inverse system of I. Proposition 1.8 gives many ways to test if I is in $\mathbb{I}_n^{[d]}(\boldsymbol{k})$. We are particularly interested in tests that involve ϕ. First of all, Proposition 1.8 shows that ϕ must be in $D_{2n-2}^{\boldsymbol{k}}(U^*)$. The other condition that ϕ must satisfy involves a map $\boldsymbol{p}_{n-1}^{\phi}$ or a matrix T_ϕ. These notions are defined whenever ϕ is a homogeneous element of $D_\bullet^R(V^*)$ of even degree, R is a commutative Noetherian ring, and V is a free R-module of finite rank.

Definition 1.5. Let R be a commutative Noetherian ring, V be a free R-module of finite rank d, and ϕ be an element of $D_{2e}^R(V^*)$, for some positive integer e.

(1) For each integer i, with $0 \le i \le 2e$, define the homomorphism

$$\boldsymbol{p}_i^{\phi} : \operatorname{Sym}_i^R V \to D_{2e-i}^R(V^*)$$

by $\boldsymbol{p}_i^{\phi}(v_i) = v_i(\phi)$, for all $v_i \in \operatorname{Sym}_i^R V$.

(2) Fix a basis $x_1, \dots, x_d$ for V. Let $N = \binom{e+d-1}{e}$ and $m_1, \dots, m_N$ be a list of the monomials of degree e in $\operatorname{Sym}_e^R V$. Define T_ϕ to be the $N \times N$ matrix $T_\phi = [\phi(m_i m_j)]$; that is, the entry of T_ϕ in row i and column j is the element $\phi(m_i m_j)$ of R.

Remarks 1.6. (1) The matrix T_ϕ is the matrix for $\boldsymbol{p}_e^{\phi}$ with respect to the basis $m_1, \dots, m_N$ for $\operatorname{Sym}_e^R V$ and the dual basis $m_1^*, \dots, m_N^*$ for $D_e^R(V^*)$.

(2) The entries of T_ϕ are the coefficients of ϕ as an element of $D_{2e}^R(V^*)$. Let ν equal $\binom{2e+d-1}{2e}$. If $M_1, \dots, M_\nu$ is a list of the monomials of degree $2e$ in $\operatorname{Sym}_{2e}^R V$

and $M_1^*, \dots, M_\nu^*$ is the basis for $D_{2n-2}^R(V^*)$ which is dual to $M_1, \dots, M_\nu$, then $\phi = \sum_i \phi(M_i) \cdot M_i^*$. Of course, each entry of T_ϕ is a coefficient of ϕ because each $m_i m_j$ is equal to some M_k.

(3) The element $\det T_\phi$ of R is known as the "determinant of the symmetric bilinear form"

$$\operatorname{Sym}_e^R V \times \operatorname{Sym}_e^R V \to R,$$

which sends (v_e, v_e') to $\phi(v_e v_e')$. A change of basis for V changes $\det T_\phi$ by a unit of R. However, in practice, we only use $\det T_\phi$, up to unit. In particular, the phrases "provided $\det T_\phi$ is a unit of R" and "R localized at the element $\det T_\phi$" are meaningful even in a coordinate-free context.

Example 1.7. Let BE_2 be the ideal of $\boldsymbol{k}[x, y, z]$ which is generated by the maximal order Pfaffians of the matrix H_2 from (0.9). We have $\mathrm{BE}_2 = (x^2, y^2, xz, yz, xy + z^2)$ and the Macaulay inverse system for BE_2 is generated by $\phi = x^* y^* - z^{*(2)}$. If $m_1 = x$, $m_2 = y$, $m_3 = z$, then

$$T_\phi = \begin{bmatrix} \phi(x^2) & \phi(xy) & \phi(xz) \\ \phi(yx) & \phi(y^2) & \phi(yz) \\ \phi(zx) & \phi(zy) & \phi(z^2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Proposition 1.8. *Let $\boldsymbol{k}$ be a field, I be a homogeneous, grade d, Gorenstein ideal in $P = \boldsymbol{k}[x_1, \dots, x_d]$, U be the d-dimensional vector space $[P]_1$, and $\phi \in D_\bullet^{\boldsymbol{k}}(U^*)$ be a homogeneous generator for the Macaulay inverse system of I. Then the following statements are equivalent*:

(1) *I is in $\mathbb{I}_n^{[d]}(\boldsymbol{k})$,*

(2) *the minimal homogeneous resolution of P/I by free P-modules has the form*

$$0 \to P(-2n-d+2) \to P(-n-d+2)^{\beta_{d-1}} \to \cdots \to P(-n-1)^{\beta_2} \to P(-n)^{\beta_1} \to P,$$

with

$$\beta_i = \frac{2n+d-2}{n+i-1} \binom{n+d-2}{i-1} \binom{n+d-i-2}{n-1},$$

for $1 \le i \le d-1$,

(3) *all of the minimal generators of I have degree n and the socle of P/I has degree $2n-2$,*

(4) *$[I]_{n-1} = 0$ and $[P/I]_{2n-1} = 0$,*

(5) *$\phi \in D_{2n-2}^{\boldsymbol{k}}(U^*)$ and the homomorphism $\boldsymbol{p}_{n-1}^\phi : \operatorname{Sym}_{n-1}^{\boldsymbol{k}} U \to D_{n-1}^{\boldsymbol{k}}(U^*)$ of Definition 1.5 is an isomorphism, and*

(6) *$\phi \in D_{2n-2}^{\boldsymbol{k}}(U^*)$ and the $\binom{d+n-2}{d-1} \times \binom{d+n-2}{d-1}$ matrix T_ϕ of Definition 1.5 is invertible.*

Remark. The Betti numbers β_i, from (2), also are equal to

$$\beta_i = \binom{d+n-1}{n+i-1}\binom{i+n-2}{i-1} - \binom{d+n-2}{i-1}\binom{d-i+n-2}{d-i},$$

for $1 \le i \le d-1$. The present formulation is given in Theorem 6.15. A quick calculation shows that the two formulations are equal.

Proof of Proposition 1.8. (1) ⇒ (2). The fact that P/I has a linear resolution is part of the definition of $\mathbb{I}_n^{[d]}(\boldsymbol{k})$. The Betti number β_i comes from the Herzog–Kühl formula [20].

(2) ⇒ (1). One can read from (2) that I is generated by forms of degree n and that P/I has a linear resolution. Thus, I is in $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, as defined in (0.1).

(2) ⇒ (3). One can read from (2) that I is generated by forms of degree n. Furthermore, the socle degree of P/I is $b + a(P)$, where $a(P)$ is the a-invariant of P and b is the “back twist” in the P-free resolution of P/I. (This observation is well-known. It amounts to computing $\operatorname{Tor}_d^P(P/I, \boldsymbol{k})$ in each component; see, for example, the proof of [23, Prop. 1.5].) In the present situation, $b = 2n + d - 2$ and $a(P) = -d$; so the socle degree of P/I is $2n - 2$.

(3) ⇒ (4). Statement (3) asserts that $[I]_i = 0$ for $i \le n-1$ and $[P/I]_i = 0$ for $2n-1 \le i$.

(4) ⇒ (2). Let $\mathbb{A}$: $0 \to A_d \to \cdots \to A_2 \to A_1 \to P \to P/I \to 0$ be a minimal homogeneous resolution of P/I by free P-modules. Write $A_\ell = \bigoplus_{0 \le i} P(-i)^{\beta_{\ell,i}}$. This resolution is self-dual; so, in particular, $A_d = P(-b)$ for some twist b. As above, it follows that the socle degree of P/I is equal to $b - d$. In the present situation, the socle degree of P/I is $2n - 2 - q$ for some non-negative integer q; so $b = 2n - 2 + d - q$. The hypothesis $I_{n-1} = 0$ ensures that

$$\beta_{1,i} = 0, \quad \text{whenever } i \le n-1. \tag{1.9}$$

Duality forces

$$\beta_{d-1,j} = 0 \quad \text{for } n-1+d-q \le j. \tag{1.10}$$

The fact that the resolution $\mathbb{A}$ is minimal ensures that

$$\text{if } \beta_{\ell,j} \ne 0 \text{ for some } 1 \le \ell, \text{ then there exists } i \le j-1 \text{ with } \beta_{\ell-1,i} \ne 0. \tag{1.11}$$

One may iterate the idea of (1.11) to see that

$$\text{if } \beta_{d-1,j} \ne 0, \text{ then there exists } i \le j+2-d \text{ with } \beta_{1,i} \ne 0. \tag{1.12}$$

Suppose $\beta_{d-1,j} \ne 0$. On the one hand, $j \le n-2+d-q$ by (1.10) and on the other hand, according to (1.12), there exits i with $i \le j+2-d$ and $\beta_{1,i} \ne 0$. Apply (1.9) to see that $n \le i$. Thus,

$$n \le i \le j+2-d \le n-q \le n;$$

hence, $q=0$, $i=n$, and $j=n+d-2$. At this point, we have shown that

$$A_d = P\bigl(-(2n-2+d)\bigr), \qquad A_{d-1} = P\bigl(-(n+d-2)\bigr)^{\beta_{d-1,n+d-2}}, \quad \text{and}$$
$$A_1 = P(-n)^{\beta_{1,n}},$$

with $\beta_{d-1,n+d-2} = \beta_{1,n}$.

We study $\beta_{\ell,j}$ for $2 \le \ell \le d-2$. Apply (1.11) to $\mathbb{A}$ repeatedly to see that $\beta_{\ell,j}=0$ for $j \le n+\ell-2$. The ideal I is Gorenstein and $\mathbb{A}$ is a minimal resolution of P/I; therefore, the complex $\mathbb{A}^*(-2n-d+2)$ is isomorphic to $\mathbb{A}$,

$$\beta_{a,b} = \beta_{a',b'} \quad \text{whenever } a+a'=d \text{ and } b+b'=2n+d-2,$$

and $\beta_{\ell,j}$ is non-zero only for $j=n+\ell-1$. One computes the Betti numbers of $\mathbb{A}$ by using the Herzog–Kühl formula [20] for the Betti numbers in a pure resolution.

(3) ⇒ (5). The ring P/I is graded and Gorenstein with socle degree $2n-2$; hence, multiplication gives a perfect pairing $[P/I]_{n-1} \otimes_{\boldsymbol{k}} [P/I]_{n-1} \to [P/I]_{2n-2}$. Statement (3) gives $[I]_{n-1}=0$; hence, $[P/I]_{n-1}$ is equal to $[P]_{n-1} = \operatorname{Sym}^{\boldsymbol{k}}_{n-1} U$. Statement (3) also gives that the socle degree of P/I is $2n-2$; thus, Theorem 1.4 yields that the function $[P/I]_{2n-2} \to \boldsymbol{k}$, which sends the class of $\overline{u_{2n-2}}$ in $[P/I]_{2n-2}$ to $\phi(u_{2n-2})$ in $\boldsymbol{k}$, is an isomorphism. (We viewed u_{2n-2} as an element of $\operatorname{Sym}^{\boldsymbol{k}}_{2n-2} U$ and $\overline{u_{2n-2}}$ as the image of u_{2n-2} in $[P/I]_{2n-2}$.) Thus, statement (3) ensures that the homomorphism

$$\operatorname{Sym}^{\boldsymbol{k}}_{n-1} U \otimes_{\boldsymbol{k}} \operatorname{Sym}^{\boldsymbol{k}}_{n-1} U \to \boldsymbol{k},$$

which is given by $u_{n-1} \otimes u'_{n-1} \mapsto \phi(u_{n-1}u'_{n-1})$, is a perfect pairing. It follows that the homomorphism $\operatorname{Sym}^{\boldsymbol{k}}_{n-1} U \to D^{\boldsymbol{k}}_{n-1}(U^*)$, which is given by $u_{n-1} \mapsto u_{n-1}(\phi)$, is an isomorphism.

(5) ⇒ (4). Apply Theorem 1.4 to $\phi \in D^{\boldsymbol{k}}_{2n-2}(U^*)$ to see that the socle degree of P/I is $2n-2$; and therefore, $[P/I]_{2n-1}=0$. If $v_{n-1} \in \operatorname{Sym}^{\boldsymbol{k}}_{n-1} U$ is in I, then $v_{n-1}(\phi)$ is the zero element of $D^{\boldsymbol{k}}_{n-1}(U^*)$; thus statement (5) guarantees that v_{n-1} is zero in $\operatorname{Sym}^{\boldsymbol{k}}_{n-1} U$ and $[I]_{n-1}=0$.

(5) ⇔ (6). It is clear that these statements are equivalent because T_ϕ is the matrix for $\boldsymbol{p}^{\phi}_{n-1}$; see Remark 1.6(1). □

The following statement is an immediate consequence of Proposition 1.8; no further proof is necessary.

Corollary 1.13. *Let U be a d-dimensional vector space over the field $\boldsymbol{k}$ and n be a positive integer. If ϕ is a homogeneous element of $D^{\boldsymbol{k}}_{\bullet}(U^*)$, then* $\operatorname{ann}\phi \in \mathbb{I}^{[d]}_n(\boldsymbol{k})$ *if and only if* $\deg\phi = 2n-2$ *and* $\det T_\phi \neq 0$. *In particular, the open subset*

$$O = D^{\boldsymbol{k}}_{2n-2}\big(U^*\big) \setminus \{\textit{the variety defined by } \det T_\phi = 0\}$$

of $D^{\boldsymbol{k}}_{2n-2}(U^*)$ *parameterizes* $\mathbb{I}^{[d]}_n(\boldsymbol{k})$.

Remark 1.14. We emphasize that, in the language of Corollary 1.13, $D^{\boldsymbol{k}}_{2n-2}(U^*)$ is a vector space of dimension $\nu = \binom{2n+d-3}{d-1}$ and, once a basis $\{b_i\}$ is chosen for this vector space, then $D^{\boldsymbol{k}}_{2n+d-3}(U^*)$ can be identified with affine ν-space: the point $(a_1, \dots, a_\nu)$ in affine space corresponds to the element $\sum a_i b_i$ of $D^{\boldsymbol{k}}_{2n-2}(U^*)$. Under this identification, $\det T_\phi$ corresponds to a homogeneous polynomial of degree $\binom{n+d-2}{d-1}$ in the coordinates of affine ν-space; see Remark 1.6(2). Thus, Corollary 1.13 parameterizes $\mathbb{I}^{[d]}_n(\boldsymbol{k})$ using an open subset of ν-space. The open subset is the complement of a hypersurface.

One further consequence of Proposition 1.8 is the following generalization of the set of ideals $\mathbb{I}^{[d]}_n(\boldsymbol{k})$.

Definition 1.15. Let R_0 be a commutative Noetherian ring, U be a non-zero free R_0-module of finite rank and n be a positive integer. Define $\mathbb{I}_n(R_0, U)$ to be the following set of ideals in $P = \operatorname{Sym}^{R_0}_\bullet U$:

$$\mathbb{I}_n(R_0, U) = \big\{\operatorname{ann}\phi \bigm| \phi \in D^{R_0}_{2n-2}\big(U^*\big) \text{ and } \det T_\phi \text{ is a unit in } R_0\big\}.$$

We see from Proposition 1.8 that if U is a vector space of dimension d over a field $\boldsymbol{k}$, then the sets of ideals $\mathbb{I}_n(R_0, U)$ and $\mathbb{I}^{[d]}_n(\boldsymbol{k})$ are equal. Corollary 4.22, which is our solution to Project 0.3, is phrased in terms of $\mathbb{I}_n(R_0, U)$.

2. The complexes $\mathbb{L}(\Psi, n)$ and $\mathbb{K}(\Psi, n)$

The following data is in effect throughout this section.

Data 2.1. Let V be a non-zero free module of rank d over the commutative Noetherian ring R.

Let $\Psi : V \to R$ be an R-module homomorphism and n be a fixed positive integer. In Theorem 2.12 we describe a complexes $\mathbb{L}(\Psi, n)$ and $\mathbb{K}(\Psi, n)$, dual to one another, so that $\mathbb{L}(\Psi, n)$ resolves $R/(\operatorname{im}\Psi)^n$ whenever the image of Ψ has grade d. Our description of these complexes is very explicit and coordinate free. We use our explicit descriptions in the proof of Theorem 4.7. Of course, the complexes of Buchsbaum and Eisenbud in [7] resolve $R/(\operatorname{im}\Psi)^n$; so, in some sense, our complex $\mathbb{L}(\Psi, n)$ "is in" [7]. Our proof of the exactness of $\mathbb{L}(\Psi, n)$ and $\mathbb{K}(\Psi, n)$ is self-contained, very explicit, and, as far as we can tell, different from the proof found in [7].

Retain Data 2.1 and let a and b be integers. Define the R-module homomorphisms

$$\begin{aligned}&\kappa^V_{a,b}: \textstyle\bigwedge^a_R V \otimes_R \operatorname{Sym}^R_b V \to \bigwedge^{a-1}_R V \otimes_R \operatorname{Sym}^R_{b+1} V \quad \text{and}\\ &\eta^V_{a,b}: \textstyle\bigwedge^a_R V \otimes_R D^R_b(V^*) \to \bigwedge^{a-1}_R V \otimes_R D^R_{b-1}(V^*)\end{aligned} \tag{2.2}$$

to be the compositions

$$\begin{aligned}&\textstyle\bigwedge^a_R V \otimes_R \operatorname{Sym}^R_b V \xrightarrow{\Delta\otimes 1} \bigwedge^{a-1}_R V \otimes_R V \otimes_R \operatorname{Sym}^R_b V\\ &\xrightarrow{1\otimes\text{mult}} \textstyle\bigwedge^{a-1}_R V \otimes_R \operatorname{Sym}^R_{b+1} V \quad \text{and}\\ &\textstyle\bigwedge^a_R V \otimes_R D^R_b(V^*) \xrightarrow{\Delta\otimes 1} \bigwedge^{a-1}_R V \otimes_R V \otimes_R D^R_b(V^*)\\ &\xrightarrow{1\otimes\text{module-action}} \textstyle\bigwedge^{a-1}_R V \otimes_R D^R_{b-1}(V^*),\end{aligned}$$

respectively; and define the R-modules

$$L^R_{a,b}(V) = \ker \kappa^V_{a,b} \quad \text{and} \quad K^R_{a,b}(V) = \ker \eta^V_{a,b}.$$

(In the future, we will often write κ and η in place of $\kappa^V_{a,b}$ and $\eta^V_{a,b}$.) The R-modules $L^R_{a,b}(V)$ and $K^R_{a,b}(V)$ have been used by many authors in many contexts. In particular, they are studied extensively in [7]; although our indexing conventions are different than the conventions of [7]; that is,

the module we call $L^R_{a,b}(V)$ is called ${L_b}^{a+1}(V)$ in [7].

The complex

$$\begin{aligned}&0 \to \textstyle\bigwedge^d_R V \otimes_R \operatorname{Sym}^R_{b-d} V \xrightarrow{\kappa^V_{d,b-d}} \bigwedge^{d-1}_R V \otimes_R \operatorname{Sym}^R_{b-d+1} V\\ &\xrightarrow{\kappa^V_{d-1,b-d+1}} \cdots \xrightarrow{\kappa^V_{2,b-2}} \textstyle\bigwedge^1_R V \otimes_R \operatorname{Sym}^R_{b-1} V \xrightarrow{\kappa^V_{1,b-1}} \bigwedge^0_R V \otimes_R \operatorname{Sym}^R_b V \to 0,\end{aligned}$$

which is a homogeneous strand of an acyclic Koszul complex, is split exact for all positive integers b; hence, $L^R_{a,b}(V)$ is a projective R-module. In fact, $L^R_{a,b}(V)$ is a free R-module of rank

$$\operatorname{rank}_R L^R_{a,b}(V) = \binom{d+b-1}{a+b}\binom{a+b-1}{a}; \tag{2.3}$$

see [7, Prop. 2.5]. The perfect pairing

$$\left(\textstyle\bigwedge^a_R V \otimes_R \operatorname{Sym}^R_b V\right) \otimes_R \left(\bigwedge^{d-a}_R V \otimes_R D^R_b(V^*)\right) \to \bigwedge^d_R V,$$

which is given by

$$(\theta_a \otimes v_b) \otimes (\theta_{d-a} \otimes w_b) \mapsto v_b(w_b) \cdot \theta_a \wedge \theta_{d-a},$$

induces a perfect pairing

$$L^R_{a,b}(V) \otimes K^R_{d-a-1,b-1}(V) \to \textstyle\bigwedge^d_R(V). \tag{2.4}$$

The indices in (2.4) are correct because the dual of the presentation

$$\textstyle\bigwedge^{a+2}_R V \otimes_R \operatorname{Sym}^R_{b-2} V \xrightarrow{\kappa^V_{a+2,b-2}} \bigwedge^{a+1}_R V \otimes_R \operatorname{Sym}^R_{b-1} V \xrightarrow{\kappa^V_{a+1,b-1}} L^R_{a,b}(V) \to 0$$

is

$$0 \to L^R_{a,b}(V)^* \to \Big(\textstyle\bigwedge^{a+1}_R V \otimes_R \operatorname{Sym}^R_{b-1} V\Big)^* \to \Big(\bigwedge^{a+2}_R V \otimes_R \operatorname{Sym}^R_{b-2} V\Big)^*,$$

which is isomorphic to

$$\begin{aligned} 0 \to K^R_{d-a-1,b-1}(V) \otimes_R \textstyle\bigwedge^d_R V^* &\to \Big(\textstyle\bigwedge^{d-a-1}_R V \otimes_R D^R_{b-1}(V^*)\Big) \otimes_R \bigwedge^d_R V^* \\ &\to \Big(\textstyle\bigwedge^{d-a-2}_R V \otimes_R D^R_{b-2}(V^*)\Big) \otimes_R \bigwedge^d_R V^*. \end{aligned}$$

Remark 2.5. The modules $L^R_{a,b}(V)$ and $K^R_{a,b}(V)$ may also be thought of as the Schur modules $L_\lambda(V)$ and Weyl modules $K_\lambda(V^*)$ which correspond to certain hooks λ. We use the notation of Examples 2.1.3.h and 2.1.17.h in Weyman [34] to see that the module we call $L^R_{a,b}(V)$ is also the Schur module $L_{(a+1,1^{b-1})}(V)$ and the module we call $K^R_{a,b}(V)$ is also $K^R_{(b+1,1^{d-a-1})}(V^*) \otimes \bigwedge^d V$, where $K^R_{(b+1,1^{d-a-1})}(V^*)$ is a Weyl module. We pursue this line of reasoning in Section 5.

Definition 2.6. Let V be a non-zero free module of rank d over the commutative Noetherian ring R, n be a positive integer, and $\Psi : V \to R$ be an R-module homomorphism. We define the complexes

$$\begin{aligned} &\mathbb{L}(\Psi,n) : 0 \to L^R_{d-1,n}(V) \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} L^R_{d-2,n}(V) \\ &\quad \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} \dots \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} L^R_{0,n}(V) \xrightarrow{\widehat{\Psi}} R, \quad \text{and} \\ &\mathbb{K}(\Psi,n) : 0 \to \textstyle\bigwedge^d_R V \to K^R_{d-1,n-1}(V) \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} K^R_{d-2,n-1}(V) \\ &\quad \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} \dots \xrightarrow{\operatorname{Kos}^\Psi \otimes 1} K^R_{0,n-1}(V) \end{aligned}$$

which appear in Theorems 2.12 and 4.7.

The ordinary Koszul complex associated to Ψ is

$$0 \to \textstyle\bigwedge^d_R V \xrightarrow{\operatorname{Kos}^\Psi} \bigwedge^{d-1}_R V \xrightarrow{\operatorname{Kos}^\Psi} \dots \xrightarrow{\operatorname{Kos}^\Psi} \bigwedge^2_R V \xrightarrow{\operatorname{Kos}^\Psi} \bigwedge^1_R V \xrightarrow{\operatorname{Kos}^\Psi} R, \tag{2.7}$$

where for each index i, Kos^Ψ is the composition

$$\textstyle\bigwedge^i_R V \xrightarrow{\Delta} V \otimes_R \bigwedge^{i-1}_R V \xrightarrow{\Psi \otimes 1} R \otimes_R \bigwedge^{i-1}_R V = \bigwedge^{i-1}_R V.$$

(The co-multiplication map Δ is discussed in (1.1).) The maps of (2.2) combine with the maps Kos^{Ψ} to form double complexes

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \\ \cdots & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^a V \otimes_R \mathrm{Sym}_b^R V & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-1} V \otimes_R \mathrm{Sym}_b^R V & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \cdots \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \\ \cdots & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-1} V \otimes_R \mathrm{Sym}_{b+1}^R V & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-2} V \otimes_R \mathrm{Sym}_{b+1}^R V & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \cdots \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \\ & & \vdots & & \vdots & & \end{array} \tag{2.8}$$

and

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\ \cdots & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^a V \otimes_R D_b^R(V^*) & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-1} V \otimes_R D_b^R(V^*) & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \cdots \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\ \cdots & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-1} V \otimes_R D_{b-1}^R(V^*) & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \bigwedge_R^{a-2} V \otimes_R D_{b-1}^R(V^*) & \xrightarrow{\mathrm{Kos}^{\Psi}\otimes 1} & \cdots \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\ & & \vdots & & \vdots & & \end{array} \tag{2.9}$$

Most of the complex $\mathbb{L}(\Psi, n)$ is induced by the double complex (2.8). (Keep in mind that $L_{d,n}^R(V)$, which is equal to $\ker(\kappa : \bigwedge_R^d V \otimes_R \mathrm{Sym}_n^R V \to \bigwedge_R^{d-1} V \otimes_R \mathrm{Sym}_{n+1}^R V)$, is zero.) It remains to describe the right-most map from $L_{0,n}^R(V) = \mathrm{Sym}_n^R(V)$ to R. According to the definition of symmetric algebra, the R-module homomorphism $\Psi : V \to R$ induces an R-algebra homomorphism $\mathrm{Sym}_\bullet^R(V) \to R$. We denote this algebra homomorphism by $\widehat{\Psi}$. Most of the complex $\mathbb{K}(\Psi, n)$ is induced by the double complex (2.9). It remains to describe the left most map:

$$\begin{aligned} \textstyle\bigwedge_R^d V = \bigwedge_R^d V \otimes_R R &\xrightarrow{1\otimes_R \mathrm{ev}_n^*} \textstyle\bigwedge_R^d V \otimes_R \mathrm{Sym}_n^R V \otimes_R D_n^R(V^*) \\ &\xrightarrow{1\otimes_R \widehat{\Psi}\otimes_R 1} \textstyle\bigwedge_R^d V \otimes_R R \otimes_R D_n^R(V^*) = \bigwedge_R^d V \otimes_R D_n^R(V^*) \underset{\simeq}{\xrightarrow{\eta}} K_{d-1,n-1}^R(V), \end{aligned} \tag{2.10}$$

where $\mathrm{ev}_n : D_n^R(V^*) \otimes_R \mathrm{Sym}_n^R V \to R$ is the natural evaluation map, see (1.3).

Example 2.11. Retain the notation and hypotheses of Definition 2.6. The complex $\mathbb{K}(\Psi,1)$ is equal to the ordinary Koszul complex (2.7) and the complex $\mathbb{L}(\Psi,1)$ is isomorphic to the ordinary Koszul complex (2.7) by way of the isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge_R^d V & \longrightarrow & \dots & \longrightarrow & \bigwedge_R^1 V & \xrightarrow{\mathrm{Kos}^{\Psi}} & \bigwedge_R^0 V \\ & & \simeq \downarrow \kappa & & & & \simeq \downarrow \kappa & & \simeq \downarrow = \\ 0 & \longrightarrow & L_{d-1,1}^R(V) & \longrightarrow & \dots & \longrightarrow & L_{0,1}^R(V) & \xrightarrow{\widehat{\Psi}} & R. \end{array}$$

Complexes "$\mathbb{K}(\Psi,0)$" and "$\mathbb{L}(\Psi,0)$" have not been defined. The recipe of Definition 2.6 would produce

$$0 \to \bigwedge_R^d V \to 0$$

for $\mathbb{K}(\Psi,0)$ and

$$0 \to \bigwedge_R^{d-1} V \xrightarrow{\mathrm{Kos}^{\Psi}} \dots \xrightarrow{\mathrm{Kos}^{\Psi}} \bigwedge_R^2 V \xrightarrow{\mathrm{Kos}^{\Psi}} \bigwedge_R^1 V \xrightarrow{\mathrm{Kos}^{\Psi}} \bigwedge_R^0 V \xrightarrow{1} R$$

for $\mathbb{L}(\Psi,0)$; neither of these objects is very satisfactory.

Theorem 2.12. *(See Buchsbaum–Eisenbud [7, Thm. 3.1].) Let V be a non-zero free module of rank d over the commutative Noetherian ring R, n be a positive integer, $\Psi : V \to R$ be an R-module homomorphism, and J be the image of Ψ. Let $\mathbb{L}(\Psi,n)$ and $\mathbb{K}(\Psi,n)$ be the complexes of Definition 2.6. Then, the following statements hold.*

(1) *The complexes $\mathbb{K}(\Psi,n)$ and $[\mathrm{Hom}_R(\mathbb{L}(\Psi,n),R) \otimes_R \bigwedge^d V][-d]$ are isomorphic, where "$[-d]$" describes a shift in homological degree.*
(2) *If the ideal J has grade at least d, then $\mathbb{L}(\Psi,n)$ is a resolution of R/J^n by free R-modules and $\mathbb{K}(\Psi,n)$ is a resolution of $\mathrm{Ext}_R^d(R/J^n,R)$ by free R-modules,*
(3) *If R is a graded ring and Ψ is homogeneous homomorphism with V equal to $R(-1)^d$, then $\mathbb{L}(\Psi,n)$ is the homogeneous linear complex*

$$\begin{aligned} 0 \to R(-n-d+1)^{\beta_d} \to R(-n-d+2)^{\beta_{d-1}} \to \cdots \\ \to R(-n-1)^{\beta_2} \to R(-n)^{\beta_1} \to R, \end{aligned}$$

with $\beta_i = \binom{n+d-1}{n+i-1}\binom{n+i-2}{i-1}$, for $1 \le i \le d$.

Proof. Most of the proof of assertion (1) is contained in (2.4). One can use (2.3) to prove (3); or one can prove (2) and then appeal to the Herzog–Kühl formula [20]. We focus on the proof of (2). Assume that J has grade d. We first show that $\mathbb{K}(\Psi,n)$ is

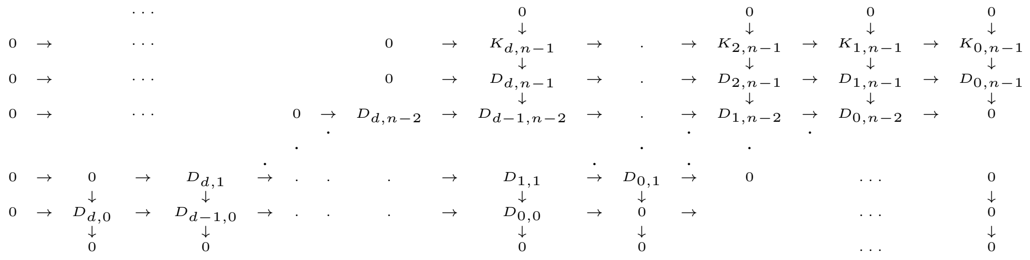


Fig. 2.13. The double complex $\mathbb{D}$ which is used in the proof of Theorem 2.12 to show that $\mathbb{K}(\Psi, n)$ is a resolution for $1 \le n$. The modules $D_{a,b}$ and $K_{a,b}$ represent $\bigwedge_R^a V \otimes_R D_b^R(V^*)$ and $K_{a,b}^R(V)$, respectively. The maps are given in (2.9).

a resolution. Let $\mathbb{K}'$ be the sub-complex of $\mathbb{K}(\Psi, n)$ which is obtained by deleting the left-most non-trivial module $\bigwedge_R^d V$. We prove that

$$\mathrm{H}_j(\mathbb{K}') = \begin{cases} 0 & \text{if } 1 \le j \le d-2 \\ \bigwedge_R^d V & \text{if } j = d-1 \end{cases} \tag{2.14}$$

and that the image of the map $\bigwedge_R^d V \to K_{d-1,n-1}^R$, which is given in (2.10), is exactly equal to $\mathrm{H}_{d-1}(\mathbb{K}')$.

Consider the double complex $\mathbb{D}$ which is obtained from (2.9) by keeping the modules $\bigwedge_R^a V \otimes_R D_b^R(V^*)$ for $b \le n-1$, and adjoining the row of kernels

$$0 \to K_{d,n-1}^R(V) \to \cdots \to K_{0,n-1}^R(V) \to 0.$$

Keep in mind that each $K_{i,n-1}^R(V)$ is the kernel at the top of a column of our truncation of (2.9). In other words, the maps

$$\begin{array}{c} 0 \\ \downarrow \\ K_{a,n-1} \\ \Big\downarrow{\scriptstyle \text{incl}} \\ \bigwedge_R^a V \otimes_R D_{n-1}^R(V^*) \\ \Big\downarrow{\scriptstyle \eta} \\ \bigwedge_R^{a-1} V \otimes_R D_{n-2}^R(V^*) \end{array}$$

form an exact sequence for each a. We have recorded a picture of $\mathbb{D}$ in Fig. 2.13, where $D_{a,b}$ represents $\bigwedge_R^a V \otimes_R D_b^R(V^*)$ and $K_{a,b}$ represents $K_{a,b}^R(V)$. Index $\mathbb{D}$ so that the total complex $\mathbb{T}(\mathbb{D})$ has the module $\sum_{j=1}^{n-1} D_{0,j}$ in position 0.

Each column of $\mathbb{D}$, except for the left-most non-trivial column, is exact. The hypothesis that J has grade d guarantees that the rows of $\mathbb{D}$, with the exception of the top-most non-trivial row, have homology concentrated in the position of the module $D_{0,*}$. We consider two sub-complexes $\mathbb{D}'$ and $\mathbb{D}''$ of $\mathbb{D}$. Let $\mathbb{D}'$ be the sub-complex of $\mathbb{D}$ which is obtained by deleting the left-most non-trivial column and $\mathbb{D}''$ be the sub-complex which is obtained by deleting the top-most non-trivial row. Observe that $\mathbb{D}/\mathbb{D}'$ is the left-most non-trivial column of $\mathbb{D}$; hence,

$$\mathbb{T}\big(\mathbb{D}/\mathbb{D}'\big)_j = \begin{cases} 0 & \text{if } j \neq d \\ D_{d,0} = \bigwedge_R^d V & \text{if } j = d. \end{cases}$$

Observe also that $\mathbb{D}/\mathbb{D}''$ is the top-most non-trivial row of $\mathbb{D}$; so, in particular,

$$\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)_j = \left\{\begin{array}{ll} K_{-1,n-1} = 0 & \text{for } j = 0 \\ K_{j-1,n-1} & \text{for } 1 \le j \le d \\ K_{d,n-1} = \ker \eta_{d,n-1} = 0 & \text{for } j = d+1 \end{array}\right\} = \mathbb{K}'_{j-1}$$

and $\mathbb{T}(\mathbb{D}/\mathbb{D}'') = \mathbb{K}'[-1]$. In light of (2.14), we show that $\mathbb{K}(\Psi, n)$ is a resolution by showing that

$$\mathrm{H}_j\big(\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)\big) = \begin{cases} 0 & \text{if } 2 \le j \le d-1 \\ \bigwedge_R^d V & \text{if } j = d \end{cases} \tag{2.15}$$

and showing that the isomorphism $\bigwedge_R^d V \to \mathrm{H}_d(\mathbb{T}(\mathbb{D}/\mathbb{D}''))$ is given by (2.10).

Every column of $\mathbb{D}'$ is split exact; so, the total complex of $\mathbb{D}'$, denoted $\mathbb{T}(\mathbb{D}')$, is also split exact. The short exact sequence of total complexes

$$0 \to \mathbb{T}\big(\mathbb{D}'\big) \to \mathbb{T}(\mathbb{D}) \to \mathbb{T}\big(\mathbb{D}/\mathbb{D}'\big) \to 0$$

yields that $\mathrm{H}_\bullet(\mathbb{T}(\mathbb{D})) \simeq \mathrm{H}_\bullet(\mathbb{T}(\mathbb{D}/\mathbb{D}'))$. Thus,

$$\mathrm{H}_j\big(\mathbb{T}(\mathbb{D})\big) \simeq \begin{cases} 0 & \text{if } j \neq d \\ \bigwedge_R^d V & \text{if } j = d; \end{cases} \tag{2.16}$$

furthermore, the isomorphism of (2.16), when $j = d$, is obtained by lifting each element θ in $\bigwedge^d V = D_{d,0}$ back to a cycle in $\mathbb{T}(\mathbb{D})$. Recall the canonical map

$$\mathrm{ev}_j^* : R \to \mathrm{Sym}_j^R(V) \otimes_R D_j^R\big(V^*\big),$$

as described in (1.3). Observe that $(1 \otimes \widehat{\Psi} \otimes 1)(\theta \otimes \mathrm{ev}_j^*(1))$ is an element of

$$\textstyle\bigwedge_R^d V \otimes_R D_j^R\big(V^*\big) = D_{d,j}.$$

It is easy to see that there exists signs $\sigma_j \in \{1,-1\}$ so that

$$\sum_{0\le j}\sigma_j\cdot(1\otimes\widehat{\Psi}\otimes 1)\big(\theta\otimes \mathrm{ev}_j^*(1)\big)$$

is the unique cycle in (2.9) which lifts $\theta\in D_{d,0}$. The complex $\mathbb{D}$ is obtained by truncating the complex (2.9) and then adjoining a row of kernels. We conclude that the isomorphism of (2.16), when $j=d$, is given by

$$\theta\mapsto\sum_{j=0}^{n-1}\sigma_j\cdot(1\otimes\widehat{\Psi}\otimes 1)\big(\theta\otimes \mathrm{ev}_j^*(1)\big)+\sigma_n\cdot\eta\big((1\otimes\widehat{\Psi}\otimes 1)\big(\theta\otimes \mathrm{ev}_n^*(1)\big)\big), \tag{2.17}$$

for $\theta\in\bigwedge_R^d(V)$.

The rows of $\mathbb{D}''$ all have homology concentrated in position $D_{0,*}$; and therefore, the homology of $\mathbb{T}(\mathbb{D}'')$ is concentrated in the position 0. The long exact sequence of homology which corresponds to the short exact sequence of complexes

$$0\to\mathbb{T}\big(\mathbb{D}''\big)\to\mathbb{T}(\mathbb{D})\to\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)\to 0$$

yields that

$$\mathrm{H}_j\big(\mathbb{T}(\mathbb{D})\big)\simeq\mathrm{H}_j\big(\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)\big)\quad\text{for } 2\le j. \tag{2.18}$$

We apply (2.16) to conclude that

$$\mathrm{H}_j\big(\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)\big)\simeq\begin{cases}0 & \text{if } 2\le j\le d-1\\ \bigwedge_R^d V & \text{if } j=d;\end{cases}$$

furthermore, the composition

$$\bigwedge\nolimits_R^d V\xrightarrow[\simeq]{(2.17)}\mathrm{H}_d\big(\mathbb{T}(\mathbb{D})\big)\xrightarrow[\simeq]{(2.18)}\mathrm{H}_d\big(\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)\big)$$

sends θ in $\bigwedge_R^d V$ to

$$\sigma_n\cdot\eta\big((1\otimes\widehat{\Psi}\otimes 1)\big(\theta\otimes \mathrm{ev}_n^*(1)\big)\big)\in K_{d-1,n-1}=\mathbb{T}\big(\mathbb{D}/\mathbb{D}''\big)_d.$$

The constant $\sigma_n\in\{1,-1\}$ is irrelevant. The map

$$\theta\mapsto\eta\big((1\otimes\widehat{\Psi}\otimes 1)\big(\theta\otimes \mathrm{ev}_n^*(1)\big)\big)$$

is exactly the map of (2.10). We have accomplished both objectives from (2.15); hence, we have shown that $\mathbb{K}(\Psi,n)$ is a resolution.

We use the same style of argument to show that $\mathbb{L}(\Psi,n)$ is a resolution. Let $\mathbb{E}$ be the double complex (2.8) truncated to include $\bigwedge_R^a V\otimes_R \mathrm{Sym}_b^R V$ for $0\le b\le n-1$ with

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & \cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & & & \cdots & & 0 & & 0 & \to & E_{d,0} & \to & E_{d-1,0} & \to & \cdot & \to & E_{1,0} & \to & E_{0,0} & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & & & \cdots & & 0 & \to & E_{d,1} & \to & E_{d-1,1} & \to & \cdot & \to & E_{1,1} & \to & E_{0,1} & \to & 0 & \to & 0 \\
 & & & & & \ddots & & \ddots & & & & \ddots & & \ddots & & \ddots & & & & & \\
0 & \to & 0 & \to & E_{d,n-2} & \to & \cdot & \to & E_{2,n-2} & \to & E_{1,n-2} & \to & E_{0,n-2} & \to & \cdots & & & & & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & & & & & \downarrow \\
0 & \to & E_{d,n-1} & \to & E_{d-1,n-1} & \to & \cdot & \to & E_{1,n-1} & \to & E_{0,n-1} & \to & 0 & & \cdots & & & & & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & & & & & \downarrow \\
0 & \to & L_{d-1,n} & \to & L_{d-2,n} & \to & \cdot & \to & L_{0,n} & \to & 0 & \to & 0 & & \cdots & & & & & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & & & & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0 & & 0 & & \cdots & & & & & & 0
\end{array}
$$

Fig. 2.19. The double complex $\mathbb{E}$ which is used in the proof of Theorem 2.12 to show that $\mathbb{L}(\Psi, n)$ is a resolution for $1 \le n$. The modules $E_{a,b}$ and $L_{a,b}$ represent $\bigwedge_R^a V \otimes_R \operatorname{Sym}_b^R V$ and $L_{a,b}^R(V)$, respectively. The maps are given in (2.8).

a row of cokernels adjoined. We have recorded a picture of $\mathbb{E}$ in Fig. 2.19 where $E_{a,b}$ represents $\bigwedge_R^a V \otimes_R \operatorname{Sym}_b^R V$ and $L_{a,b}$ represents $L_{a,b}^R(V)$. We index $\mathbb{E}$ so that $\mathbb{T}(\mathbb{E})_0$ is the module $\sum_{b=0}^{n-1} E_{0,b} \oplus L_{0,n}$. Define the sub-complexes $\mathbb{E}'$ and $\mathbb{E}''$ of $\mathbb{E}$ with $\mathbb{E}'$ equal to the right-most non-trivial column of $\mathbb{E}$ and $\mathbb{E}''$ equal to the bottom-most non-trivial row of $\mathbb{E}$. Every column of $\mathbb{E}/\mathbb{E}'$ is exact; so, $\mathrm{H}_\bullet(\mathbb{T}(\mathbb{E}/\mathbb{E}')) = 0$ and the long exact sequence of homology associated to the short exact sequences of complexes

$$0 \to \mathbb{T}(\mathbb{E}') \to \mathbb{T}(\mathbb{E}) \to \mathbb{T}(\mathbb{E}/\mathbb{E}') \to 0$$

yields that

$$\mathrm{H}_i(\mathbb{T}(\mathbb{E})) \simeq \mathrm{H}_i(\mathbb{T}(\mathbb{E}')) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i \ne 0. \end{cases}$$

Furthermore, the long exact sequence of homology yields that the isomorphism $\mathrm{H}_0(\mathbb{T}(\mathbb{E})) \to R$ is induced by the map that sends the cycle

$$\sum_{b=0}^{n} z_b \in \mathbb{T}(\mathbb{E})_0 \text{ to } z_0 \in R, \tag{2.20}$$

where $z_b \in E_{0,b}$ for $0 \le b \le n-1$ and $z_n \in L_{0,n}^R(V) = \bigwedge_R^0 V \otimes \operatorname{Sym}_n^R V$.

The rows of $\mathbb{E}/\mathbb{E}''$ have homology concentrated in position $E_{0,*}$. It follows that $\mathrm{H}_i(\mathbb{T}(\mathbb{E}/\mathbb{E}'')) = 0$ for $1 \le i$. The long exact sequence of homology associated to the short exact sequences of complexes

$$0 \to \mathbb{T}(\mathbb{E}'') \to \mathbb{T}(\mathbb{E}) \to \mathbb{T}(\mathbb{E}/\mathbb{E}'') \to 0$$

yields that

$$\mathrm{H}_i(\mathbb{T}(\mathbb{E}'')) = 0 \quad \text{for } 1 \le i \tag{2.21}$$

and that the natural map

$$\mathrm{H}_0\big(\mathbb{T}(\mathbb{E}'')\big) \to \mathrm{H}_0\big(\mathbb{T}(\mathbb{E})\big) \quad \text{is an injection.} \tag{2.22}$$

The map (2.22) sends the class of a cycle z_n from $L_{0,n}^R(V) = \mathbb{T}(\mathbb{E}'')_0$ to the class of the corresponding cycle in $\mathbb{T}(\mathbb{E})$. If

$$z_n = \prod_{i=1}^{n} \ell i \in L_{0,n}^R(V) = \textstyle\bigwedge_R^0 V \otimes_R \operatorname{Sym}_n^R V, \quad \text{with } \ell_i \text{ in } V,$$

then one lift of z_n to $\mathbb{T}(\mathbb{E})$ is $\sum_{b=0}^{n} \pm z_b$, with

$$z_b = \widehat{\Psi}(\ell_{b+1} \cdots \ell_n) \cdot \ell_1 \cdots \ell_b \in E_{0,b} \quad \text{for } 0 \le b \le n-1.$$

Combine (2.21), (2.22), and (2.20) to conclude that the augmented complex

$$\mathbb{T}\big(\mathbb{E}''\big) \xrightarrow{\widehat{\Psi}} R$$

is a resolution. This resolution is precisely equal to $\mathbb{L}(\Psi, n)$. □

3. The generators

Observation 3.9 is an important step in the present paper. In Definition 4.4 we introduce a family of complexes $\mathbb{G}(*)$. It is immediately clear that these complexes are resolutions. The main theorem of the paper is Theorem 4.7 which identifies the zeroth homology of the resolutions $\mathbb{G}(*)$. The proof of Theorem 4.7 relies on Observation 3.9.

Data 3.1. Let R be a commutative Noetherian ring, V be a free R-module of rank d, n be a positive integer, Φ be an element of $D_{2n-2}^R(V^*)$, $\operatorname{ann}\Phi$ be the ideal

$$\operatorname{ann}\Phi = \big\{\theta \in \operatorname{Sym}_\bullet^R V \;\big|\; \theta(\Phi) = 0 \in D_\bullet^R\big(V^*\big)\big\}$$

of the R-algebra $\operatorname{Sym}_\bullet^R V$, and

$$\boldsymbol{p}_{n-1}^{\Phi} : \operatorname{Sym}_{n-1}^R V \to D_{n-1}^R\big(V^*\big) \tag{3.2}$$

be the R-module homomorphism defined by

$$\boldsymbol{p}_{n-1}^{\Phi}(v_{n-1}) = v_{n-1}(\Phi),$$

for $v_{n-1} \in \operatorname{Sym}_{n-1}^R V$. Assume that $\boldsymbol{p}_{n-1}^{\Phi}$ is an isomorphism. For each integer i, let $[\operatorname{ann}\Phi]_i$ represent $\operatorname{ann}\Phi \cap \operatorname{Sym}_i^R V$. Define $\sigma_{n-1} : D_{n-1}^R(V^*) \to \operatorname{Sym}_{n-1}^R V$ to be the inverse of p^{Φ}_{n-1}. In particular,

$$\big[\sigma_{n-1}(w_{n-1})\big](\Phi) = w_{n-1} \quad \text{and} \quad \Phi\big(v_{n-1}\big[\sigma_{n-1}(w_{n-1})\big]\big) = v_{n-1}(w_{n-1}) \tag{3.3}$$

for all $w_{n-1} \in D^R_{n-1}(V^*)$ and $v_{n-1} \in \operatorname{Sym}^R_{n-1} V$.

In Observation 3.9 we identify a set of generators for $\operatorname{ann}\Phi$.

Observation 3.4. *Adopt Data 3.1 and let ρ be an integer with $0 \le \rho \le n-1$. Then the following statements hold.*

(1) *The R-module homomorphism $\Phi : \operatorname{Sym}^R_{2n-2} V \to R$ induces an isomorphism*

$$\frac{\operatorname{Sym}^R_{2n-2} V}{[\operatorname{ann}\Phi]_{2n-2}} \simeq R.$$

(2) *The pairing*

$$\operatorname{Sym}^R_{n-1-\rho} V \otimes_R \operatorname{Sym}^R_{n-1+\rho} V \to R, \tag{3.5}$$

which is given by

$$v_{n-1-\rho} \otimes v_{n-1+\rho} \mapsto (v_{n-1-\rho} v_{n-1+\rho})(\Phi),$$

induces a perfect pairing

$$\operatorname{Sym}^R_{n-1-\rho} V \otimes_R \frac{\operatorname{Sym}^R_{n-1+\rho} V}{[\operatorname{ann}\Phi]_{n-1+\rho}} \to R.$$

(3) *If $x_1, \dots, x_d$ is a basis for V, then the elements*

$$\left\{ x_1^{\rho} \sigma_{n-1}\big(x_1^{*(a_1+\rho)} x_2^{*(a_2)} \cdots x_d^{*(a_d)}\big) \;\middle|\; \sum_i a_i = n-1-\rho,\ 0 \le a_i \right\} \tag{3.6}$$

of $\operatorname{Sym}^R_{n-1+\rho} V$ are dual to the monomial basis

$$\left\{ x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} \;\middle|\; \sum_i a_i = n-1-\rho \right\}$$

of $\operatorname{Sym}^R_{n-1-\rho} V$ under the pairing (3.5).

Proof. Assertion (1) is a special case of (2); (2) is an immediate consequence of (3); and (3) is obvious. □

Remarks 3.7. (1) If $C_{n-1+\rho}$ is the R-submodule of $\operatorname{Sym}^R_{n-1+\rho} V$ which is generated by the elements of (3.6), then Observation 3.4 shows that $C_{n-1+\rho}$ is a free R-module with basis (3.6) and $\operatorname{Sym}^R_{n-1+\rho} V$ may be decomposed as the direct sum of two R-submodules:

$$\operatorname{Sym}^R_{n-1+\rho} V = [\operatorname{ann} \Phi]_{n-1+\rho} \oplus C_{n-1+\rho}.$$

(2) If $0 \le \rho \le n-1$, then Observation 3.4 shows that

$$\boldsymbol{p}^{\Phi}_{n-1+\rho} : \operatorname{Sym}^R_{n-1+\rho} V \to D^R_{n-1-\rho}(V^*)$$

is surjective. (Recall the definition of $\boldsymbol{p}^{\Phi}_i$ from Definition 1.5.) The R-module $D^R_{n-1-\rho}(V^*)$ is free; so there exists an R-module homomorphism

$$\sigma_{n-1-\rho} : D^R_{n-1-t}(V^*) \to \operatorname{Sym}^R_{n-1+\rho} V$$

which is a splitting map for $\boldsymbol{p}^{\Phi}_{n-1+\rho}$; thus, in particular, the composition

$$D^R_{n-1-\rho}(V^*) \xrightarrow{\sigma_{n-1-\rho}} \operatorname{Sym}^R_{n-1+\rho} V \xrightarrow{\boldsymbol{p}^{\Phi}_{n-1+\rho}} D^R_{n-1-\rho}(V^*) \tag{3.8}$$

is the identity map on $D^R_{n-1-\rho}(V^*)$. The map σ_{n-1} has been previously defined, in Data 3.1, in a coordinate-free manner. The maps $\sigma_{n-1-\rho}$, for $1 \le \rho \le n-1$, depend on the choice of a basis.
(3) For each ρ with $0 \le \rho \le n-1$, let $\alpha_{n-1-\rho} : K^R_{1,n-1-\rho}(V) \to \operatorname{Sym}^R_{n+\rho} V$ be the composition

$$K^R_{1,n-1-\rho}(V) \xrightarrow{1\otimes\sigma_{n-1-\rho}} V \otimes_R \operatorname{Sym}^R_{n-1+\rho} V \xrightarrow{\text{multiplication}} \operatorname{Sym}^R_{n+\rho} V,$$

and let $A[n+\rho]$ be the image of $\alpha_{n-1-\rho}$ in $\operatorname{Sym}^R_{n+\rho} V$. Notice that $A[n+\rho]$ is defined to be an R-module. This R-module is independent of the choice of coordinates when $\rho = 0$, and depends on the choice of coordinate when $1 \le \rho \le n-1$.

Observation 3.9. *Adopt Data 3.1 and the notation of Remarks 3.7. The following statements hold.*

(1) *If $0 \le \rho \le n-1$, then the R-module $A[n+\rho]$ is contained in the R-module $[\operatorname{ann} \Phi]_{n+\rho}$.*
(2) *The ideal $\operatorname{ann} \Phi$ of $\operatorname{Sym}^R_\bullet(V)$ is generated by $[\operatorname{ann} \Phi]_n$.*
(3) *The R-modules $A[n]$ and $[\operatorname{ann} \Phi]_n$ are equal.*
(4) *Let $\Psi : V \to R$ be an R-module homomorphism and $\widehat{\Psi} : \operatorname{Sym}^R_\bullet(V) \to R$ be the R-algebra homomorphism induced by Ψ. If I and J are the ideals $I = \widehat{\Psi}(\operatorname{ann} \Phi)$ and $J = \widehat{\Psi}(V)$ of R, then the following statements hold for all non-negative integers ρ,*

(a) $\widehat{\Psi}(\operatorname{ann}\Phi \cap \operatorname{Sym}^R_{n+\rho} V)$ *is equal to the ideal* $J^\rho I$ *of* R, *and*
(b) *a generating set for the ideal* $J^\rho I$ *may be obtained in a polynomial manner from the images of the maps*

$$\Psi : V \to R \quad \textit{and} \quad \Phi : \operatorname{Sym}^R_{2n-2} V \to R.$$

Proof. We first prove (1). A typical element of $A[n+\rho]$ has the form $\alpha_{n-1-\rho}(\Theta)$, where $\Theta = \sum_i \ell_i \otimes w_i$ is in $K^R_{1,n-1-\rho}$, with $\ell_i \in V$, $w_i \in D^R_{n-1-\rho}(V^*)$, and $\sum_i \ell_i(w_i) = 0$. The map $\alpha_{n-1-\rho}$ sends Θ to $\sum_i \ell_i \cdot \sigma_{n-1-\rho}(w_i)$; and therefore, (3.8) yields that

$$\big[\alpha_{n-1-\rho}(\Theta)\big](\Phi) = \sum_i \ell_i\big(\big[\sigma_{n-1-\rho}(w_i)\big](\Phi)\big) = \sum_i \ell_i(w_i) = 0,$$

and (1) is established.

We prove (2) and (3) simultaneously. Let $\mathcal{I}$ be the ideal of $\operatorname{Sym}^R_\bullet V$ which is generated by $A[n]$, and let $[\mathcal{I}]_i$ be the R-module $\mathcal{I} \cap \operatorname{Sym}^R_i V$ for each i. It is clear that $\mathcal{I}$ is generated by $[\mathcal{I}]_n$; and assertion (1) shows that $[\mathcal{I}]_n \subseteq [\operatorname{ann}\Phi]_n$; hence, $\mathcal{I} \subseteq \operatorname{ann}\Phi$. We prove that $\mathcal{I} = \operatorname{ann}\Phi$.

Both ideals $\mathcal{I}$ and $\operatorname{ann}\Phi$ are homogeneous; so it suffices to prove the equality $[\mathcal{I}]_i = [\operatorname{ann}\Phi]_i$ for one degree i at a time. It is clear that $[\mathcal{I}]_i$ is zero for $i \le n-1$. On the other hand, if v_i is an element of $[\operatorname{ann}\Phi]_i$ for some $i \le n-1$, and x is a basis element of V, then $x^{n-1-i}v_i \in \ker \boldsymbol{p}^\Phi_{n-1} = 0$. But x^{n-1-i} is a regular element in $\operatorname{Sym}^R_\bullet V$; so $v_i = 0$.

We identify a few critical elements of $[\mathcal{I}]_n$. Let $x_1, \dots, x_d$ be a basis for V and let $a_1, \dots, a_d$ be non-negative integers. Observe that

$$\begin{cases} x_i\sigma_{n-1}\big(x_1^{*(a_1)} \cdots x_i^{*(a_i+1)} \cdots x_j^{*(a_j)} \cdots x_n^{*(a_n)}\big) \\ -x_j\sigma_{n-1}\big(x_1^{*(a_1)} \cdots x_i^{*(a_i)} \cdots x_j^{*(a_j+1)} \cdots x_n^{*(a_n)}\big) \end{cases} \tag{3.10}$$

is an element of $[\mathcal{I}]_n$ for any pair $i \ne j$ when $\sum_\ell a_\ell = n-2$, and

$$x_i\sigma_{n-1}\big(x_1^{*(a_1)} \cdots x_n^{*(a_n)}\big) \tag{3.11}$$

is an element of $[\mathcal{I}]_n$ whenever $a_i = 0$ and $\sum_\ell a_\ell = n-1$.

Fix an integer ρ with $0 \le \rho \le n-1$. We prove that $[\operatorname{ann}\Phi]_{n-1+\rho} \subseteq [\mathcal{I}]_{n-1+\rho}$. Recall the R-submodule $C_{n-1+\rho}$ of $\operatorname{Sym}^R_{n-1+\rho} V$ and the direct sum decomposition

$$\operatorname{Sym}^R_{n-1+\rho} V = [\operatorname{ann}\Phi]_{n-1+\rho} \oplus C_{n-1+\rho}$$

from Remark 3.7. Use the fact that

$$\sigma_{n-1} : D^R_{n-1}\big(V^*\big) \to \operatorname{Sym}^R_{n-1} V$$

is an isomorphism of free R-modules to see that $\sigma_{n-1}(D^R_{n-1}(V^*)) = \operatorname{Sym}^R_{n-1} V$; and hence, $\operatorname{Sym}^R_\rho V \cdot \sigma_{n-1}(D^R_{n-1}(V^*)) = \operatorname{Sym}^R_{n-1+\rho} V$. We next show that

$$\operatorname{Sym}^R_{n-1+\rho} V \subseteq [\mathcal{I}]_{n-1+\rho} + C_{n-1+\rho}. \tag{3.12}$$

Fix non-negative integers $A_1, \dots, A_d$ and $a_1, \dots, a_d$, with $\sum_i A_i = \rho$ and $\sum_i a_i$ equal to $n-1$. Let "$\equiv$" mean congruent mod $\mathcal{I}$. Apply (3.10) and (3.11) repeatedly to see that $x_1^{A_1} \cdots x_d^{A_d} \sigma_{n-1}(x_1^{*(a_1)} \cdots x_d^{*(a_d)})$ is

$$\equiv \begin{cases} x_1^{A_1-a_1} x_2^{A_2+a_1} x_3^{A_3} \cdots x_d^{A_d} \sigma_{n-1}(x_2^{*(a_2+a_1)} \cdots x_d^{*(a_d)}) \equiv 0 & \text{if } a_1 < A_1 \\ x_1^{A_1+a_i} \cdots x_i^{A_i-a_i} \cdots x_d^{A_d} \sigma_{n-1}(x_1^{*(a_1+a_i)} \cdots x_i^{*(0)} \cdots x_d^{*(a_d)}) \equiv 0 & \begin{cases} \text{if } a_i < A_i, \text{ for} \\ \text{some } i \neq 1 \end{cases} \\ x_1^\rho \sigma_{n-1}(x_1^{*(a_1-A_1+\rho)} x_2^{*(a_2-A_2)} \cdots x_d^{*(a_d-A_d)}) \in C_{n-1+\rho} & \begin{cases} \text{if } A_i \le a_i \\ \text{for all } i. \end{cases} \end{cases} \tag{3.13}$$

Thus, (3.12) holds and

$$\operatorname{Sym}^R_{n-1+\rho} V \subseteq [\mathcal{I}]_{n-1+\rho} + C_{n-1+\rho} \subseteq [\operatorname{ann} \Phi]_{n-1+\rho} \oplus C_{n-1+\rho} = \operatorname{Sym}^R_{n-1+\rho} V.$$

It follows that

$$[\mathcal{I}]_{n-1+\rho} + C_{n-1+\rho} = [\operatorname{ann} \Phi]_{n-1+\rho} + C_{n-1+\rho}, \qquad [\operatorname{ann} \Phi]_{n-1+\rho} \cap C_{n-1+\rho} = 0,$$
$$\text{and} \quad [\mathcal{I}]_{n-1+\rho} \subseteq [\operatorname{ann} \Phi]_{n-1+\rho};$$

and therefore, $[\mathcal{I}]_{n-1+\rho} = [\operatorname{ann} \Phi]_{n-1+\rho}$.

Finally, we consider $i > 2n-2$. It is clear that $[\operatorname{ann} \Phi]_i = \operatorname{Sym}^R_i V$ and, if one makes the calculation analogous to (3.13), then it is not possible for the bottom case to occur, so $[\mathcal{I}]_i$ is also equal to $\operatorname{Sym}^R_i V$.

(4.a) One consequence of (2) is that the R sub-modules

$$[\operatorname{ann} \Phi]_{n+\rho}, \qquad [\operatorname{ann} \Phi]_n \cdot \operatorname{Sym}^R_\rho V \quad \text{and} \quad [\operatorname{ann} \Phi]_n \cdot \left(\operatorname{Sym}^R_1 V\right)^\rho \tag{3.14}$$

of $\operatorname{Sym}^R_{n+\rho} V$ are equal. The R-algebra homomorphism $\widehat{\Psi}$ carries this R-module to the ideal $\widehat{\Psi}([\operatorname{ann} \Phi]_{n+\rho}) = \widehat{\Psi}([\operatorname{ann} \Phi]_n) \cdot \widehat{\Psi}((\operatorname{Sym}^R_1 V)^\rho) = IJ^\rho$ of R.

(4.b) We will identify a set polynomials $\{p_\alpha\}$ in $\mathbb{Z}[\{X_1, \dots, X_d\}, \{t_M\}]$, where

$$M \text{ roam over the monomials of degree } 2n-2 \text{ in } d \text{ variables.} \tag{3.15}$$

We have already picked a basis $x_1, \dots, x_d$ for V. If M is a monomial from (3.15), then let $M|x$ represent the corresponding element of $\operatorname{Sym}^R_{2n-2} V$; we think of "$M|x$" as "$M$

evaluated at $x_1, \dots, x_d$." We will choose the p_α so that the set of p_α, with X_i evaluated at $\Psi(x_i)$ and t_M evaluated at $\Phi(M|x)$, generates $J^\rho I$.

Our proof of (2) shows that $[\operatorname{ann}\Phi]_n$ is generated by the elements of (3.10) and (3.11). Thus, (3.14) shows that $J^\rho I$ is generated by $\widehat{\Psi}$ applied to the elements of $(\operatorname{Sym}_1^R V)^\rho$ times the elements of (3.10) and (3.11).

Recall from Remark 1.6 that the matrix of the map $\boldsymbol{p}_{n-1}^{\Phi}$ of (3.2) is $T_\Phi = [\Phi(m_i m_j)]$, where $m_1, \dots, m_N$ is a list of the monomials of degree $n-1$ in $x_1, \dots, x_d$. Each entry in the matrix T_Φ is $\Phi(M|x)$ for some M from (3.15). The matrix T_Φ is invertible over R by the hypothesis that $\boldsymbol{p}_{n-1}^{\Phi}$ is an isomorphism; thus, $\det T_\Phi$ is a unit in R and the matrix for σ_{n-1} is $T_\Phi^{-1} = \frac{1}{\det T_\Phi}\operatorname{Adj}(T_\Phi)$, where $\operatorname{Adj}(T_\Phi)$ is the classical adjoint of T_Φ. The classical adjoint of T_Φ is built in a polynomial manner from the entires of T_Φ; the classical adjoint of T_Φ is the matrix for the map $(\det T_\Phi)\sigma_{n-1}$; and the map $(\det T_\Phi)\sigma_{n-1}$ differs from the map σ_{n-1} by a unit of R – this unit does not affect the ideal generated by the image. Thus, the ideal $J^\rho I$ is generated by the elements

$$\widehat{\Psi}\left((x_1, \dots, x_d)^\rho \left\{ \begin{array}{l} x_i(\det T_\Phi)\sigma_{n-1}(x_1^{*(a_1)} \cdots x_i^{*(a_i+1)} \cdots x_j^{*(a_j)} \cdots x_n^{*(a_n)}) \\ -x_j(\det T_\Phi)\sigma_{n-1}(x_1^{*(a_1)} \cdots x_i^{*(a_i)} \cdots x_j^{*(a_j+1)} \cdots x_n^{*(a_n)}) \end{array}\right.\right),$$

for $i \neq j$ and $\sum_\ell a_\ell = n-2$, together with the elements

$$\widehat{\Psi}\big((x_1, \dots, x_d)^\rho x_i(\det T_\Phi)\sigma_{n-1}\big(x_1^{*(a_1)} \cdots x_n^{*(a_n)}\big)\big),$$

with $a_i = 0$ and $\sum_\ell a_\ell = n-1$. Each of these elements of R is built in a polynomial manner from $\{\Psi(x_i)\} \cup \{\Phi(M|x)\}$. □

4. The main theorem

Data 4.1. Consider the data (R, V, n, Ψ, Φ), where R is a commutative Noetherian ring, V is a non-zero free R-module of finite rank, n is a positive integer, $\Psi : V \to R$ is an R-module homomorphism, and Φ is an element of $D_{2n-2}^R(V^*)$. Let d be the rank of V, $\widehat{\Psi} : \operatorname{Sym}_\bullet^R(V) \to R$ be the R-algebra homomorphism induced by Ψ, and I and J be the ideals $I = \widehat{\Psi}(\operatorname{ann}\Phi)$ and $J = \widehat{\Psi}(V)$ of R. For each integer i with $0 \le i \le 2n-2$, let $\boldsymbol{p}_i^{\Phi} : \operatorname{Sym}_i^R V \to D_{2n-2-i}^R(V^*)$ be the R-module homomorphism defined by $\boldsymbol{p}_i^{\Phi}(v_i) = v_i(\Phi)$, for $v_i \in \operatorname{Sym}_i^R V$.

In this section we produce a family of complexes $\mathbb{G}(R, V, n, \Psi, \Phi; r)$, one for each integer r with $0 \le r \le 2n-2$. The main result is Theorem 4.7, where we prove that if the grade of J is at least d and $\boldsymbol{p}_{n-1}^{\Phi}$ is an isomorphism, then each complex $\mathbb{G}(R, V, n, \Psi, \Phi; r)$ is a resolution and the complexes $\mathbb{G}(R, V, n, \Psi, \Phi; n-1)$ and $\mathbb{G}(R, V, n, \Psi, \Phi; n)$ both resolve R/I.

Corollaries 4.16 and 4.22 are applications of Theorem 4.7. Corollary 4.16 treats the generic resolution $\widetilde{\mathbb{G}}(r)$ and Corollary 4.22 specializes the generic resolution in order to accomplish Project 0.3. Section 5 and the beginning of Example 6.21 give examples of $\widetilde{\mathbb{G}}$.

Recall the complexes $\mathbb{L}$ and $\mathbb{K}$ from Definition 2.6.

Observation 4.2. *Adopt Data 4.1 and let r be an integer with $1 \le r \le 2n-2$. Then the R-module homomorphism $\boldsymbol{p}_r^{\Phi} : \operatorname{Sym}_r^R(V) \to D_{2n-2-r}^R(V^*)$ induces a map of complexes*

$$E(R, V, n, \Psi, \Phi; r) : \mathbb{L}(\Psi, r) \to \mathbb{K}(\Psi, 2n-1-r)[-1]$$

as described:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & L_{d-1,r} & \longrightarrow \cdots \longrightarrow & L_{1,r} & \longrightarrow & L_{0,r} & \longrightarrow & R \\ & & \downarrow & & \downarrow{\scriptstyle 1\otimes \boldsymbol{p}_r^{\Phi}} & & \downarrow{\scriptstyle 1\otimes \boldsymbol{p}_r^{\Phi}} & & \downarrow{\scriptstyle 1\otimes \boldsymbol{p}_r^{\Phi}} & & \\ 0 & \longrightarrow & \bigwedge_R^d V & \longrightarrow & K_{d-1,2n-2-r} & \longrightarrow \cdots \longrightarrow & K_{1,2n-2-r} & \longrightarrow & K_{0,2n-2-r}, & & \end{array}$$

where $L_{a,b}$ and $K_{a,b}$ represent $L_{a,b}^R(V)$ and $K_{a,b}^R(V)$, respectively.

Proof. The proof is straightforward and short. □

Remark 4.3. If the ring R is bi-graded and the R-homomorphisms

$$\Psi : R(-1,0)^d \to R \quad \text{and} \quad \Phi : R(0,-1)^{\binom{2n+d-3}{d-1}} \to R$$

are bi-homogeneous, then the double complex $E(R, V, n, \Psi, \Phi; r)$ of Observation 4.2 is bi-homogeneous. The entries of the matrix $L_{0,r} \to R$ have degree $(r, 0)$; the entries of the matrices $L_{\ell,r} \to L_{\ell,r-1}$ and $K_{\ell,2n-2-r} \to K_{\ell-1,2n-2-r}$ have degree $(1, 0)$; the entries of the matrices for $1 \otimes \boldsymbol{p}_r^{\Phi}$ have degree $(0, 1)$; and the entries of the matrix $\bigwedge_R^d V \to K_{d-1,2n-2-r}$ have degree $(2n-1-r, 0)$. The ranks of $L_{a,b}$ and $K_{a',b'}$ are given in (2.3) and (2.4). (See, also, (5.8).) The double complex $E(R, V, n, \Psi, \Phi; r)$ is bi-homogeneous with

$$\begin{aligned} L_{a,r} &= R(-r-a, 0)^{\binom{d+r-1}{a+r}\binom{a+r-1}{a}} \\ K_{a,2n-2-r} &= R(-r-a, -1)^{\binom{d+2n-2-r}{a}\binom{2n+d-3-r-a}{d-1-a}} \\ \textstyle\bigwedge_R^d V &= R(-2n-d+2, -1)^1. \end{aligned}$$

Definition 4.4. Adopt Data 4.1 and let r be an integer with $1 \le r \le 2n-2$. Let $\mathbb{G}(R, V, n, \Psi, \Phi; r)$ be the total complex of the double complex $E(R, V, n, \Psi, \Phi; r)$, from Observation 4.2, with $R \oplus K_{0,2n-2-r}^R$ in position 0.

Observation 4.5. *Adopt Data 4.1 and let r be an integer with $1 \le r \le 2n-2$ and $\mathbb{G}(R, V, n, \Psi, \Phi; r)$ be the complex of Definition 4.4. Then the complexes*

$$\mathbb{G}(R, V, n, \Psi, \Phi; r) \quad \text{and} \quad \Big[\operatorname{Hom}_R\big(\mathbb{G}(R, V, n, \Psi, \Phi; 2n-1-r), R\big) \otimes_R \textstyle\bigwedge^d V\Big][-d]$$

of Definition 4.4 are isomorphic, where "$[-d]$" describes a shift in homological degree.

Proof. Apply assertion (1) of Theorem 2.12. □

Observation 4.6. *Adopt Data 4.1 and let r be an integer with $1 \le r \le 2n-2$. If J has grade at least d, then the complex $\mathbb{G}(R,V,n,\Psi,\Phi;r)$ of Definition 4.4 is a resolution.*

Proof. Apply assertion (2) of Theorem 2.12 to see that the complexes

$$\mathbb{L}(\Psi,r) \quad \text{and} \quad \mathbb{K}(\Psi,2n-1-r)$$

have homology concentrated in position zero, then use the long exact sequence of homology associated to a mapping cone to conclude that $\mathbb{G}(R,V,n,\Psi,\Phi;r)$ also has homology concentrated in position zero. □

Theorem 4.7. *Adopt Data 4.1. Assume that J is a proper ideal with grade at least d and*

$$\boldsymbol{p}_{n-1}^{\Phi} : \operatorname{Sym}_{n-1}^R(V) \to D_{n-1}^R\bigl(V^*\bigr)$$

is an isomorphism. For each index r, with $1 \le r \le 2n-2$, let $\mathbb{G}(r)$ represent the complex $\mathbb{G}(R,V,n,\Psi,\Phi;r)$, of Definition 4.4. The following statements hold.

(1) *Each of the ideals $J^\rho I$, with $0 \le \rho \le n-2$, is a perfect, grade d, ideal of R.*
(2) *The ideal I is a perfect, grade d, Gorenstein ideal of R.*
(3) *The complex $\mathbb{G}(n+\rho)$ is a resolution of $R/J^\rho I$ by free R-modules, for $0 \le \rho \le n-2$.*
(4) *The complexes $\mathbb{G}(n)$ and $\mathbb{G}(n-1)$ both resolve $R/I = \operatorname{Ext}_R^d(R/I,R)$.*
(5) *The complex $\mathbb{G}(n-1-\rho)$ is a resolution of $\operatorname{Ext}_R^d(R/J^\rho I,R)$ by free R-modules, for $0 \le \rho \le n-2$.*

Remark. We are primarily interested in the resolution $\mathbb{G}(n+\rho)$ of $R/J^\rho I$ when $\rho = 0$. The argument which produces this resolution of $R/J^\rho I$ for $\rho = 0$ also resolves $R/J^\rho I$ for $0 \le \rho \le n-2$. The ideal $J^\rho I$ is a "truncation" of I.

Proof of Theorem 4.7. First notice that the ideals I and J of R have the same radical. Recall that $I = \widehat{\Psi}(\operatorname{ann}\Phi)$ and $J = \widehat{\Psi}(V)$. It is clear that, as ideals of $\operatorname{Sym}_\bullet^R(V)$,

$$(\operatorname{ann}\Phi) \subseteq \bigl(\operatorname{Sym}_1^R V\bigr) \quad \text{and} \quad \bigl(\operatorname{Sym}_{2n-1}^R V\bigr) \subseteq (\operatorname{ann}\Phi).$$

It follows that

$$J^{2n-1} = \bigl(\widehat{\Psi}(V)\bigr)^{2n-1} = \widehat{\Psi}\bigl(\operatorname{Sym}_{2n-1}^R(V)\bigr) \subseteq \widehat{\Psi}\bigl(\operatorname{ann}(\Phi)\bigr) \subseteq \widehat{\Psi}(V) = J;$$

and therefore, $J^{2n-1} \subseteq I \subseteq J$. In particular,

$$\text{the ideals } J \text{ and } J^\rho I \text{ have the same grade for all non-negative integers } \rho. \tag{4.8}$$

The hypothesis that $d \le \operatorname{grade} J$ also guarantees that $d \le \operatorname{grade} J^\rho I$, for $0 \le \rho$.

We saw in Observation 4.6 that $\mathbb{G}(r)$ is a resolution for every integer r between 1 and $2n-2$. The heart of the argument is the proof that

$$\mathrm{H}_0\bigl(\mathbb{G}(n-1+\lambda)\bigr) = \begin{cases} R/I & \text{if } \lambda = 0 \\ R/J^{\lambda-1}I & \text{if } 1 \le \lambda \le n-1. \end{cases} \tag{4.9}$$

We assume (4.9) for the time-being and complete the proof. Assertion (3) and the part of assertion (4) that $\mathbb{G}(n)$ and $\mathbb{G}(n-1)$ both resolve R/I follow immediately from (4.9). Fix an index ρ, with $0 \le \rho \le n-2$. We have exhibited that $\mathbb{G}(n+\rho)$ is a free resolution of $R/J^\rho I$ of length d; thus,

$$d \le \operatorname{grade} J^\rho I \le \text{the projective dimension of the } R\text{-module } R/J^\rho I \le d,$$

and $J^\rho I$ is a perfect, grade d, ideal of R. Assertion (1) is established and

$$\operatorname{Hom}_R\bigl(\mathbb{G}(n+\rho), R\bigr) \to \operatorname{Ext}_R^d\bigl(R/J^\rho I, R\bigr) \to 0$$

is a resolution. Observation 4.5 yields assertion (5). We know from (4) that $\mathbb{G}(n-1)$ resolves R/I and from (5) that $\mathbb{G}(n-1)$ resolves $\operatorname{Ext}_R^d(R/I,R)$. It follows that $R/I = \operatorname{Ext}_R^d(R/I,R)$, and the proof of (2) and (4) are complete.

We now prove (4.9). Write $\mathbb{G}$ for $\mathbb{G}(n-1+\lambda)$. We compute $\mathrm{H}_0(\mathbb{G})$. (The same ideas appear again in the proof of Lemma 6.16.) The homology $\mathrm{H}_0(\mathbb{G})$ is defined to be the cokernel of

$$\begin{matrix} L^R_{0,n-1+\lambda}(V) \\ \oplus \\ K^R_{1,n-1-\lambda}(V) \end{matrix} \xrightarrow{\begin{bmatrix} \widehat{\Psi} & 0 \\ \boldsymbol{p}^\Phi_{n-1+\lambda} & \Psi\otimes 1 \end{bmatrix}} \begin{matrix} R \\ \oplus \\ K^R_{0,n-1-\lambda}(V). \end{matrix}$$

Recall that $L^R_{0,n-1+\lambda}(V) = \operatorname{Sym}^R_{n-1+\lambda} V$ and $K^R_{0,n-1-\lambda} = D^R_{n-1-\lambda}(V^*)$. Recall, also, the splitting map

$$\sigma_{n-1-\lambda} : D^R_{n-1-\lambda}\bigl(V^*\bigr) \to \operatorname{Sym}^R_{n-1+\lambda} V$$

of Remark 3.7(2). We decompose $L^R_{0,n-1+\lambda}(V) = \operatorname{Sym}^R_{n-1+\lambda} V$ as

$$\ker \boldsymbol{p}^\Phi_{n-1+\lambda} \oplus \operatorname{im} \sigma_{n-1-\lambda}$$

and we employ the isomorphism

$$\begin{matrix} \ker \boldsymbol{p}^\Phi_{n-1+\lambda} \\ \oplus \\ K^R_{0,n-1-\lambda}(V) \end{matrix} \xrightarrow[\simeq]{\begin{bmatrix} 1 & 0 \\ 0 & \sigma_{n-1-\lambda} \end{bmatrix}} \left.\begin{matrix} \ker \boldsymbol{p}^\Phi_{n-1+\lambda} \\ \oplus \\ \operatorname{im} \sigma_{n-1-\lambda} \end{matrix}\right\} = L^R_{0,n-1+\lambda}(V)$$

to see that $\mathrm{H}_0(\mathbb{G})$ is the cokernel of

$$\begin{array}{c} \ker \boldsymbol{p}^{\Phi}_{n-1+\lambda} \\ \oplus \\ K^R_{0,n-1-\lambda}(V) \\ \oplus \\ K^R_{1,n-1-\lambda}(V) \end{array} \xrightarrow{\left[\begin{smallmatrix} \widehat{\Psi} & \widehat{\Psi}\circ\sigma_{n-1-\lambda} & 0 \\ 0 & 1 & \Psi\otimes 1 \end{smallmatrix}\right]} \begin{array}{c} R \\ \oplus \\ K^R_{0,n-1-\lambda}(V). \end{array}$$

Row and column operations yield that $\mathrm{H}_0(\mathbb{G})$ is the cokernel of

$$\begin{bmatrix} \widehat{\Psi} & 0 & -\widehat{\Psi}\circ\sigma_{n-1-\lambda}\circ(\Psi\otimes 1) \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.10}$$

It is not difficult to see that the diagram

$$\begin{array}{ccccc} K^R_{1,n-1-\lambda}(V) & \xrightarrow{\Psi} & D^R_{n-1-\lambda}(V^*) & \xrightarrow{\sigma_{n-1-\lambda}} & \mathrm{Sym}^R_{n-1+\lambda} V \\ {\scriptstyle 1\otimes\sigma_{n-1-\lambda}}\downarrow & & & & \downarrow{\scriptstyle \widehat{\Psi}} \\ V\otimes_R \mathrm{Sym}^R_{n-1+\lambda} V & \xrightarrow{\text{mult}} & \mathrm{Sym}^R_{n+\lambda} V & \xrightarrow{\widehat{\Psi}} & R \end{array}$$

commutes. The clockwise composition from $K^R_{1,n-1-\lambda}(V)$ to R is the map in the upper right-hand corner of (4.10), up to sign. The counter-clockwise composition from $K^R_{1,n-1-\lambda}(V)$ to $\mathrm{Sym}^R_{n+\lambda} V$ is called $\alpha_{n-1-\lambda}$ (see Remark 3.7(3)) and the image of $\alpha_{n-1-\lambda}$ is called $A[n+\lambda]$. It follows that

$$\mathrm{H}_0(\mathbb{G}) = \frac{R}{\widehat{\Psi}(\ker(\boldsymbol{p}^{\Phi}_{n-1+\lambda})) + \widehat{\Psi}(A[n+\lambda])}.$$

Use Observation 3.9 to see that

$$\widehat{\Psi}\big(\ker\big(\boldsymbol{p}^{\Phi}_{n-1+\lambda}\big)\big) = \widehat{\Psi}\big([\operatorname{ann}\Phi]_{n-1+\lambda}\big) = \begin{cases} 0 & \text{if } \lambda = 0, \\ J^{\lambda-1}I & \text{if } 1\le\lambda\le n-1, \end{cases}$$

$$\begin{array}{ll} \widehat{\Psi}\big(A[n+\lambda]\big) = \widehat{\Psi}(\operatorname{ann}\Phi) = I & \text{if } \lambda = 0, \quad \text{and} \\ \widehat{\Psi}\big(A[n+\lambda]\big) \subseteq \widehat{\Psi}([\operatorname{ann}\Phi]_{n+\lambda}) = J^{\lambda}I \subseteq J^{\lambda-1}I & \text{if } 1\le\lambda\le n-1. \end{array}$$

Thus, (4.9) holds and the proof is complete. □

The complexes of Definition 4.4 can be built generically. (When we consider generic data, there would be no loss of generality if we had written "$\mathbb{Z}$" every where we have written "R_0" because, the generic objects constructed over $\mathbb{Z}$ would become generic objects constructed over R_0 once we applied the base change $R_0 \otimes_{\mathbb{Z}}$ __.)

Definition 4.11. Consider (R_0, U, n), where R_0 is a commutative Noetherian ring, U is a free R_0-module of rank d, and n is a positive integer. We define the *generic ring* $\widetilde{R} = \widetilde{R}(R_0, U, n)$, the *generic ideals* $\widetilde{I} = \widetilde{I}(R_0, U, n)$ and $\widetilde{J} = \widetilde{J}(R_0, U, n)$ of $\widetilde{R}$, and, for each integer r with $1 \le r \le 2n-2$, the *generic complex* $\widetilde{\mathbb{G}}(R_0, U, n; r)$. View $U \oplus \operatorname{Sym}^{R_0}_{2n-2} U$ as a bi-graded free R_0-module, where the elements of U have degree $(1,0)$ and the elements of $\operatorname{Sym}^{R_0}_{2n-2} U$ have degree $(0,1)$. Let $\widetilde{R}$ be the bi-graded R_0-algebra $\operatorname{Sym}^{R_0}_\bullet(U \oplus \operatorname{Sym}^{R_0}_{2n-2} U)$. Define the R_0-module inclusion homomorphisms $i_1 : U \to \widetilde{R}$ and $i_2 : \operatorname{Sym}^{R_0}_{2n-2} U \to \widetilde{R}$ by

$$U \to U \oplus 0 = [\widetilde{R}]_{(1,0)} \subseteq \widetilde{R} \quad \text{and} \quad \operatorname{Sym}^{R_0}_{2n-2} U \to 0 \oplus \operatorname{Sym}^{R_0}_{2n-2} U = [\widetilde{R}]_{(0,1)} \subseteq \widetilde{R},$$

respectively. Let V be the free $\widetilde{R}$-module $V = \widetilde{R} \otimes_{R_0} U$ and define $\Psi : V \to \widetilde{R}$ and $\Phi : \operatorname{Sym}^{\widetilde{R}}_{2n-2} V \to \widetilde{R}$ to be the $\widetilde{R}$-module homomorphisms

$$\begin{array}{rcl} V = \widetilde{R} \otimes_{R_0} U & \xrightarrow{1 \otimes i_1} \widetilde{R} \otimes_{R_0} \widetilde{R} \xrightarrow{\text{multiplication}} \widetilde{R} & \text{and} \\ \operatorname{Sym}^{\widetilde{R}}_{2n-2} V = \widetilde{R} \otimes_{R_0} \operatorname{Sym}^{R_0}_{2n-2} U & \xrightarrow{1 \otimes i_2} \widetilde{R} \otimes_{R_0} \widetilde{R} \xrightarrow{\text{multiplication}} \widetilde{R}, & \end{array} \tag{4.12}$$

respectively. Let $\widehat{\Psi} : \operatorname{Sym}^{\widetilde{R}}_\bullet(V) \to \widetilde{R}$ be the $\widetilde{R}$-algebra homomorphism induced by Ψ, $\widetilde{I}$ and $\widetilde{J}$ be the ideals $\widetilde{I} = \widehat{\Psi}(\operatorname{ann} \Phi)$ and $\widetilde{J} = \widehat{\Psi}(V)$ of $\widetilde{R}$. For each integer r with $1 \le r \le 2n-2$, define $\widetilde{\mathbb{G}}(R_0, U, n; r)$ to be the complex $\mathbb{G}(\widetilde{R}, V, n, \Psi, \Phi; r)$ of Definition 4.4.

Remark 4.13. Continue with the notation and hypotheses of Definition 4.11. Suppose that U has basis $x_1, \dots, x_d$. In this case, we may view $\widetilde{R}$ as the bi-graded polynomial ring $\widetilde{R} = R_0[x_1, \dots, x_d, \{t_M\}]$, where

$$M \text{ roams over all monomials of degree } 2n-2 \text{ in } \{x_1, \dots, x_d\}. \tag{4.14}$$

The variables x_i have bi-degree $(1,0)$ and each variable t_M has bi-degree $(0,1)$. The map $\Phi : \operatorname{Sym}^{\widetilde{R}}_{2n-2} V \to \widetilde{R}$ from (4.12) is the same as the element

$$\sum_{(4.14)} t_{x_1^{a_1} \cdots x_d^{a_d}} \otimes x_1^{*(a_1)} \cdots x_d^{*(a_d)} \in \widetilde{R} \otimes_{R_0} D^{R_0}_{2n-2} U^* = D^{\widetilde{R}}_{2n-2}(V^*), \tag{4.15}$$

where * in U^* means R_0-dual and * in V^* means $\widetilde{R}$-dual.

Our first application of Theorem 4.7 is to the generic situation.

Corollary 4.16. *Consider the data (R_0, U, n), where R_0 is a commutative Noetherian ring, U is a free R_0-module of positive rank d, n is a positive integers. Let $\widetilde{R} = \widetilde{R}(R_0, U, n)$, $\widetilde{I} = \widetilde{I}(R_0, U, n)$, $\widetilde{J} = \widetilde{J}(R_0, U, n)$, and for each integer r, with $1 \le r \le 2n-2$, $\widetilde{\mathbb{G}}(r) = \widetilde{\mathbb{G}}(R_0, U, n; r)$, be the generic ring, ideals, and complexes of Definition 4.11, and let $\boldsymbol{\delta}$ be* $\det T_\Phi$ *for the $\widetilde{R}$-module Φ of (4.12). The following statements hold.*

(1) *Each of the ideals* $\widetilde{J}^\rho\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$*, with* $0 \le \rho \le n-2$*, is a perfect, grade* d*, ideal of* $\widetilde{R}_{\boldsymbol{\delta}}$*.*
(2) *The ideal* $\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ *is a perfect, grade* d*, Gorenstein ideal of* $\widetilde{R}_{\boldsymbol{\delta}}$*.*
(3) *The complex* $\widetilde{\mathbb{G}}(n+\rho)_{\boldsymbol{\delta}}$ *is a resolution of* $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^\rho\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ *by free* $\widetilde{R}_{\boldsymbol{\delta}}$*-modules, for* $0 \le \rho \le n-2$*.*
(4) *The complexes* $\widetilde{\mathbb{G}}(n)_{\boldsymbol{\delta}}$ *and* $\widetilde{\mathbb{G}}(n-1)_{\boldsymbol{\delta}}$ *both are resolutions of*

$$\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}} = \operatorname{Ext}^d_{\widetilde{R}_{\boldsymbol{\delta}}}(\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}, \widetilde{R}_{\boldsymbol{\delta}})$$

by free $\widetilde{R}_{\boldsymbol{\delta}}$*-modules.*
(5) *The complex* $\widetilde{\mathbb{G}}(n-1-\rho)_{\boldsymbol{\delta}}$ *is a resolution of* $\operatorname{Ext}^d(\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^\rho\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}, \widetilde{R}_{\boldsymbol{\delta}})$ *by free* $\widetilde{R}_{\boldsymbol{\delta}}$*-modules, for* $0 \le \rho \le n-2$*.*

Remark 4.17. The sense in which "$\boldsymbol{\delta} = \det T_\Phi$" is coordinate-free is explained in Remark 1.6(3). In the language of Remark 4.13, if $m_1, \dots, m_N$ is a list of the monomials of degree $n-1$ in $x_1, \dots, x_d$, then $\boldsymbol{\delta}$ is equal to $\det(t_{m_i m_j})$. Each $t_{m_i m_j}$ is an indeterminate of the polynomial ring $\widetilde{R}$ from (4.14).

Proof of Corollary 4.16. We apply Theorem 4.7. In the language of Remark 4.13, the ring $\widetilde{R}$ is the polynomial ring $R_0[x_1, \dots, x_d, \{t_M\}]$ and the ideal $\widetilde{J}$ is generated by the variables $x_1, \dots, x_d$; thus, $\widetilde{J}$ is a proper ideal of grade d. Furthermore, the map

$$\boldsymbol{p}^\Phi_{n-1} : \operatorname{Sym}^{\widetilde{R}_{\boldsymbol{\delta}}}_{n-1} V_{\boldsymbol{\delta}} \to D^{\widetilde{R}_{\boldsymbol{\delta}}}_{n-1}(V^*_{\boldsymbol{\delta}})$$

is invertible because its determinant is the unit $\boldsymbol{\delta}$; see Definition 1.5 and Remark 1.6(1). The hypotheses of Theorem 4.7 are satisfied. The conclusions follow. □

Our second application of Theorem 4.7 is to the study of the ideals in the set $\mathbb{I}_n(R_0, U)$ of Definition 1.15. Fix the data (R_0, U, n), where R_0 is a commutative Noetherian ring, U is a free R_0-module of positive rank d, and n is a positive integer. Let $\widetilde{R}$, $\widetilde{I}$, and $\widetilde{\mathbb{G}}(n)_{\boldsymbol{\delta}}$, be the generic ring, ideal, and resolution created for the data (R_0, U, n) as described in Definition 4.11 and Corollary 4.16. Let P be the polynomial ring $P = \operatorname{Sym}^{R_0}_\bullet U$ and $I = \operatorname{ann}\phi$ be an ideal in P from the set $\mathbb{I}_n(R_0, U)$; so, in particular, $\phi \in D^{R_0}_{2n-2}(U^*)$ and $\det T_\phi$ (as described in Definition 1.5) is a unit of R_0. We prove in Corollary 4.22 that ϕ naturally induces a P-algebra homomorphism $\widehat{\phi} : \widetilde{R} \to P$ so that

(1) $\widehat{\phi}(\widetilde{I}) = I$, and
(2) $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)_{\boldsymbol{\delta}}$ is a resolution of P/I by free P-modules.

We first give the coordinate-free, official, description of $\widehat{\phi}$. The map $\widehat{\phi}$ exists for any $\phi \in D^{R_0}_{2n-2}(U^*)$. The condition that $\det T_\phi$ is a unit in R_0 is not needed until we establish properties of $\widehat{\phi}$. The rings P and $\widetilde{R}$ are defined to be

$$P = \operatorname{Sym}^{R_0}_\bullet U \quad \text{and} \quad \widetilde{R} = \operatorname{Sym}^{R_0}_\bullet\big(U \oplus \operatorname{Sym}^{R_0}_{2n-2} U\big);$$

thus, $\widetilde{R}$ is also equal to

$$\widetilde{R} = \operatorname{Sym}_\bullet^P\bigl(P \otimes_{R_0} \operatorname{Sym}_{2n-2}^{R_0} U\bigr).$$

The element ϕ of $D_{2n-2}^{R_0}(U^*)$ **IS** an R_0-module homomorphism $\operatorname{Sym}_{2n-2}^{R_0} U \to R_0$; therefore ϕ induces a P-module homomorphism

$$1 \otimes \phi : P \otimes_{R_0} \operatorname{Sym}_{2n-2}^{R_0} U \to P \otimes_{R_0} R_0 = P;$$

and, according to the defining property of symmetric algebra, $1 \otimes \phi$ induces a P-algebra homomorphism

$$\widehat{1 \otimes \phi} : \widetilde{R} = \operatorname{Sym}_\bullet^P\bigl(P \otimes_{R_0} \operatorname{Sym}_{2n-2}^{R_0} U\bigr) \to P, \tag{4.18}$$

which we abbreviate as $\widehat{\phi} : \widetilde{R} \to P$. In practice, $\widetilde{R} = P[\{t_M\}]$ for indeterminates t_M as described in (4.14) and $\widehat{\phi}$ is the P-algebra homomorphism

$$\widehat{\phi} : \widetilde{R} = P\bigl[\{t_M\}\bigr] \to P, \quad \text{with } \widehat{\phi}(t_M) = \phi(M) \in R_0. \tag{4.19}$$

Notice that the element ϕ of $D_{2n-2}^{R_0}(U^*)$ has the form

$$\phi = \sum_{(4.14)} \tau_{x_1^{a_1}\cdots x_d^{a_d}} x_1^{*(a_1)} \cdots x_d^{*(a_d)} \in D_{2n-2}^{R_0} U^*, \tag{4.20}$$

for some elements τ_M in R_0. Apply ϕ to the monomial M from (4.14) to see that $\tau_M = \phi(M)$. It follows that if $\Phi \in \widetilde{R} \otimes_{R_0} D_{2n-2}^{R_0}(U^*)$ is the element of (4.15) and (4.12), and P is an $\widetilde{R}$-algebra by way of $\widehat{\phi} : \widetilde{R} \to P$, then

$$P \otimes_{\widetilde{R}} \Phi \quad \text{is equal to} \quad 1 \otimes \phi \in P \otimes_{R_0} D_{2n-2}^{R_0}\bigl(U^*\bigr). \tag{4.21}$$

Corollary 4.22. *Fix (R_0, U, n), where R_0 is a commutative Noetherian ring, U is a free R_0-module of positive rank d, and n is a positive integer. Let $\widetilde{R} = \widetilde{R}(R_0, U, n)$, $\widetilde{I} = \widetilde{I}(R_0, U, n)$, $\widetilde{J} = \widetilde{J}(R_0, U, n)$, and, for each integer r with $1 \le r \le 2n-2$, $\widetilde{\mathbb{G}}(r) = \widetilde{\mathbb{G}}(R_0, U, n; r)$ be the generic ring, ideals, and complexes of Definition 4.11. Let $\boldsymbol{\delta}$ be the element $\det T_\Phi$ in $\widetilde{R}$ as described in Definition 4.11 and Remark 4.17, P be the polynomial ring $P = \operatorname{Sym}_\bullet^{R_0} U$, and J be the ideal $(\operatorname{Sym}_1^{R_0}(U))$ of P. Let $I = \operatorname{ann}\phi$ be an ideal of P from the set $\mathbb{I}_n(R_0, U)$ of Definition 1.15 with $\phi \in D_{2n-2}^{R_0}(U^*)$ and $\det T_\phi$ a unit of R_0. View P as an $\widetilde{R}$-algebra by way of the P-algebra homomorphism $\widehat{\phi} : \widetilde{R} \to P$ of (4.18) and (4.19). The following statements hold.*

(1) *The ideals $\widehat{\phi}(\widetilde{J}^\rho \widetilde{I})$ and $J^\rho I$ of P are equal, for all non-negative integers ρ.*
(2) *The complex $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n+\rho)_{\boldsymbol{\delta}}$ is a resolution of $P/J^\rho I$ by free P-modules, for $0 \le \rho \le n-2$.*

(3) *The complex* $P\otimes_{\widetilde{R}}\widetilde{\mathbb{G}}(n-1-\rho)_{\boldsymbol{\delta}}$ *is a resolution of* $\operatorname{Ext}^d_P(P/J^\rho I, P)$ *by free* P*-modules, for* $0 \le \rho \le n-2$.
(4) *The complexes* $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n)_{\boldsymbol{\delta}}$ *and* $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(n-1)_{\boldsymbol{\delta}}$ *both resolve*

$$P/I = \operatorname{Ext}^d_P(P/I, P).$$

(5) *Each of the ideals* $J^\rho I$*, with* $0 \le \rho \le n-2$*, is a perfect, grade* d*, ideal of* P.
(6) *The ideal* I *is a perfect, grade* d*, Gorenstein ideal of* P.
(7) *Each of the natural ring homomorphisms*

$$\operatorname{Sym}^{R_0}_{\bullet}\left(\operatorname{Sym}^{R_0}_{2n-2} U\right)_{\boldsymbol{\delta}} \to \left(\frac{\widetilde{R}}{\widetilde{J}^\rho \widetilde{I}}\right)_{\boldsymbol{\delta}}, \tag{4.23}$$

for $0 \le \rho \le n-2$ *is flat.*

Remark 4.24. In the language of Remark 4.13 the ring homomorphism (4.23) is

$$R_0\left[\{t_M\}\right]_{\boldsymbol{\delta}} \to \left(\frac{R_0[x_1, \dots, x_d, \{t_M\}]}{\widetilde{J}^\rho \widetilde{I}}\right)_{\boldsymbol{\delta}}. \tag{4.25}$$

In particular, when $\rho = 0$ and $R_0 = \boldsymbol{k}$ is a field, then

$$\boldsymbol{k}\left[\{t_M\}\right]_{\boldsymbol{\delta}} \to \left(\frac{\widetilde{R}}{\widetilde{I}}\right)_{\boldsymbol{\delta}} \tag{4.26}$$

is a flat family of $\boldsymbol{k}$-algebras parameterized by $\mathbb{I}^{[d]}_n(\boldsymbol{k})$ in the sense that every algebra $\boldsymbol{k}[x_1, \dots, x_d]/I$, with $I \in \mathbb{I}^{[d]}_n(\boldsymbol{k})$, is a fiber of (4.26).

Proof of Corollary 4.22. The element Φ of $D^{\widetilde{R}}_{2n-2}(\widetilde{R} \otimes_{R_0} U)$ is defined in (4.12). Recall from (4.21) that $P \otimes_{\widetilde{R}} \Phi = 1 \otimes \phi$ in $P \otimes_{\widetilde{R}} D^{R_0}_{2n-2}(U^*)$. One consequence is that $\widehat{\phi}$ carries the element $\boldsymbol{\delta}$ of $\widetilde{R}$ to the unit $\det T_\phi$ of R_0; therefore, $\widehat{\phi} : \widetilde{R} \to P$ automatically induces a well-defined P-algebra homomorphism $\widehat{\phi} : \widetilde{R}_{\boldsymbol{\delta}} \to P$. The complexes $\widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ are resolutions by Corollary 4.16. The complexes $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ are resolutions by Theorem 4.7 because the two hypotheses of Theorem 4.7 are satisfied. The first hypothesis concerns the grade of the ideal $\widehat{\phi}(\widetilde{J})$. We see that $\widehat{\phi}(\widetilde{J})$ is equal to the ideal J of P and that this ideal has grade at least d. (Indeed, when one uses the language of Remark 4.13, J is the ideal $(x_1, \dots, x_d)$ in the polynomial ring $P = R_0[x_1, \dots, x_d]$.) The second hypothesis concerns the element $\det T_{P\otimes_{\widetilde{R}}\Phi}$ of P. We have already observed that $P \otimes_{\widetilde{R}} \Phi = 1 \otimes \phi$. It follows that $\det T_{P\otimes_{\widetilde{R}}\Phi}$ is equal to the unit $\det T_\phi$ of P.

We use the language of Remark 4.13 and (4.20) to prove (1). Let ρ be a non-negative integer. We saw in Observation 3.9(4b) that the ideal $\widetilde{J}^\rho \widetilde{I}$ of $\widetilde{R}$ is built in a polynomial manner from the data $\{x_i\} \cup \{t_M\}$ and that the ideal $J^\rho I$ of P is built using the same polynomials from the data $\{x_i\} \cup \{\tau_M\}$. The P-algebra homomorphism $\widehat{\phi} : \widetilde{R} \to P$ carries t_M to τ_M. Thus, $\widehat{\phi}(\widetilde{J}^\rho \widetilde{I}) = J^\rho I$ and (1) is established. Assertions (2)–(6) now follow immediately from Theorem 4.7.

Assertion (7) is essentially obvious. Fix ρ, with $0 \le \rho \le n-2$. We use the language of Remark 4.13. To prove that (4.25) is flat, it suffices to prove the result locally and therefore, according to the local criterion for flatness, (see, for example, [27, Thm. 49] or [15, Thm. 6.8]) it suffices to prove that

$$\operatorname{Tor}_1^A\left(\frac{A}{\mathfrak{p}A}, \frac{B}{\widetilde{J}^\rho \widetilde{I} B}\right) = 0, \tag{4.27}$$

where $A = R_0[\{t_M\}]_{\mathfrak{p}}$, $B = R_0[\{x_i\}, \{t_M\}]_{\mathfrak{P}}$, $\mathfrak{P}$ is a prime ideal of $R_0[\{x_i\}, \{t_M\}]$ which contains $\widetilde{J}^\rho \widetilde{I}$ and does not contain $\boldsymbol{\delta}$, and $\mathfrak{p} = R_0[\{t_M\}] \cap \mathfrak{P}$. Apply Theorem 4.7 to

$$\frac{A}{\mathfrak{p}A} \otimes_{\widetilde{R}_{\mathfrak{P}}} \widetilde{\mathbb{G}}(n+\rho)_{\mathfrak{P}}, \tag{4.28}$$

which is the complex $\mathbb{G}(n+\rho)$ built over the field $\frac{A}{\mathfrak{p}A}$ with Macaulay inverse system $\frac{A}{\mathfrak{p}A} \otimes_{\widetilde{R}_{\mathfrak{P}}} \Phi$. The Macaulay inverse system still induces an isomorphism from Sym_{n-1} to D_{n-1} (because a ring homomorphism cannot send a unit to zero), and the image of $\widetilde{J}$ in B is still a proper ideal (recall, from the proof of Theorem 4.7, that $\widetilde{J}$ and $\widetilde{I}$ have the same radical) of grade at least d. Thus, (4.28) is a resolution and (4.27) holds. □

5. Examples of the resolution $\widetilde{\mathbb{G}}(n)$

In Theorem 5.2 we prove that the complex $\widetilde{\mathbb{G}}(r)$ is a monomial complex. To do this we introduce the standard bases for the Schur and Weyl modules that we call $L_{p,q}$ and $K_{p,q}$. In Example 5.15, we use these bases to exhibit the matrices of $\widetilde{\mathbb{G}}(r)$, when $d = n = r = 3$. The matrices of $\widetilde{\mathbb{G}}(r)$, when $d = 3$ and $n = r = 2$ are given at the beginning of Example 6.21.

Data 5.1. Consider (R_0, U, n), where R_0 is a commutative Noetherian ring, U is a free R_0-module of rank d, and n is a positive integer. Let $\widetilde{R} = \widetilde{R}(R_0, U, n)$, $\widetilde{I} = \widetilde{I}(R_0, U, n)$, $\widetilde{J} = \widetilde{J}(R_0, U, n)$, and $\widetilde{\mathbb{G}}(r) = \widetilde{\mathbb{G}}(R_0, U, n; r)$, for $1 \le r \le 2n-2$, be the generic ring, the generic ideals, and the generic complexes of Definition 4.11. Write $L_{p,q}$ and $K_{p,q}$ for $L_{p,q}^{R_0} U$ and $K_{p,q}^{R_0} U$, respectively. Let Φ be the element of $\widetilde{R} \otimes_{R_0} D_{2n-2}^{R_0}(U^*)$ which is described in (4.12) and (4.15), and let $\boldsymbol{\delta}$ be the element $\det T_\Phi$ of $\widetilde{R}$ as described in Corollary 4.16 and Remark 4.17.

Theorem 5.2. *Adopt Data 5.1. Then one can choose bases for the free modules of $\widetilde{\mathbb{G}}(r)$ so every entry of every matrix is a signed monomial.*

Proof. Let $x_1,\dots,x_d$ be a basis for U and think of $\widetilde{R}$ as the polynomial ring $R_0[x_1,\dots,x_d,\{t_M\}]$ as described in Remark 4.13. We see from Definition 4.4 that $\widetilde{\mathbb{G}}(r)$ is the mapping cone of

$$\begin{array}{ccccccccccc} & & \widetilde{R}\otimes_{R_0} L_{d-1,r} & \to & \dots & \to & \widetilde{R}\otimes_{R_0} L_{p,r} & \to & \dots & \to & \widetilde{R}\otimes_{R_0} L_{0,r} & \to & \widetilde{R} \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ \widetilde{R}\otimes_{R_0} \bigwedge^d_{R_0} U & \to & \widetilde{R}\otimes_{R_0} K_{d-1,2n-2-r} & \to & \dots & \to & \widetilde{R}\otimes_{R_0} K_{p,2n-2-r} & \to & \dots & \to & \widetilde{R}\otimes_{R_0} K_{0,2n-2-r}. & & \end{array} \tag{5.3}$$

The top complex in the above diagram is a minimal resolution of $\widetilde{R}/(x_1,\dots,x_d)^r$ by free $\widetilde{R}$-modules and the bottom complex is the dual of a minimal resolution of $\widetilde{R}/(x_1,\dots,x_d)^{2n-1-r}$. Both of these complexes are naturally monomial complexes. We will use well-understood bases for each of the modules; and therefore, it will not be difficult to demonstrate that the matrices for the horizontal maps have monomial entries. (We consider the horizontal maps at the end of the proof.) The interesting part of the proof involves the vertical maps.

Let $\omega = x_1\wedge\dots\wedge x_d$ be the basis for $\bigwedge^d_{R_0} U$. There are standard bases for $L_{p,q}$ and $K_{p,q}$. We will define these bases and make a few remarks. More details may be found in Remark 2.5, [34], [4, Sect. III.1], and elsewhere. These bases are usually exhibited as tableau; our tableau are simply hooks, so we will simply record the information without distinguishing between the row and the column. The basis for $L_{p,q}$ is

$$\left\{ \ell_{\boldsymbol{a};\boldsymbol{b}} \;\middle|\; \begin{array}{l} \boldsymbol{a} \text{ is } a_1<\dots<a_{p+1}, \\ \boldsymbol{b} \text{ is } b_1\le\dots\le b_{q-1}, \text{ and} \\ a_1\le b_1 \end{array} \right\} \tag{5.4}$$

and the basis for $K_{p,q}$ is

$$\left\{ k_{\boldsymbol{a};\boldsymbol{b}} \;\middle|\; \begin{array}{l} \boldsymbol{a} \text{ is } a_1<\dots<a_{d-p-1}, \\ \boldsymbol{b} \text{ is } b_1\le\dots\le b_{q+1}, \text{ and} \\ b_1< a_1 \end{array} \right\}, \tag{5.5}$$

where

$$\begin{aligned} \ell_{\boldsymbol{a};\boldsymbol{b}} &= \kappa(x_{a_1}\wedge\dots\wedge x_{a_{p+1}}\otimes x_{b_1}\cdot\ldots\cdot x_{b_{q-1}}) \in L_{p,q}\subseteq \textstyle\bigwedge^p_{R_0}U\otimes_{R_0}\operatorname{Sym}^{R_0}_q U \quad \text{and} \\ k_{\boldsymbol{a};\boldsymbol{b}} &= \eta\big(\big(x^*_{a_1}\wedge\dots\wedge x^*_{a_{d-p-1}}\big)(\omega)\otimes x_1^{*(\beta_1)}\cdot\ldots\cdot x_d^{*(\beta_d)}\big)\in K_{p,q}\subseteq \textstyle\bigwedge^p_{R_0}U\otimes_{R_0} D^{R_0}_q(U^*), \end{aligned}$$

for

$$\boldsymbol{b} = (\underbrace{1,\dots,1}_{\beta_1},\underbrace{2,\dots,2}_{\beta_2},\dots,\underbrace{d,\dots,d}_{\beta_d}), \quad \text{with } \sum\beta_i = q+1. \tag{5.6}$$

The maps κ and η are defined in (2.2). To show that (5.4) is a basis for $L_{p,q}$ one can verify that (5.4) contains $\operatorname{rank} L_{p,q}$ elements (see (2.3)) and that (5.4) spans $\kappa(\bigwedge^{p+1}_{R_0} U\otimes_{R_0}$

$\mathrm{Sym}^{R_0}_{q-1} U) = L_{p,q}$. The assertion about spanning is obvious. Indeed, if $X = x_{a_1} \wedge \cdots \wedge x_{a_{p+1}} \otimes x_{b_1} \cdots x_{b_{q-1}} \in \bigwedge^{p+1}_{R_0} U \otimes_{R_0} \mathrm{Sym}^{R_0}_{q-1} U$ with $a_1 < \cdots < a_{p+1}$, $b_1 \le \cdots \le b_{q-1}$, **but** $b_1 < a_1$, then

$$\begin{aligned} 0 &= \kappa\kappa(x_{b_1} \wedge x_{a_1} \wedge \cdots \wedge x_{a_{p+1}} \otimes x_{b_2} \cdots x_{b_{q-1}}) \\ &= \kappa(X) + \text{a linear combination of elements of (5.4) with coefficients from } \{+1, -1\}. \end{aligned} \tag{5.7}$$

In a similar manner, one shows that (5.5) is a basis for $K_{p,q}$ by showing that (5.5) contains

$$\operatorname{rank} K_{p,q}(U) = \binom{d+q}{p}\binom{d+q-p-1}{q} \tag{5.8}$$

elements (we used (2.3) and (2.4) to compute this number) and if

$$X = \left(x^*_{a_1} \wedge \cdots \wedge x^*_{a_{d-p-1}}\right)(\omega) \otimes x_1^{*(\beta_1)} \cdot \ldots \cdot x_d^{*(\beta_d)} \in \textstyle\bigwedge^{p+1}_{R_0} U \otimes_{R_0} D^{R_0}_{q+1}(U^*)$$

with $a_1 < \cdots < a_{d-p-1}$, $b_1 \le \cdots \le b_{q+1}$ for $(b_1, \ldots, b_{q+1})$ given in (5.6), **but** $a_1 \le b_1$, then

$$\begin{aligned} 0 &= \begin{cases} \eta\eta((x^*_{a_2} \wedge \cdots \wedge x^*_{a_{d-p-1}})(\omega) \otimes x^*_{a_1} x^*_{b_1}{}^{(\beta_{b_1})} \cdot \ldots \cdot x_d^{*(\beta_d)}) & \text{if } a_1 < b_1 \\ \eta\eta((x^*_{a_2} \wedge \cdots \wedge x^*_{a_{d-p-1}})(\omega) \otimes x^*_{b_1}{}^{(\beta_{b_1}+1)} \cdot \ldots \cdot x_d^{*(\beta_d)}) & \text{if } a_1 = b_1 \end{cases} \\ &= \eta(X) + \text{a linear combination of elements of (5.5) with coefficients from } \{+1, -1\}. \end{aligned} \tag{5.9}$$

The critical calculation involves a careful analysis of (5.9).

Claim 5.10. *Fix positive integers P, Q, and $a_1 < \cdots < a_P$, with $P \le d-1$, $Q \le 2n-2$, and $a_P \le d$. Let $\beta_1, \ldots, \beta_d$ vary over all choices of non-negative integers with $\sum \beta_i = Q$. We claim that when all elements of the form*

$$\eta\left(\left(x^*_{a_1} \wedge \cdots \wedge x^*_{a_P}\right)(\omega) \otimes x_1^{*(\beta_1)} \ldots x_d^{*(\beta_d)}\right) \tag{5.11}$$

are written in terms of the basis elements $\{k_{\boldsymbol{a};\boldsymbol{b}}\}$ of $K_{d-P-1,Q-1}$, as given in (5.5), then any given basis element $k_{\boldsymbol{a};\boldsymbol{b}}$ appears ***at most once****.*

Proof. Consider $\beta_1, \ldots, \beta_d$, with $\sum \beta_i = Q$. Let b_1 be the least index with $\beta_{b_1} \ne 0$.

If $b_1 < a_1$, then the expression (5.11) is the basis element $k_{\boldsymbol{a};\boldsymbol{b}}$ for

$$\boldsymbol{a} \text{ equal to } a_1 < \cdots < a_P \quad \text{and} \quad \boldsymbol{b} \text{ equal to } \underbrace{b_1, \ldots, b_1}_{\beta_{b_1}}, \cdots, \underbrace{d, \ldots, d}_{\beta_d}. \tag{5.12}$$

If $a_1 \le b_1$, then the expression (5.11) involves the basis elements $k_{\boldsymbol{a};\boldsymbol{b}}$ for:

$$\left\{ \begin{array}{l} \boldsymbol{a} \text{ equal to } j, a_2, \dots, a_P \text{ written in strictly ascending order} \\ \boldsymbol{b} \text{ equal to } a_1, \underbrace{b_1, \dots, b_1}_{\beta_{b_1}}, \dots \underbrace{j, \dots, j}_{\beta_j - 1}, \cdots, \underbrace{d, \dots, d}_{\beta_d} \end{array} \middle| \begin{array}{c} a_1 < j \\ \text{and} \\ 1 \le \beta_j \end{array} \right\}. \tag{5.13}$$

Now one notices that given $k_{\boldsymbol{a};\boldsymbol{b}}$ as described in (5.12) or (5.13), one can recreate the unique d-tuple $(\beta_1, \dots, \beta_d)$ so that (5.11) involves the basis element $k_{\boldsymbol{a};\boldsymbol{b}}$. This completes the proof of Claim 5.10. □

We show that the matrix for the vertical map

$$1 \otimes \boldsymbol{p}_r^{\Phi} : \widetilde{R} \otimes_{R_0} L_{p,r} \to \widetilde{R} \otimes_{R_0} K_{p,2n-2-r}$$

has monomial entries. (In fact, each non-zero entry in the matrix is plus or minus a variable from $\widetilde{R}$.) In this calculation, $0 \le p \le d-1$ and $1 \le r \le 2n-2$. Let

$$\ell = \kappa\big(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes x_1^{\beta_1} \dots x_d^{\beta_d}\big)$$

be a basis element in $L_{p,r}$; so, in particular, $a_1 < \dots < a_{p+1}$ and $\sum \beta_i = r-1$. (Let b_1 be the least index with $\beta_{b_1} \ne 0$. The inequality $a_1 \le b_1$ also holds; but this inequality will play no role in the present calculation.) It is easy to see that

$$\big(1 \otimes \boldsymbol{p}_r^{\Phi}\big) \circ \kappa = \eta \circ \big(1 \otimes \boldsymbol{p}_{r-1}^{\Phi}\big).$$

Recall that

$$\Phi = \sum_{\sum C_i = 2n-2} t_{x_1^{C_1} \dots x_d^{C_d}} x_1^{*(C_1)} \cdots x_d^{*(C_d)},$$

where the sum is taken over all non-integers $C_1, \dots C_d$ with $\sum C_i = 2n-2$. Of course, t_M is a monomial (indeed, even a variable) in $\widetilde{R}$ for all monomials $M = x_1^{C_1} \cdots x_d^{C_d}$ of degree $2n-2$. It follows that

$$\begin{aligned} \big(1 \otimes \boldsymbol{p}_r^{\Phi}\big)(\ell) &= \big(1 \otimes \boldsymbol{p}_r^{\Phi}\big)\big(\kappa\big(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes x_1^{\beta_1} \cdots x_d^{\beta_d}\big)\big) \\ &= \eta\big(\big(1 \otimes \boldsymbol{p}_{r-1}^{\Phi}\big)\big(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes x_1^{\beta_1} \cdots x_d^{\beta_d}\big)\big) \\ &= \eta\big(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes \big(x_1^{\beta_1} \cdots x_d^{\beta_d}\big)(\Phi)\big) \\ &= \eta\Bigg(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes \sum_{\substack{\sum C_i = 2n-2 \\ 0 \le C_i - \beta_i}} t_{x_1^{C_1} \dots x_d^{C_d}} x_1^{*(C_1 - \beta_1)} \cdots x_d^{*(C_d - \beta_d)}\Bigg) \\ &= \sum_{\sum c_i = 2n-1-r} t_{x_1^{\beta_1 + c_1} \dots x_d^{\beta_d + c_d}} \eta\big(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes x_1^{*(c_1)} \cdot \dots \cdot x_d^{*(c_d)}\big), \end{aligned}$$

where the most recent sum is taken over all non-negative integers $c_1, \dots, c_d$ with $\sum c_i = 2n - 1 - r$. Let $A_1 < \dots < A_{d-p-1}$ be the complement of $a_1 < \dots < a_{p+1}$ in $\{1, \dots, d\}$. Observe that

$$x_{a_1} \wedge \dots \wedge x_{a_{p+1}} = \pm\big(x^*_{A_1} \wedge \dots \wedge x^*_{A_{d-p-1}}\big)(\omega).$$

We have shown that the vertical map $1 \otimes \boldsymbol{p}_r^{\Phi}$ sends the basis vector ℓ to a sum of terms of the form

$$\text{monomial} \cdot \eta\big(\big(x^*_{A_1} \wedge \dots \wedge x^*_{A_{d-p-1}}\big)(\omega) \otimes x_1^{*(c_1)} \cdot \dots \cdot x_d^{*(c_d)}\big), \tag{5.14}$$

with $A_1, \dots A_{d-p-1}$ fixed and $c_1, \dots, c_d$ allowed to vary under the constraint that $\sum c_i$ is fixed. Claim 5.10 shows that when the elements of (5.14) are written in terms of the basis $\{k_{\boldsymbol{a};\boldsymbol{b}}\}$ of $K_{p,2n-2-r}$, then any given basis element $k_{\boldsymbol{a};\boldsymbol{b}}$ appears at most once. In other words, the vertical map is a monomial map: each non-zero entry in the resulting matrix is plus or minus a variable.

As promised, we now consider the horizontal maps. The maps

$$\widetilde{R} \otimes_{R_0} L_{0,r} \to R_0 \quad \text{and} \quad \widetilde{R} \otimes_{R_0} \textstyle\bigwedge^d U \to \widetilde{R} \otimes K_{d-1,2n-2-r}$$

merely list all of the monomials in $x_1, \dots, x_d$ of degree r and degree $2n-1-r$, respectively. The map $\widetilde{R} \otimes_{R_0} L_{p,r} \to \widetilde{R} \otimes_{R_0} L_{p-1,r}$ sends the basis element

$$\ell_{\boldsymbol{a};\boldsymbol{b}} = \kappa(x_{a_1} \wedge \dots \wedge x_{a_{p+1}} \otimes x_{b_1} \cdots x_{b_{r-1}})$$

of $L_{p,r}$ (with $a_1 < \dots < a_{p+1}$, $b_1 \le \dots \le b_{r-1}$, and $a_1 \le b_1$) to $A + B$, with $A = x_{a_1} \cdot \kappa(x_{a_2} \wedge \dots \wedge x_{a_{p+1}} \otimes x_{b_1} \cdots x_{b_{r-1}})$ and

$$B = \sum_{i=2}^{p+1} (-1)^{i+1} x_{a_i} \cdot \kappa(x_{a_1} \wedge \dots \wedge \widehat{x_{a_i}} \wedge \dots \wedge x_{a_{p+1}} \otimes x_{b_1} \cdots x_{b_{r-1}})$$

The sum B is already written in terms of basis elements of $L_{p-1,r}$. If necessary, one can use the technique of (5.7) to write A in terms of basis elements. It is not difficult to see that the basis elements needed for A are distinct from the basis elements used in B.

Finally, we consider the map $\widetilde{R} \otimes_{R_0} K_{p,r} \to \widetilde{R} \otimes_{R_0} K_{p-1,r}$ applied to the basis element

$$k_{\boldsymbol{a};\boldsymbol{b}} = \eta\big(\big(x^*_{a_1} \wedge \dots \wedge x^*_{a_{d-p-1}}\big)(\omega) \otimes x_1^{*(\beta_1)} \cdot \dots \cdot x_d^{*(\beta_d)}\big)$$

of $K_{p,r}$ with $a_1 < \dots < a_{d-p-1}$, $b_1 \le \dots \le b_{r+1}$, and $b_1 < a_1$ for

$$(b_1, \dots, b_{r+1}) = (\underbrace{1, \dots, 1}_{\beta_1}, \underbrace{2, \dots, 2}_{\beta_2}, \dots, \underbrace{d, \dots, d}_{\beta_d}).$$

The basis element $k_{\boldsymbol{a};\boldsymbol{b}}$ is sent to $A+B$ with

$$A = \sum_{i \le b_1} x_i \eta\big(\big(x_i^* \wedge x_{a_1}^* \wedge \cdots \wedge x_{a_{d-p-1}}^*\big)(\omega) \otimes x_1^{*(\beta_1)} \cdot \ldots \cdot x_d^{*(\beta_d)}\big)$$

$$B = \sum_{b_1 < i} x_i \eta\big(\big(x_i^* \wedge x_{a_1}^* \wedge \cdots \wedge x_{a_{d-p-1}}^*\big)(\omega) \otimes x_1^{*(\beta_1)} \cdot \ldots \cdot x_d^{*(\beta_d)}\big).$$

The sum B is already written in terms of basis elements of $K_{p-1,r}$. One can use the technique of (5.9) to write A in terms of basis elements. It is not difficult to see that the basis elements needed for A are distinct from the basis elements used in B. $\square$

Example 5.15. Adopt Data 5.1 with $d=3$. Let x, y, z be a basis for the free R_0-module U. We exhibit the resolution $\widetilde{\mathbb{G}}(3) = \widetilde{\mathbb{G}}(R_0, U, 3; 3)$ over $\widetilde{R} = R_0[z, y, z, \{t_M\}]$, as M roams over the fifteen monomials of degree 4 in x, y, z. The resolution $\widetilde{\mathbb{G}}(3)$ is the mapping cone of

$$\begin{array}{ccccccccc} & & 0 & \to & \widetilde{R} \otimes_{R_0} L_{2,3} & \to & \widetilde{R} \otimes_{R_0} L_{1,3} & \to & \widetilde{R} \otimes_{R_0} L_{0,3} & \to & \widetilde{R} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{R} \otimes_{R_0} \bigwedge^3_{R_0} U & \to & \widetilde{R} \otimes_{R_0} K_{2,1} & \to & \widetilde{R} \otimes_{R_0} K_{1,1} & \to & \widetilde{R} \otimes_{R_0} K_{0,1}. & & \end{array}$$

We record the matrices using the bases $\ell_{\boldsymbol{a};\boldsymbol{b}}$ and $k_{\boldsymbol{a};\boldsymbol{b}}$ of (5.4) and (5.5):

$L_{2,3}$	$L_{1,3}$	$L_{0,3}$	$K_{2,1}$	$K_{1,1}$	$K_{0,1}$
$\ell_{1,2,3;1,1}$	$\ell_{1,2;1,1}$	$\ell_{1;1,1}$	$k_{\_;1,1}$	$k_{2;1,1}$	$k_{2,3;1,1}$
$\ell_{1,2,3;1,2}$	$\ell_{1,2;1,2}$	$\ell_{1;1,2}$	$k_{\_;1,2}$	$k_{2;1,2}$	$k_{2,3;1,2}$
$\ell_{1,2,3;1,3}$	$\ell_{1,2;1,3}$	$\ell_{1;1,3}$	$k_{\_;1,3}$	$k_{2;1,3}$	$k_{2,3;1,3}$
$\ell_{1,2,3;2,2}$	$\ell_{1,2;2,2}$	$\ell_{1;2,2}$	$k_{\_;2,2}$	$k_{3;1,1}$	
$\ell_{1,2,3;2,3}$	$\ell_{1,2;2,3}$	$\ell_{1;2,3}$	$k_{\_;2,3}$	$k_{3;1,2}$	
$\ell_{1,2,3;3,3}$	$\ell_{1,2;3,3}$	$\ell_{1;3,3}$	$k_{\_;3,3}$	$k_{3;1,3}$	
	$\ell_{1,3;1,1}$	$\ell_{2;2,2}$		$k_{3;2,2}$	
	$\ell_{1,3;1,2}$	$\ell_{2;2,3}$		$k_{3;2,3}$	
	$\ell_{1,3;1,3}$	$\ell_{2;3,3}$			
	$\ell_{1,3;2,2}$	$\ell_{3;3,3}$			
	$\ell_{1,3;2,3}$				
	$\ell_{1,3;3,3}$				
	$\ell_{2,3;2,2}$				
	$\ell_{2,3;2,3}$				
	$\ell_{2,3;3,3}$				

We identify x_1 with x, x_2 with y, and x_3 with z. We take $x \wedge y \wedge z$ to be the basis for $\bigwedge^3_{R_0} U$. The resolution $\widetilde{\mathbb{G}}(3)$ then is the mapping cone of

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \widetilde{R}(-5,0)^6 & \xrightarrow{h_3} & \widetilde{R}(-4,0)^{15} & \xrightarrow{h_2} & \widetilde{R}(-3,0)^{10} & \xrightarrow{h_1} & \widetilde{R} \\
 & & \downarrow & & \downarrow{\scriptstyle v_3} & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_1} & & \\
0 & \longrightarrow & \widetilde{R}(-7,-1) & \xrightarrow{h_3'} & \widetilde{R}(-5,-1)^6 & \xrightarrow{h_2'} & \widetilde{R}(-4,-1)^8 & \xrightarrow{h_1'} & \widetilde{R}(-3,-1)^3 & &
\end{array}$$

with $h_1 = \left[\,x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3\,\right]$,

$$h_2 = \begin{bmatrix}
-y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & -y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -y & 0 & 0 & 0 & x & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & -y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & -y & 0 & 0 & x & 0 & 0 & -z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -y & 0 & 0 & x & 0 & 0 & -z & 0 & 0 & 0 \\
0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 \\
0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & x & 0 & 0 & y & -z & 0 \\
0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & x & 0 & 0 & y & -z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & y
\end{bmatrix},$$

$$h_3 = \begin{bmatrix}
z & 0 & 0 & 0 & 0 & 0 \\
0 & z & 0 & 0 & 0 & 0 \\
-x & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & z & 0 & 0 \\
0 & -x & 0 & 0 & z & 0 \\
0 & 0 & -x & 0 & 0 & z \\
-y & 0 & 0 & 0 & 0 & 0 \\
x & -y & 0 & 0 & 0 & 0 \\
0 & 0 & -y & 0 & 0 & 0 \\
0 & x & 0 & -y & 0 & 0 \\
0 & 0 & x & 0 & -y & 0 \\
0 & 0 & 0 & 0 & 0 & -y \\
0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & x
\end{bmatrix},$$

$$v_1 = \begin{bmatrix}
-t_{x^4} & -t_{x^3y} & -t_{x^3z} & -t_{x^2y^2} & -t_{x^2yz} & -t_{x^2z^2} & -t_{xy^3} & -t_{xy^2z} & -t_{xyz^2} & -t_{xz^3} \\
-t_{x^3y} & -t_{x^2y^2} & -t_{x^2yz} & -t_{xy^3} & -t_{xy^2z} & -t_{xyz^2} & -t_{y^4} & -t_{y^3z} & -t_{y^2z^2} & -t_{yz^3} \\
-t_{x^3z} & -t_{x^2yz} & -t_{x^2z^2} & -t_{xy^2z} & -t_{xyz^2} & -t_{xz^3} & -t_{y^3z} & -t_{y^2z^2} & -t_{yz^3} & -t_{z^4}
\end{bmatrix},$$

v_2 is

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -t_{x^4} & -t_{x^3y} & -t_{x^3z} & -t_{x^2y^2} & -t_{x^2yz} & -t_{x^2z^2} & -t_{xy^3} & -t_{xy^2z} & -t_{xyz^2} \\
0 & 0 & 0 & 0 & 0 & 0 & -t_{x^3y} & -t_{x^2y^2} & -t_{x^2yz} & -t_{xy^3} & -t_{xy^2z} & -t_{xyz^2} & -t_{y^4} & -t_{y^3z} & -t_{y^2z^2} \\
0 & 0 & 0 & 0 & 0 & 0 & -t_{x^3z} & -t_{x^2yz} & -t_{x^2z^2} & -t_{xy^2z} & -t_{xyz^2} & -t_{xz^3} & -t_{y^3z} & -t_{y^2z^2} & -t_{yz^3} \\
t_{x^4} & t_{x^3y} & t_{x^3z} & t_{x^2y^2} & t_{x^2yz} & t_{x^2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & -t_{xy^2z} & -t_{xyz^2} & -t_{xz^3} \\
t_{x^3y} & t_{x^2y^2} & t_{x^2yz} & t_{xy^3} & t_{xy^2z} & t_{xyz^2} & 0 & 0 & 0 & 0 & 0 & 0 & -t_{y^3z} & -t_{y^2z^2} & -t_{yz^3} \\
t_{x^3z} & t_{x^2yz} & t_{x^2z^2} & t_{xy^2z} & t_{xyz^2} & t_{xz^3} & 0 & 0 & 0 & 0 & 0 & 0 & -t_{y^2z^2} & -t_{yz^3} & -t_{z^4} \\
t_{x^2y^2} & t_{xy^3} & t_{xy^2z} & t_{y^4} & t_{y^3z} & t_{y^2z^2} & t_{x^2yz} & t_{xy^2z} & t_{xyz^2} & t_{y^3z} & t_{y^2z^2} & t_{yz^3} & 0 & 0 & 0 \\
t_{x^2yz} & t_{xy^2z} & t_{xyz^2} & t_{y^3z} & t_{y^2z^2} & t_{yz^3} & t_{x^2z^2} & t_{xyz^2} & t_{xz^3} & t_{y^2z^2} & t_{yz^3} & t_{z^4} & 0 & 0 & 0
\end{bmatrix},$$

$$v_3 = \begin{bmatrix} t_{x^4} & t_{x^3y} & t_{x^3z} & t_{x^2y^2} & t_{x^2yz} & t_{x^2z^2} \\ t_{x^3y} & t_{x^2y^2} & t_{x^2yz} & t_{xy^3} & t_{xy^2z} & t_{xyz^2} \\ t_{x^3z} & t_{x^2yz} & t_{x^2z^2} & t_{xy^2z} & t_{xyz^2} & t_{xz^3} \\ t_{x^2y^2} & t_{xy^3} & t_{xy^2z} & t_{y^4} & t_{y^3z} & t_{y^2z^2} \\ t_{x^2yz} & t_{xy^2z} & t_{xyz^2} & t_{y^3z} & t_{y^2z^2} & t_{yz^3} \\ t_{x^2z^2} & t_{xyz^2} & t_{xz^3} & t_{y^2z^2} & t_{yz^3} & t_{z^4} \end{bmatrix},$$

$$h_1' = \begin{bmatrix} -z & 0 & x & y & -x & 0 & 0 & 0 \\ 0 & -z & 0 & 0 & y & 0 & -x & 0 \\ 0 & 0 & -z & 0 & 0 & y & 0 & -x \end{bmatrix},$$

$$h_2' = \begin{bmatrix} y & -x & 0 & 0 & 0 & 0 \\ 0 & y & 0 & -x & 0 & 0 \\ 0 & 0 & y & 0 & -x & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ 0 & z & 0 & 0 & -x & 0 \\ 0 & 0 & z & 0 & 0 & -x \\ 0 & 0 & 0 & z & -y & 0 \\ 0 & 0 & 0 & 0 & z & -y \end{bmatrix}, \quad \text{and} \quad h_3' = \begin{bmatrix} x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{bmatrix}.$$

6. The minimal resolution

In Theorem 6.15 we turn the resolution $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ of assertion (2) from Corollary 4.22 into a minimal resolution (when P is a polynomial ring over a field). Our calculations are made at the level of $\widetilde{R}$; and, when $r = n$, our calculations are coordinate-free. The explicit resolution $\widetilde{\mathbb{G}}'(2)_{\boldsymbol{\delta}}$ with $d = 3$ and $n = r = 2$ is recorded as Example 6.21.

Data 6.1. Consider (R_0, U, n, r), where R_0 is a commutative Noetherian ring, U is a free R_0-module of rank d, n is a positive integer, and r is an integer with $n \le r \le 2n-2$. Let $\widetilde{R} = \widetilde{R}(R_0, U, n)$, $\widetilde{I} = \widetilde{I}(R_0, U, n)$, $\widetilde{J} = \widetilde{J}(R_0, U, n)$, and $\widetilde{\mathbb{G}}(r) = \widetilde{\mathbb{G}}(R_0, U, n; r)$ be the generic ring, the generic ideals, and the generic complexes of Definition 4.11. Write $L_{p,q}$ and $K_{p,q}$ for $L_{p,q}^{R_0}U$ and $K_{p,q}^{R_0}U$, respectively. Let Φ be the element of $\widetilde{R} \otimes_{R_0} D_{2n-2}^{R_0}(U^*)$ which is described in (4.12) and (4.15), $\boldsymbol{\delta}$ be the element $\det T_{\Phi}$ of $\widetilde{R}$ as described in Corollary 4.16 and Remark 4.17, $\Psi : \widetilde{R} \otimes_{R_0} U \to \widetilde{R}$ be the multiplication map of (4.12) and $\widehat{\Psi} : \widetilde{R} \otimes_{R_0} \operatorname{Sym}_{\bullet}^{R_0} U \to \widetilde{R}$ be the $\widetilde{R}$-algebra map induced by the $\widetilde{R}$-module homomorphism $\widetilde{R} \otimes_{R_0} U \to \widetilde{R}$.

Retain Data 6.1. Apply $\widetilde{R}_{\boldsymbol{\delta}} \otimes_{\widetilde{R}} \_\_$ to the double complex (5.3) to obtain the double complex

$$\begin{array}{ccccccccccc} & & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} L_{d-1,r} & \to & \dots & \to & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} L_{p,r} & \to & \dots & \to & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} L_{0,r} & \to & \widetilde{R}_{\boldsymbol{\delta}} \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \bigwedge_{R_0}^d U & \to & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} K_{d-1,2n-2-r} & \to & \dots & \to & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} K_{p,2n-2-r} & \to & \dots & \to & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} K_{0,2n-2-r}. & & \end{array} \tag{6.2}$$

We know from assertion (3) of Corollary 4.16 that the mapping cone of (6.2) is the resolution $\widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^{r-n}\widetilde{I}$ by free $\widetilde{R}_{\boldsymbol{\delta}}$-modules. In this section we show that each of the vertical maps

$$\begin{array}{c} \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} L_{p,r} \\ {\scriptstyle 1\otimes \boldsymbol{p}_r^{\Phi}}\Big\downarrow \\ \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} K_{p,2n-2-r} \end{array} \tag{6.3}$$

in (6.2) splits and we split each of these vertical maps from $\widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ thereby producing a smaller resolution $\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$ of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^{r-n}\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$.

The $\widetilde{R}$-module homomorphisms which constitute $\widetilde{\mathbb{G}}'(r)$ are introduced in Definition 6.14. The modules $X_{p,r}$ which form the bulk of $\widetilde{\mathbb{G}}'(r)$ may be found in Definition 6.6. Most of the maps of $\widetilde{\mathbb{G}}'(r)$ are induced by maps of the form "Kos^{Ψ}"; these maps are shown to be legitimate in Observation 6.8. The final ingredient, a map called $\mathfrak{L}_r$, may be found in Definition 6.9 and in Remarks 6.11. The properties of $\widetilde{\mathbb{G}}'(r)$, and especially $\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$, are established in Theorem 6.15.

Claim 6.4. *Adopt Data 6.1. For each index p, with $0 \le p \le d-1$, the $\widetilde{R}_{\boldsymbol{\delta}}$-module homomorphism (6.3) is surjective.*

Proof. Observe first that the diagram

$$\begin{array}{ccc} \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \bigwedge_{R_0}^{p+1} U \otimes_{R_0} \mathrm{Sym}_{r-1}^{R_0} U & \xrightarrow{\ \kappa\ } & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \bigwedge_{R_0}^{p} U \otimes_{R_0} \mathrm{Sym}_{r}^{R_0} U \\ {\scriptstyle 1\otimes \boldsymbol{p}_{r-1}^{\Phi}}\Big\downarrow & & {\scriptstyle 1\otimes \boldsymbol{p}_{r}^{\Phi}}\Big\downarrow \\ \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \bigwedge_{R_0}^{p+1} U \otimes_{R_0} D_{2n-r-1}^{R_0}(U^*) & \xrightarrow{\ \eta\ } & \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \bigwedge_{R_0}^{p} U \otimes_{R_0} D_{2n-2-r}^{R_0}(U^*) \end{array} \tag{6.5}$$

commutes. The parameter $r-1$ is guaranteed to be at least $n-1$; so Remark 3.7(2) ensures that the vertical maps in (6.5) are surjective. The domain of (6.3) is the image of κ in (6.5), and the target of (6.3) is the image of η in (6.5). $\square$

Definition 6.6. Adopt Data 6.1. For each index p, with $0 \le p \le d-1$, define the $\widetilde{R}$-module $X_{p,r}$ to be the kernel of

$$\begin{array}{c} \widetilde{R} \otimes_{R_0} L_{p,r} \\ {\scriptstyle 1\otimes \boldsymbol{p}_r^{\Phi}}\Big\downarrow \\ \widetilde{R} \otimes_{R_0} K_{p,2n-2-r}. \end{array}$$

Remarks 6.7. 1. Notice that the $\widetilde{R}$-module $X_{p,r}$ is defined in a coordinate-free manner.
2. The R_0-module $K_{p,2n-2-r}$ is free; so it follows, from Claim 6.4, that $(X_{p,r})_{\boldsymbol{\delta}}$ is a projective $\widetilde{R}_{\boldsymbol{\delta}}$-module of rank:

$$\operatorname{rank} L_{p,r}U - \operatorname{rank} K_{p,2n-2-r}U.$$

Formulas for these ranks are given in (2.3) and (5.8).
3. The module $X_{d-1,n}$ is equal to zero because the diagram

$$\begin{array}{ccc} \widetilde{R}\otimes_{R_0}\bigwedge^{d}_{R_0}U\otimes_{R_0}\operatorname{Sym}^{R_0}_{n-1}U & \xrightarrow{\kappa} & \widetilde{R}\otimes_{R_0}\bigwedge^{d-1}_{R_0}U\otimes_{R_0}\operatorname{Sym}^{R_0}_{n}U \\ {\scriptstyle 1\otimes\boldsymbol{p}^{\Phi}_{n-1}}\big\downarrow \simeq & & {\scriptstyle 1\otimes\boldsymbol{p}^{\Phi}_{n}}\big\downarrow \\ \widetilde{R}\otimes_{R_0}\bigwedge^{d}_{R_0}U\otimes_{R_0}D^{R_0}_{n-1}(U^*) & \xrightarrow{\eta} & \widetilde{R}\otimes_{R_0}\bigwedge^{d}_{R_0}U\otimes_{R_0}D^{R_0}_{n-2}(U^*) \end{array}$$

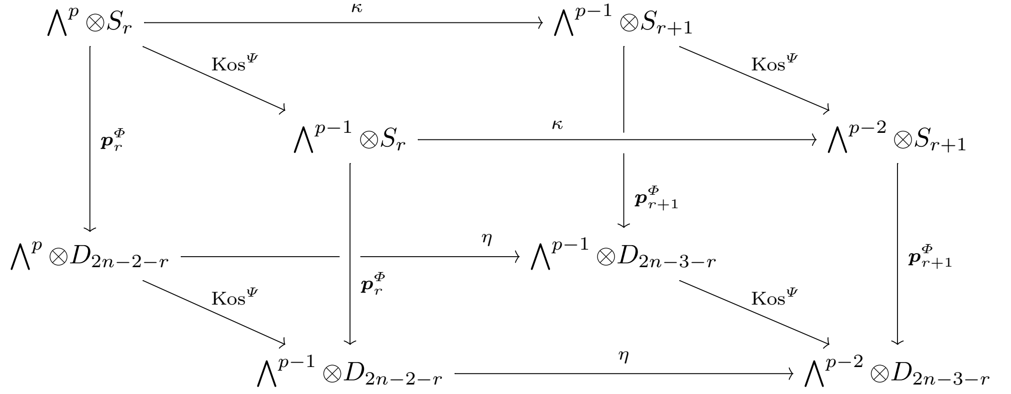
commutes.

Observation 6.8. *Adopt Data 6.1. Then the map*

$$\operatorname{Kos}^{\Psi}:\widetilde{R}\otimes_{R_0}\bigwedge^{p}_{R_0}U\to\widetilde{R}\otimes_{R_0}\bigwedge^{p-1}_{R_0}U,$$

from (2.6), induces an $\widetilde{R}$-module homomorphism $\operatorname{Kos}^{\Psi}:X_{p,r}\to X_{p-1,r}$.

Proof. The cube



commutes, where "$\bigwedge^a\otimes S_b$" represents $\widetilde{R}\otimes_{R_0}\bigwedge^{a}_{R_0}U\otimes_{R_0}\operatorname{Sym}^{R_0}_{b}U$ and "$\bigwedge^a\otimes D_b$" represents $\widetilde{R}\otimes_{R_0}\bigwedge^{a}_{R_0}U\otimes_{R_0}D^{R_0}_{b}(U^*)$. The $\widetilde{R}$-module $X_{p,r}$ is the set of elements in the back, left, top module which are sent to 0 in the back face of the cube. The elements of $X_{p,r}$

are sent to elements in the front, left, top module which go to zero in the front face of the cube. Such elements are in $X_{p-1.r}$. $\square$

A snake-like map $\mathfrak{L}_r : \widetilde{R}\otimes_{R_0}\bigwedge_{R_0}^d U \to \widetilde{R}\otimes_{R_0} L_{d-2,r}$ from the module on the bottom left of (5.3) to the second (from the left) non-zero module on the top complex of (5.3) plays a very important role in the complex $\widetilde{\mathbb{G}}'(r)$. Recall that for each index i, with $n-1\le i\le 2n-2$, the $\widetilde{R}$-module homomorphism

$$\boldsymbol{p}_i^{\Phi} : \widetilde{R}_{\boldsymbol{\delta}}\otimes_{R_0}\operatorname{Sym}_i^{R_0} U \to \widetilde{R}_{\boldsymbol{\delta}}\otimes_{R_0} D_{2n-2-i}^{R_0}\bigl(U^*\bigr)$$

is a surjection. A splitting map

$$\sigma_{2n-2-i} : \widetilde{R}_{\boldsymbol{\delta}}\otimes_{R_0} D_{2n-2-i}^{R_0}\bigl(U^*\bigr) \to \widetilde{R}_{\boldsymbol{\delta}}\otimes_{R_0}\operatorname{Sym}_i^{R_0} U$$

has been named in Remark 3.7(2). Careful examination of Remark 3.7(2) shows that the only denominator that occurs in the map σ_{2n-2-i} is $\boldsymbol{\delta}$; so, in fact,

$$\boldsymbol{\delta}\sigma_{2n-2-i} : \widetilde{R}\otimes_{R_0} D_{2n-2-i}^{R_0}\bigl(U^*\bigr) \to \widetilde{R}\otimes_{R_0}\operatorname{Sym}_i^{R_0} U$$

is an $\widetilde{R}$-module homomorphism.

Definition 6.9. Retain Data 6.1. Define the map

$$\mathfrak{L}_r : \widetilde{R}\otimes_{R_0}\bigwedge\nolimits_{R_0}^d U \to \widetilde{R}\otimes_{R_0} L_{d-2,r}$$

to be the following composition of $\widetilde{R}$-module homomorphisms:

$$\begin{array}{ccc}
 & \widetilde{R}\otimes_{R_0} L_{d-1,r} & \xrightarrow{\operatorname{Kos}^{\Psi}} \widetilde{R}\otimes_{R_0} L_{d-2,r}. \\
 & \kappa \Big\uparrow \simeq & \\
 & \widetilde{R}\otimes_{R_0}\bigwedge_{R_0}^d U\otimes_{R_0}\operatorname{Sym}_{r-1}^{R_0} U & \\
 & 1\otimes 1\otimes\boldsymbol{\delta}\sigma_{2n-1-r} \Big\uparrow & \\
 & \widetilde{R}\otimes_{R_0}\bigwedge_{R_0}^d U\otimes_{R_0} D_{2n-1-r}^{R_0}(U^*) & \\
 & \eta^{-1} \Big\uparrow \simeq & \\
\widetilde{R}\otimes_{R_0}\bigwedge_{R_0}^d U \xrightarrow{(2.10)} & \widetilde{R}\otimes_{R_0} K_{d-1,2n-2-r} &
\end{array} \tag{6.10}$$

Remarks 6.11. 1. It is clear from the definition of $\mathfrak{L}_r$ that the image of $\mathfrak{L}_r$ is actually contained in $X_{d-2,r}$. Indeed, $X_{d-2,r}$ is the kernel of the vertical map in (5.3) emanating from $\widetilde{R}\otimes_{R_0} L_{d-2,r}$ and (5.3) is a map of complexes.

2. In practice,

$$\mathfrak{L}_r : \widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \to X_{d-2,r}$$

does not really involve η^{-1} because the map (2.10) is the composition

$$\begin{aligned} &\widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \otimes_{R_0} R_0 \xrightarrow{1\otimes 1\otimes \mathrm{ev}^*} \widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \otimes_{R_0} \mathrm{Sym}_{2n-1-r}^{R_0} U \otimes_{R_0} D_{2n-1-r}^{R_0}(U^*) \\ &\xrightarrow{\widehat{\Psi}} \widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \otimes_{R_0} D_{2n-1-r}^{R_0}(U^*) \xrightarrow{\eta} \widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \otimes_{R_0} D_{2n-2-r}^{R_0}(U^*), \end{aligned} \tag{6.12}$$

so there is no reason to compute $\eta^{-1} \circ \eta$. The map ev^* is described in (1.3).

3. It is important to notice that $\mathfrak{L}_n$ can be made an equivariant map because σ_{n-1} is the uniquely determined inverse of

$$\boldsymbol{p}_{n-1}^{\Phi} : \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} \mathrm{Sym}_{n-1}^{R_0} U \to \widetilde{R}_{\boldsymbol{\delta}} \otimes_{R_0} D_{n-1}^{R_0}(U^*);$$

furthermore, $\mathfrak{L}_n$ can be defined over $\widetilde{R}$ in a polynomial and equivariant manner by using the classical adjoint of $\boldsymbol{p}_{n-1}^{\Phi}$ in place of its inverse. We do not know an equivariant description of $\mathfrak{L}_i$ for $n+1 \le i \le 2n-2$. An equivariant description of the classical adjoint in the present situation is a little tricky; so we will record the notion completely at this point. Usually, we will be a little cavalier on this issue. An interested reader should always think of the domain of $\mathfrak{L}_n$ as

$$\widetilde{R} \otimes_{R_0} \textstyle\bigwedge_{R_0}^d U \otimes_{R_0} \Big(\textstyle\bigwedge_{R_0}^{\mathrm{top}} \big(D_{n-1}^{R_0}(U^*)\big)\Big)^{\otimes 2}. \tag{6.13}$$

The classical adjoint of $\boldsymbol{p}_{n-1}^{\Phi} : \mathrm{Sym}_{n-1}^{R_0} U \to D_{n-1}^{R_0}(U^*)$ is the map

$$\mathrm{Adj}\big(\boldsymbol{p}_{n-1}^{\Phi}\big) : D_{n-1}^{R_0}(U^*) \otimes_{R_0} \Big(\textstyle\bigwedge_{R_0}^{\mathrm{top}} \big(\mathrm{Sym}_{n-1}^{R_0} U\big)\Big)^{\otimes 2} \to \mathrm{Sym}_{n-1}^{R_0} U,$$

with

$$\big[\mathrm{Adj}\big(\boldsymbol{p}_{n-1}^{\Phi}\big)\big](w_{n-1} \otimes \Theta_1 \otimes \Theta_2) = \Big[\Big(\textstyle\bigwedge^{\mathrm{top}-1} {\boldsymbol{p}_{n-1}^{\Phi}}^*\Big)\big(w_{n-1}(\Theta_1)\big)\Big](\Theta_2),$$

for $w_{n-1} \in D_{n-1}^{R_0}(U^*)$ and $\Theta_i \in \bigwedge_{R_0}^{\mathrm{top}}(\mathrm{Sym}_{n-1}^{R_0} U)$, where "top" is equal to the rank of $\mathrm{Sym}_{n-1}^{R_0} U$. That is, "top" is equal to $\binom{n+d-2}{n-1}$. One easily checks that

$$\begin{aligned} \boldsymbol{p}_{n-1}^{\Phi}\big(\big[\mathrm{Adj}\big(\boldsymbol{p}_{n-1}^{\Phi}\big)\big](w_{n-1} \otimes \Theta_1 \otimes \Theta_2)\big) &= \Big[\Big(\textstyle\bigwedge^{\mathrm{top}} \boldsymbol{p}_{n-1}^{\Phi}\Big)(\Theta_1)\Big](\Theta_2) \cdot w_{n-1} \\ \big(\big[\mathrm{Adj}\big(\boldsymbol{p}_{n-1}^{\Phi}\big)\big]\big(\big(\boldsymbol{p}_{n-1}^{\Phi}\big)^*(u_{n-1}) \otimes \Theta_1 \otimes \Theta_2\big)\big) &= \Big[\Big(\textstyle\bigwedge^{\mathrm{top}} \boldsymbol{p}_{n-1}^{\Phi}\Big)(\Theta_1)\Big](\Theta_2) \cdot u_{n-1} \end{aligned}$$

for $u_{n-1} \in \mathrm{Sym}_{n-1}^{R_0} U$. Thus, $\mathrm{Adj}(\boldsymbol{p}_{n-1}^{\Phi}) \circ \boldsymbol{p}_{n-1}^{\Phi} : \mathrm{Sym}_{n-1}^{R_0} U \to \mathrm{Sym}_{n-1}^{R_0} U$ and $\boldsymbol{p}_{n-1}^{\Phi} \circ \mathrm{Adj}(\boldsymbol{p}_{n-1}^{\Phi}) : D_{n-1}^{R_0}(U^*) \to D_{n-1}^{R_0}(U^*)$ are "both equal to multiplication by the determinant of $\boldsymbol{p}_{n-1}^{\Phi}$", once one picks a basis element for the rank one free module $\bigwedge_{R_0}^{\text{top}}(\operatorname{Sym}_{n-1}^{R_0} U)$.

Definition 6.14. Adopt Data 6.1. Let $\widetilde{\mathbb{G}}'(r)$ be the following sequence of $\widetilde{R}$-modules and $\widetilde{R}$-module homomorphisms:

$$0 \to \begin{matrix} X_{d-1,r} \\ \oplus \\ \widetilde{R} \otimes_{R_0} \bigwedge_{R_0}^d U \end{matrix} \xrightarrow{[\,\mathrm{Kos}^{\Psi}\ \ \mathfrak{L}_r\,]} X_{d-2,r} \xrightarrow{\mathrm{Kos}^{\Psi}} X_{d-3,r} \xrightarrow{\mathrm{Kos}^{\Psi}} \dots \xrightarrow{\mathrm{Kos}^{\Psi}} X_{0,r} \xrightarrow{\widehat{\Psi}} \widetilde{R}.$$

Of course, according to Remark 6.7(3), $\widetilde{\mathbb{G}}'(n)$ is equal to

$$0 \to \widetilde{R} \otimes_{R_0} \bigwedge_{R_0}^d U \xrightarrow{\mathfrak{L}_n} X_{d-2,n} \xrightarrow{\mathrm{Kos}^{\Psi}} X_{d-3,n} \xrightarrow{\mathrm{Kos}^{\Psi}} \dots \xrightarrow{\mathrm{Kos}^{\Psi}} X_{0,n} \xrightarrow{\widehat{\Psi}} \widetilde{R}.$$

(The modification of (6.13) should be applied if one wants $\widetilde{\mathbb{G}}'(n)$ to be one hundred percent equivariant.)

Theorem 6.15. *Adopt Data 6.1.*

(1) *The $\widetilde{R}$-module homomorphisms $\widetilde{\mathbb{G}}'(r)$ of Definition 6.14 form a complex.*
(2) *The complex $\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$ is a resolution of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^{r-n}\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ by projective $\widetilde{R}_{\boldsymbol{\delta}}$-modules.*
(3) *If $\boldsymbol{k}$ is a field and an R_0-algebra, P is the polynomial ring $\operatorname{Sym}_{\bullet}^{\boldsymbol{k}} U$, $I = \operatorname{ann}\phi$ is an ideal of the set $\mathbb{I}_n(\boldsymbol{k}, U)$ of Definition 1.15, J is the ideal of P generated by $\operatorname{Sym}_1^{\boldsymbol{k}} U$, and P is an $\widetilde{R}$-algebra by way of*

$$\widetilde{R} \to \boldsymbol{k} \otimes_{R_0} \widetilde{R} \xrightarrow{\widehat{\phi}} P,$$

where $\widehat{\phi} : \boldsymbol{k} \otimes_{R_0} \widetilde{R} \to P$ is the P-algebra homomorphism of (4.18) and (4.19), then $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$ is a minimal homogeneous resolution of $P/J^{r-n}I$ by free P-modules.
(4) *In the situation of (3), with $r = n$, $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(n)_{\boldsymbol{\delta}}$ is an equivariant minimal homogeneous resolution of P/I by free P-modules.*
(5) *In the situation of (3), the graded Betti numbers of $P/J^{r-n}I$ are described by*

$$0 \to \begin{matrix} P(-r-d+1)^{\beta_d} \\ \oplus \\ P(-2n-d+2) \end{matrix} \to P(-r-d+2)^{\beta_{d-1}} \to \cdots$$
$$\to P(-r-1)^{\beta_2} \to P(-r)^{\beta_1} \to P,$$

with $\beta_i = \binom{d+r-1}{i-1+r}\binom{i+r-2}{i-1} - \binom{d+2n-r-2}{i-1}\binom{d+2n-r-i-2}{d-i}$. In particular, the strand with the exponents labeled by β_i is linear.

Proof. Assertion (1) is clear from the construction of $\widetilde{\mathbb{G}}'(r)$. The point of this assertion is that $\widetilde{\mathbb{G}}'(r)$ is a well-defined complex over $\widetilde{R}$. It is clear from Lemma 6.16 that $\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$ is a well-defined complex over $\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$; but in fact, we carefully defined $\widetilde{\mathbb{G}}'(r)$ to actually be a complex over $\widetilde{R}$, without any denominators. Assertion (2) is a consequence of Lemma 6.16 applied to the resolution $\widetilde{\mathbb{G}}(r)_{\boldsymbol{\delta}}$ of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{J}^{r-n}\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$, as found in Corollary 4.16. Claim 6.4 establishes that the vertical maps of (6.2) are surjective. The targets of these maps are free; hence the maps split and the hypotheses of Lemma 6.16 are satisfied.

(3) and (5). We know from (2) that the complex $P\otimes_{\widetilde{R}}\widetilde{\mathbb{G}}'(r)_{\boldsymbol{\delta}}$ is a homogeneous resolution of $P/J^{r-n}I$ by free P-modules. One readily reads that the graded Betti numbers of this complex are given in (5); see also Remark 6.7(2). We conclude that $P\otimes_{\widetilde{R}}\widetilde{\mathbb{G}}'(r)$ is a minimal resolution.

(4) All of the maps and modules of $\widetilde{\mathbb{G}}'(r)$ are obviously equivariant, except, possibly, $\mathfrak{L}_r$. We have explained in Remark 6.11(3) how to make $\mathfrak{L}_n$ become an equivariant map. □

Lemma 6.16. *Let R be a commutative Noetherian ring and let*

$$\begin{array}{ccccccccccccc} & & 0 & \longrightarrow & T_d & \xrightarrow{h_d} & T_{d-1} & \xrightarrow{h_{d-1}} & \dots & \xrightarrow{h_2} & T_1 & \xrightarrow{h_1} & T_0 \\ & & & & \downarrow{\scriptstyle v_d} & & \downarrow{\scriptstyle v_{d-1}} & & & & \downarrow{\scriptstyle v_1} & & \\ 0 & \longrightarrow & B_d & \xrightarrow{h'_d} & B_{d-1} & \xrightarrow{h'_{d-1}} & B_{d-2} & \xrightarrow{h'_{d-2}} & \dots & \xrightarrow{h'_1} & B_0 & & \end{array} \tag{6.17}$$

be a map of complexes of R-modules. Suppose that for each i, with $0\le i\le d-1$, $\sigma_i: B_i\to T_{i+1}$ is an R-module homomorphism which splits $v_{i+1}: T_{i+1}\to B_i$ in the sense that the composition $v_{i+1}\circ\sigma_i$ is the identity map on B_i. Then the following statements hold.

(1) *The R-module homomorphisms*

$$0\to\begin{array}{c}\ker v_d\\ \oplus\\ B_d\end{array}\xrightarrow{[\,v_d\ \ h_d\circ\sigma_{d-1}\circ h'_d\,]}\ker v_{d-1}\xrightarrow{h_{d-1}}\dots$$
$$\xrightarrow{h_3}\ker v_2\xrightarrow{h_2}\ker v_1\xrightarrow{h_1}T_0 \tag{6.18}$$

form a complex.

(2) *There is a quasi-isomorphism from the total complex of (6.17) to the complex (6.18).*

Proof. Assertion (1) is obvious as soon as one observes that $h_i(\ker v_i)$ is contained in $\ker v_{i-1}$. We prove assertion (2). (We follow ideas used in the proof of Theorem 4.7.) Let $(\mathbb{M}_\bullet, m_\bullet)$ be the mapping cone of (6.17). Fix i with $1\le i\le d$. Observe that $m_i:\mathbb{M}_i\to\mathbb{M}_{i-1}$ is

$$\begin{array}{c}T_i\\ \oplus\\ B_i\end{array}\xrightarrow{\begin{bmatrix}h_i & 0\\ v_i & -h'_i\end{bmatrix}}\begin{array}{c}T_{i-1}\\ \oplus\\ B_{i-1}.\end{array} \tag{6.19}$$

Take advantage of the direct sum decomposition $T_i = \ker v_i \oplus \operatorname{im} \sigma_{i-1}$, which is induced by the equation

$$\theta = (1 - \sigma_{i-1} \circ v_i)(\theta) + (\sigma_{i-1} \circ v_i)(\theta),$$

for all θ in T_i to write (6.19) in the form

$$\begin{array}{ccc}
\begin{matrix}\ker v_i\\ \oplus\\ \operatorname{im}\sigma_{i-1}\\ \oplus\\ B_i\end{matrix} & \xrightarrow{\begin{bmatrix}(1-\sigma_{i-2}\circ v_{i-1})\circ h_i & (1-\sigma_{i-2}\circ v_{i-1})\circ h_i & 0\\ \sigma_{i-2}\circ v_{i-1}\circ h_i & \sigma_{i-2}\circ v_{i-1}\circ h_i & 0\\ 0 & v_i & -h'_i\end{bmatrix}} & \begin{matrix}\ker v_{i-1}\\ \oplus\\ \operatorname{im}\sigma_{i-2}\\ \oplus\\ B_{i-1}\end{matrix}\\
{=}\big\downarrow & & {=}\big\downarrow\\
\begin{matrix}\ker v_i\\ \oplus\\ \operatorname{im}\sigma_{i-1}\\ \oplus\\ B_i\end{matrix} & \xrightarrow{\begin{bmatrix}h_i & h_i-\sigma_{i-2}h'_{i-1}\circ v_i & 0\\ 0 & \sigma_{i-2}\circ h'_{i-1}\circ v_i & 0\\ 0 & v_i & -h'_i\end{bmatrix}} & \begin{matrix}\ker v_{i-1}\\ \oplus\\ \operatorname{im}\sigma_{i-2}\\ \oplus\\ B_{i-1}\end{matrix}
\end{array}$$

Use the isomorphisms $\sigma_j : B_j \to \operatorname{im}\sigma_j$ and $v_{j+1} : \operatorname{im}\sigma_j \to B_j$ to see that (6.19) is isomorphic to Fig. 6.20. This calculation works for $1 \le i \le d-1$, but it must be modified for $i = d$, because in the isomorphism of Fig. 6.20, the image of $h_{i+1} \circ \sigma_i - \sigma_{i-1} \circ h'_i$ is contained in $\ker v_i$, provided $i \le d-1$, since $v_{d+1} \circ \sigma_d$ is not defined (or is zero); but it certainly is not the identity map on B_d. The modification when $i = d$ is

$$\begin{array}{ccc}
\begin{matrix}\ker v_d\\ \oplus\\ B_{d-1}\\ \oplus\\ B_d\end{matrix} & \xrightarrow{\begin{bmatrix}h_d & h_d\circ\sigma_{d-1}-\sigma_{d-2}\circ h'_{d-1} & 0\\ 0 & h'_{d-1} & 0\\ 0 & 1 & -h'_d\end{bmatrix}} & \begin{matrix}\ker v_{d-1}\\ \oplus\\ B_{d-2}\\ \oplus\\ B_{d-1}\end{matrix}\\
{\simeq}\Big\downarrow\begin{bmatrix}1&0&0\\0&1&-h'_d\\0&0&1\end{bmatrix} & & {\simeq}\Big\downarrow\begin{bmatrix}1&0&-(h_d\circ\sigma_{d-1}-\sigma_{d-2}\circ h'_{d-1})\\0&1&-h'_{d-1}\\0&0&1\end{bmatrix}\\
\begin{matrix}\ker v_d\\ \oplus\\ B_{d-1}\\ \oplus\\ B_d\end{matrix} & \xrightarrow{\begin{bmatrix}h_d & 0 & h_d\circ\sigma_{d-1}\circ h'_d\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}} & \begin{matrix}\ker v_{d-1}\\ \oplus\\ B_{d-2}\\ \oplus\\ B_{d-1}.\end{matrix}
\end{array}$$

It is now clear that (6.18) is obtained from $\mathbb{M}$ by splitting off the extraneous summands. □

$$
\begin{array}{ccc}
\begin{matrix}\ker v_i\\ \oplus\\ B_{i-1}\\ \oplus\\ B_i\end{matrix} & \xrightarrow{\begin{bmatrix} h_i & h_i-\sigma_{i-2}h'_{i-1}\circ v_i\circ\sigma_{i-1} & 0\\ 0 & v_{i-1}\circ\sigma_{i-2}\circ h'_{i-1}\circ v_i\circ\sigma_{i-1} & 0\\ 0 & v_i\circ\sigma_{i-1} & -h'_i\end{bmatrix}} & \begin{matrix}\ker v_{i-1}\\ \oplus\\ B_{i-2}\\ \oplus\\ B_{i-1}\end{matrix}\\
{=}\big\downarrow & & {=}\big\downarrow\\
\begin{matrix}\ker v_i\\ \oplus\\ B_{i-1}\\ \oplus\\ B_i\end{matrix} & \xrightarrow{\begin{bmatrix} h_i & h_i\circ\sigma_{i-1}-\sigma_{i-2}\circ h'_{i-1} & 0\\ 0 & h'_{i-1} & 0\\ 0 & 1 & -h'_i\end{bmatrix}} & \begin{matrix}\ker v_{i-1}\\ \oplus\\ B_{i-2}\\ \oplus\\ B_{i-1}\end{matrix}\\
{\simeq}\Big\downarrow\begin{bmatrix} 1 & 0 & -(h_{i+1}\circ\sigma_i-\sigma_{i-1}\circ h'_i)\\ 0 & 1 & -h'_i\\ 0 & 0 & 1\end{bmatrix} & & {\simeq}\Big\downarrow\begin{bmatrix} 1 & 0 & -(h_i\circ\sigma_{i-1}-\sigma_{i-2}\circ h'_{i-1})\\ 0 & 1 & -h'_{i-1}\\ 0 & 0 & 1\end{bmatrix}\\
\begin{matrix}\ker v_i\\ \oplus\\ B_{i-1}\\ \oplus\\ B_i\end{matrix} & \xrightarrow{\begin{bmatrix} h_i & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}} & \begin{matrix}\ker v_{i-1}\\ \oplus\\ B_{i-2}\\ \oplus\\ B_{i-1}.\end{matrix}
\end{array}
$$

Fig. 6.20. A complex isomorphic to (6.19). This complex is used in the proof of Lemma 6.16.

Example 6.21. Adopt Data 6.1 with $R_0 = \mathbb{Z}$, $d = 3$, and $n = r = 2$. In this example we record and explain the generic resolution $\widetilde{\mathbb{G}}'(2)_{\boldsymbol{\delta}}$ of Theorem 6.15 when $(d, n, r) = (3, 2, 2)$. Theorem 6.15 only promises a resolution by projective $\widetilde{R}_{\boldsymbol{\delta}}$-modules; however, in fact, this is a resolution by **free** $\widetilde{R}_{\boldsymbol{\delta}}$-modules. Recall that $\widetilde{R}$ is the polynomial ring $\widetilde{R} = \mathbb{Z}[x, y, z, t_{x^2}, t_{xy}, t_{xz}, t_{y^2}, t_{yz}, t_{z^2}]$, T is the matrix

$$
T = \begin{bmatrix} t_{x^2} & t_{xy} & t_{xz}\\ t_{xy} & t_{y^2} & t_{yz}\\ t_{xz} & t_{yz} & t_{z^2}\end{bmatrix},
$$

and $\boldsymbol{\delta} = \det T$. Let Q be the classical adjoint of T. We note for future reference that Q is a symmetric matrix and

$$
TQ = QT = \boldsymbol{\delta} I_3,
$$

where I_r represents the $r \times r$ identity matrix. Define the elements $\lambda_1, \lambda_2, \lambda_3$ of $\widetilde{R}$ by

$$
[\lambda_1, \lambda_2, \lambda_3] = [x, y, z]Q \quad \text{or} \quad Q\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} \lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}.
$$

View $\widetilde{R}_{\boldsymbol{\delta}}$ as a graded ring where x, y, z all have degree one and the t's have degree zero.

Claim 6.22. *The generic resolution $\widetilde{\mathbb{G}}'(2)_{\boldsymbol{\delta}}$ is equal to*

$$0 \to \widetilde{R}_{\boldsymbol{\delta}}(-5) \xrightarrow{d_3} \widetilde{R}_{\boldsymbol{\delta}}(-3)^5 \xrightarrow{d_2} \widetilde{R}_{\boldsymbol{\delta}}(-2)^5 \xrightarrow{d_1} \widetilde{R}_{\boldsymbol{\delta}},$$

with

$$d_1 = d_3^{\mathrm{T}} = [\, -t_{z^2}x\lambda_1 + \boldsymbol{\delta} z^2 \quad -t_{y^2}x\lambda_1 + \boldsymbol{\delta} y^2 \quad -t_{yz}x\lambda_1 + \boldsymbol{\delta} yz \quad x\lambda_2 \quad x\lambda_3 \,], \tag{6.23}$$

and d_2 is the alternating matrix which is obtaining by completing:

$$\begin{bmatrix} 0 & -xQ_{2,3} & -xQ_{3,3} & -xt_{y^2}Q_{1,3} & -xt_{yz}Q_{3,1} + \boldsymbol{\delta} y \\ * & 0 & xQ_{2,2} & xt_{yz}Q_{2,1} - \boldsymbol{\delta} z & xt_{z^2}Q_{2,1} \\ * & * & 0 & xt_{yz}Q_{3,1} - xt_{y^2}Q_{2,1} + \boldsymbol{\delta} y & xt_{z^2}Q_{3,1} - xt_{yz}Q_{2,1} - \boldsymbol{\delta} z \\ * & * & * & 0 & xQ_{1,1}^2 \\ * & * & * & * & 0 \end{bmatrix}. \tag{6.24}$$

Remark. We write d_3^{T} to mean the transpose of the matrix d_3. In the present formulation, the matrix d_1 is equal to $-\boldsymbol{\delta}$ times the row vector of signed maximal order Pfaffians of d_2.

Proof of Claim 6.22. The resolution $\widetilde{\mathbb{G}}'(2)_{\boldsymbol{\delta}}$ is obtained using our techniques as described in Example 5.15. The resolution $\widetilde{\mathbb{G}}(2)$ is the mapping cone of

$$\begin{array}{ccccccccc} & & 0 & \to & \widetilde{R}\otimes_{\mathbb{Z}} L_{2,2} & \to & \widetilde{R}\otimes_{\mathbb{Z}} L_{1,2} & \to & \widetilde{R}\otimes_{\mathbb{Z}} L_{0,2} & \to & \widetilde{R} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{R}\otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^3 U & \to & \widetilde{R}\otimes_{\mathbb{Z}} K_{2,0} & \to & \widetilde{R}\otimes_{\mathbb{Z}} K_{1,0} & \to & \widetilde{R}\otimes_{\mathbb{Z}} K_{0,0}. & & \end{array}$$

We record the matrices of $\widetilde{\mathbb{G}}(2)$ using the bases $\ell_{\boldsymbol{a};\boldsymbol{b}}$ and $k_{\boldsymbol{a};\boldsymbol{b}}$ of (5.4) and (5.5):

$$\begin{array}{llllll} L_{2,2} & L_{1,2} & L_{0,2} & K_{2,0} & K_{1,0} & K_{0,0} \\ \ell_{1,2,3;1} & \ell_{1,2;1} & \ell_{1;1} & k_{\_;1} & k_{2;1} & k_{2,3;1} \\ \ell_{1,2,3;2} & \ell_{1,2;2} & \ell_{1;2} & k_{\_;2} & k_{3;1} & \\ \ell_{1,2,3;3} & \ell_{1,2;3} & \ell_{1;3} & k_{\_;3} & k_{3;2} & \\ & \ell_{1,3;1} & \ell_{2;2} & & & \\ & \ell_{1,3;2} & \ell_{2;3} & & & \\ & \ell_{1,3;3} & \ell_{3;3} & & & \\ & \ell_{2,3;2} & & & & \\ & \ell_{2,3;3} & & & & \end{array} \tag{6.25}$$

We identify x_1 with x, x_2 with y, and x_3 with z. We take $\omega = x \wedge y \wedge z$ to be the basis for $\bigwedge_{\mathbb{Z}}^3 U$. The resolution $\widetilde{\mathbb{G}}(2)$ then is the mapping cone of

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \widetilde{R}(-4)^3 & \xrightarrow{h_3} & \widetilde{R}(-3)^8 & \xrightarrow{h_2} & \widetilde{R}(-2)^6 & \xrightarrow{h_1} & \widetilde{R} \\ & & \downarrow & & \downarrow{\scriptstyle v_3} & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_1} & & \\ 0 & \longrightarrow & \widetilde{R}(-5) & \xrightarrow{h_3'} & \widetilde{R}(-4)^3 & \xrightarrow{h_2'} & \widetilde{R}(-3)^3 & \xrightarrow{h_1'} & \widetilde{R}(-2) & & \end{array}$$

with

$$h_1 = [\,x^2, xy, xz, y^2, yz, z^2\,] \qquad h_2 = \begin{bmatrix} -y & 0 & 0 & -z & 0 & 0 & 0 & 0 \\ x & -y & 0 & 0 & -z & 0 & 0 & 0 \\ 0 & 0 & -y & x & 0 & -z & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & x & 0 & x & 0 & y & -z \\ 0 & 0 & 0 & 0 & 0 & x & 0 & y \end{bmatrix},$$

$$h_3 = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ -x & 0 & z \\ -y & 0 & 0 \\ x & -y & 0 \\ 0 & 0 & -y \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix},$$

$$v_1 = [\,t_{x^2}, t_{xy}, t_{xz}, t_{y^2}, t_{yz}, t_{z^2}\,],$$

$$v_2 = \begin{bmatrix} 0 & 0 & 0 & -t_{x^2} & -t_{xy} & -t_{xz} & -t_{y^2} & -t_{yz} \\ t_{x^2} & t_{xy} & t_{xz} & 0 & 0 & 0 & -t_{yz} & -t_{z^2} \\ t_{xy} & t_{y^2} & t_{yz} & t_{xz} & t_{yz} & t_{z^2} & 0 & 0 \end{bmatrix}, \qquad v_3 = T,$$

$$h_1' = [\,-z \quad y \quad -x\,], \qquad h_2' = \begin{bmatrix} y & -x & 0 \\ z & 0 & -x \\ 0 & z & -y \end{bmatrix}, \quad \text{and} \quad h_3' = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

According to Definition 6.14, the resolution $\widetilde{\mathbb{G}}'(2)_{\boldsymbol{\delta}}$ has the form

$$0 \to \widetilde{R}_{\boldsymbol{\delta}} \otimes_{\mathbb{Z}} \textstyle\bigwedge_{\mathbb{Z}}^3 U \xrightarrow{\mathfrak{L}_2} (X_{1,2})_{\boldsymbol{\delta}} \xrightarrow{\mathrm{Kos}^{\Psi}} (X_{0,2})_{\boldsymbol{\delta}} \xrightarrow{\widehat{\Psi}} \widetilde{R}_{\boldsymbol{\delta}}.$$

The key step in this proof is that we are able to identify bases for $(X_{0,2})_{\boldsymbol{\delta}} = (\ker v_1)_{\boldsymbol{\delta}}$ and $(X_{1,2})_{\boldsymbol{\delta}} = (\ker v_2)_{\boldsymbol{\delta}}$. Indeed, the matrices

$$J_1 = \begin{bmatrix} Q & 0 \\ 0 & I_3 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

are invertible over $\widetilde{R}_{\boldsymbol{\delta}}$; it is easy to read the kernels of the matrices

$$v_1 J_1 = \begin{bmatrix} \boldsymbol{\delta} & 0 & 0 & t_{y^2} & t_{yz} & t_{z^2} \end{bmatrix} \quad \text{and}$$
$$v_2 J_2 = \begin{bmatrix} 0 & 0 & 0 & -\boldsymbol{\delta} & 0 & 0 & -t_{y^2} & -t_{yz} \\ \boldsymbol{\delta} & 0 & 0 & 0 & 0 & 0 & -t_{yz} & -t_{z^2} \\ 0 & \boldsymbol{\delta} & 0 & 0 & 0 & \boldsymbol{\delta} & 0 & 0 \end{bmatrix};$$

and, in particular, we conclude that $(X_{0,2})_{\boldsymbol{\delta}}$ and $(X_{1,2})_{\boldsymbol{\delta}}$ are both free $\widetilde{R}_{\boldsymbol{\delta}}$-modules and the columns of

$$B = \begin{bmatrix} -t_{z^2} Q_{*,1} & -t_{y^2} Q_{*,1} & -t_{yz} Q_{*,1} & Q_{*,2} & Q_{*,3} \\ 0 & \boldsymbol{\delta} & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\delta} & 0 & 0 \\ \boldsymbol{\delta} & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \tag{6.26}$$

$$B' = \begin{bmatrix} Q_{*,3} & 0 & Q_{*,2} & t_{yz} Q_{*,1} & t_{z^2} Q_{*,1} \\ 0 & -Q_{*,2} & -Q_{*,3} & -t_{y^2} Q_{*,1} & -t_{yz} Q_{*,1} \\ 0 & 0 & 0 & \boldsymbol{\delta} & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{\delta} \end{bmatrix} \tag{6.27}$$

represent bases for $(X_{0,2})_{\boldsymbol{\delta}}$ and $(X_{1,2})_{\boldsymbol{\delta}}$, respectively. (We write $Q_{*,j}$ to represent column j of the matrix Q.) In other words, we have proven that

$$g_1 = -t_{z^2} \sum_{j=1}^{3} Q_{j,1} \ell_{1;j} + \boldsymbol{\delta} \ell_{3;3} \qquad g_2 = -t_{y^2} \sum_{j=1}^{3} Q_{j,1} \ell_{1;j} + \boldsymbol{\delta} \ell_{2;2}$$
$$g_3 = -t_{yz} \sum_{j=1}^{3} Q_{j,1} \ell_{1;j} + \boldsymbol{\delta} \ell_{2;3} \qquad g_4 = \sum_{j=1}^{3} Q_{j,2} \ell_{1;j}$$
$$g_5 = \sum_{j=1}^{3} Q_{j,3} \ell_{1;j} \quad \text{and}$$

$$\gamma_1 = \sum_{j=1}^{3} Q_{j,3} \ell_{1,2;j} \qquad \gamma_2 = -\sum_{j=1}^{3} Q_{j,2} \ell_{1,3;j} \qquad \gamma_3 = \sum_{j=1}^{3} Q_{j,2} \ell_{1,2;j} - \sum_{j=1}^{3} Q_{j,3} \ell_{1,3;j}$$
$$\gamma_4 = t_{yz} \sum_{j=1}^{3} Q_{j,1} \ell_{1,2;j} - t_{y^2} \sum_{j=1}^{3} Q_{j,1} \ell_{1,3;j} + \boldsymbol{\delta} \ell_{2,3;2}$$
$$\gamma_5 = t_{z^2} \sum_{j=1}^{3} Q_{j,1} \ell_{1,2;j} - t_{yz} \sum_{j=1}^{3} Q_{j,1} \ell_{1,3;j} + \boldsymbol{\delta} \ell_{2,3;3}$$

are bases for the free $\widetilde{R}_{\boldsymbol{\delta}}$-modules $(X_{0,2})_{\boldsymbol{\delta}}$ and $(X_{1,2})_{\boldsymbol{\delta}}$, respectively. There is no difficulty in seeing that $[\widehat{\Psi}(g_1), \widehat{\Psi}(g_2), \widehat{\Psi}(g_3), \widehat{\Psi}(g_4), \widehat{\Psi}(g_5)]$ is equal to the matrix d_1 from (6.23). Indeed, for example,

$$\begin{aligned}\widehat{\Psi}(g_1) &= \widehat{\Psi}\left(-t_{z^2}\sum_{j=1}^{3} Q_{j,1}\ell_{1;j} + \boldsymbol{\delta}\ell_{3;3}\right)\\ &= -t_{z^2}\sum_{j=1}^{3} Q_{j,1}\widehat{\Psi}\big(\kappa(x\otimes x_j)\big) + \boldsymbol{\delta}\widehat{\Psi}\big(\kappa(z\otimes z)\big)\\ &= -t_{z^2}\sum_{j=1}^{3} Q_{j,1}\widehat{\Psi}(xx_j) + \boldsymbol{\delta}\widehat{\Psi}\big(z^2\big) = -t_{z^2}x\left(x_j\sum_{j=1}^{3} Q_{j,1}\right) + \boldsymbol{\delta}z^2\\ &= -t_{z^2}x\lambda_1 + \boldsymbol{\delta}z^2,\end{aligned}$$

as expected, where, as always, we write x for x_1, y for x_2, and z for x_3.

We next calculate the matrix for $\boldsymbol{\delta}\operatorname{Kos}^{\Psi} : (X_{1,2})_{\boldsymbol{\delta}} \to (X_{0,2})_{\boldsymbol{\delta}}$ with respect to the bases $\{g_1,\dots,g_5\}$ and $\{\gamma_1,\dots,\gamma_5\}$. Use the fact that $\ell_{2;1} = \ell_{1;2}$ to see that the column vector for

$$\boldsymbol{\delta}\operatorname{Kos}^{\Psi}(\gamma_1) = \boldsymbol{\delta}\sum_{j=1}^{3} Q_{j,3}\operatorname{Kos}^{\Psi}(\ell_{1,2;j}) = \boldsymbol{\delta}\sum_{j=1}^{3} Q_{j,3}(x\ell_{2;j} - y\ell_{1;j}),$$

with respect to the basis for $L_{0,2}$ in (6.25), is

$$\boldsymbol{\delta}\begin{bmatrix} -yQ_{1,3}\\ -yQ_{2,3}+xQ_{1,3}\\ -yQ_{3,3}\\ xQ_{2,3}\\ xQ_{3,3}\\ 0\end{bmatrix}.$$

Recall the matrix B of (6.26) which expresses the basis of $(X_{0,2})_{\boldsymbol{\delta}}$ in terms of the basis for $L_{0,2}$. We claim that

$$\boldsymbol{\delta}\begin{bmatrix} -yQ_{1,3}\\ -yQ_{2,3}+xQ_{1,3}\\ -yQ_{3,3}\\ xQ_{2,3}\\ xQ_{3,3}\\ 0\end{bmatrix} = B\begin{bmatrix} 0\\ xQ_{2,3}\\ xQ_{3,3}\\ xt_{y^2}Q_{1,3}\\ xt_{yz}Q_{1,3}-\boldsymbol{\delta}y\end{bmatrix}. \tag{6.28}$$

Once (6.28) is established, then one reads that

$$\boldsymbol{\delta}\operatorname{Kos}^{\Psi}(\gamma_1) = 0g_1 + xQ_{2,3}g_2 + xQ_{3,3}g_3 + xt_{y^2}Q_{1,3}g_4 + (xt_{yz}Q_{1,3}-\boldsymbol{\delta}y)g_5,$$

as expected. The main trick in the calculation of (6.28) involves the coefficient of x in the top three rows. On the right side, this coefficient is

$$
\begin{aligned}
&\begin{bmatrix} -t_{z^2}Q_{*,1} & -t_{y^2}Q_{*,1}-t_{yz}Q_{*,1} & Q_{*,2} & Q_{*,3} \end{bmatrix} \begin{bmatrix} 0 \\ Q_{2,3} \\ Q_{3,3} \\ t_{y^2}Q_{1,3} \\ t_{yz}Q_{1,3} \end{bmatrix} \\
&\quad = Q \begin{bmatrix} -t_{y^2}Q_{2,3}-t_{yz}Q_{3,3} \\ t_{y^2}Q_{1,3} \\ t_{yz}Q_{1,3} \end{bmatrix} = Q \begin{bmatrix} 0 & -t_{y^2} & -t_{yz} \\ t_{y^2} & 0 & 0 \\ t_{yz} & 0 & 0 \end{bmatrix} Q_{*,3} \\
&\quad = Q \left(\begin{bmatrix} t_{xy} & 0 & 0 \\ t_{y^2} & 0 & 0 \\ t_{yz} & 0 & 0 \end{bmatrix} - \begin{bmatrix} t_{xy} & t_{y^2} & t_{yz} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) Q_{*,3} \\
&\quad = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{\delta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q_{*,3} - Q \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\delta} Q_{1,3} \\ 0 \end{bmatrix},
\end{aligned}
$$

which is equal to the coefficient of x in the top three rows on the left side of (6.28). When $\operatorname{Kos}^{\Psi}(\gamma_2)$, $\operatorname{Kos}^{\Psi}(\gamma_3)$, $\operatorname{Kos}^{\Psi}(\gamma_4)$, and $\operatorname{Kos}^{\Psi}(\gamma_5)$ are written in terms of the basis for $L_{0,2}$, then the result is the columns of

$$
\begin{bmatrix}
zQ_{1,2} & -yQ_{1,2}+zQ_{1,3} & -yt_{yz}Q_{1,1}+zt_{y^2}Q_{1,1} & -yt_{z^2}Q_{1,1}+zt_{yz}Q_{1,1} \\
zQ_{2,2} & xQ_{1,2}-yQ_{2,2}+zQ_{2,3} & xt_{yz}Q_{1,1}-yt_{yz}Q_{2,1}+zt_{y^2}Q_{2,1} & xt_{z^2}Q_{1,1}-yt_{z^2}Q_{2,1}+zt_{yz}Q_{2,1} \\
zQ_{3,2}-xQ_{1,2} & -yQ_{3,2}-xQ_{1,3}+zQ_{3,3} & -yt_{yz}Q_{3,1}-xt_{y^2}Q_{1,1}+zt_{y^2}Q_{3,1} & -yt_{z^2}Q_{3,1}-xt_{yz}Q_{1,1}+zt_{yz}Q_{3,1} \\
0 & xQ_{2,2} & xt_{yz}Q_{2,1}-z\boldsymbol{\delta} & xt_{z^2}Q_{2,1} \\
-xQ_{2,2} & 0 & xt_{yz}Q_{3,1}-xt_{y^2}Q_{2,1}+y\boldsymbol{\delta} & xt_{z^2}Q_{3,1}-xt_{yz}Q_{2,1}-\boldsymbol{\delta}z \\
-xQ_{3,2} & -xQ_{3,3} & -xt_{y^2}Q_{3,1} & -xt_{yz}Q_{3,1}+\boldsymbol{\delta}y
\end{bmatrix}.
$$

Calculations similar to the one we just made show that $\boldsymbol{\delta}$ times the above matrix is equal to

$$
B \begin{bmatrix}
-xQ_{2,3} & -xQ_{3,3} & -xt_{y^2}Q_{1,3} & -xt_{yz}Q_{3,1}+\boldsymbol{\delta}y \\
0 & xQ_{2,2} & xt_{yz}Q_{2,1}-\boldsymbol{\delta}z & xt_{z^2}Q_{2,1} \\
-xQ_{2,2} & 0 & xt_{yz}Q_{3,1}-xt_{y^2}Q_{2,1}+\boldsymbol{\delta}y & xt_{z^2}Q_{3,1}-xt_{yz}Q_{2,1}-\boldsymbol{\delta}z \\
-xt_{yz}Q_{1,2}+z\boldsymbol{\delta} & -xt_{yz}Q_{3,1}+xt_{y^2}Q_{2,1}-\boldsymbol{\delta}y & 0 & xQ_{1,1}^2 \\
-xt_{z^2}Q_{2,1} & -xt_{z^2}Q_{3,1}+xt_{yz}Q_{2,1}+\boldsymbol{\delta}z & -xQ_{1,1}^2 & 0
\end{bmatrix};
$$

and therefore the matrix d_2 from (6.24) is the matrix for $\boldsymbol{\delta}\operatorname{Kos}^{\Psi}: (X_{1,2})_{\boldsymbol{\delta}} \to (X_{0,2})_{\boldsymbol{\delta}}$, with respect to the bases $\{g_1,\dots,g_5\}$ and $\{\gamma_1,\dots,\gamma_5\}$.

We use (6.10) and (6.12) to compute $\mathfrak{L}_2(\omega)$ for $\omega = x\wedge y\wedge z \in \bigwedge_{\mathbb{Z}}^3 U$. Thus, $\mathfrak{L}_2(\omega) = (\operatorname{Kos}^{\Psi} \circ \kappa)(\Theta)$ for

$$
\begin{aligned}
\Theta &= x\otimes\omega\otimes\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(x^*\big) + y\otimes\omega\otimes\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(y^*\big) + z\otimes\omega\otimes\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(z^*\big) \\
&\in \widetilde{R}\otimes_{\mathbb{Z}} \textstyle\bigwedge_{\mathbb{Z}}^3 U \otimes_{\mathbb{Z}} U.
\end{aligned}
$$

The map $\boldsymbol{\delta}(\boldsymbol{p}_1^{\Phi})^{-1}: U^* \to \widetilde{R}\otimes_{\mathbb{Z}} U$ is given by the classical adjoint Q of T:

$$\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(x^*\big) = Q_{1,1}x + Q_{2,1}y + Q_{3,1}z,$$
$$\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(y^*\big) = Q_{1,2}x + Q_{2,2}y + Q_{3,2}z, \quad \text{and}$$
$$\boldsymbol{\delta}\big(\boldsymbol{p}_1^{\Phi}\big)^{-1}\big(z^*\big) = Q_{1,3}x + Q_{2,3}y + Q_{3,3}z;$$

so, Θ is equal to

$$\begin{cases} x \otimes \omega \otimes (Q_{1,1}x + Q_{2,1}y + Q_{3,1}z) \\ +y \otimes \omega \otimes (Q_{1,2}x + Q_{2,2}y + Q_{3,2}z) \\ +z \otimes \omega \otimes (Q_{1,3}x + Q_{2,3}y + Q_{3,3}z) \end{cases} = \begin{cases} (xQ_{1,1} + yQ_{1,2} + zQ_{1,3}) \cdot (\omega \otimes x) \\ +(xQ_{2,1} + yQ_{2,2} + zQ_{2,3}) \cdot (\omega \otimes y) \\ +(xQ_{3,1} + yQ_{3,2} + zQ_{3,3}) \cdot (\omega \otimes z) \end{cases}$$
$$= \lambda_1 \cdot (\omega \otimes x) + \lambda_2 \cdot (\omega \otimes y) + \lambda_3 \cdot (\omega \otimes z).$$

The composition $\operatorname{Kos}^{\Psi} \circ \kappa$ is equal to $\kappa \circ \operatorname{Kos}^{\Psi}$:

$$\begin{array}{ccc} \widetilde{R} \otimes_{\mathbb{Z}} L_{2,2} & \xrightarrow{\operatorname{Kos}^{\Psi}} & \widetilde{R} \otimes_{\mathbb{Z}} L_{1,2} \\ \kappa \uparrow & & \kappa \uparrow \\ \widetilde{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^3 U \otimes_{\mathbb{Z}} U & \xrightarrow{\operatorname{Kos}^{\Psi}} & \widetilde{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^2 U \otimes_{\mathbb{Z}} U. \end{array}$$

It follows that

$$\mathfrak{L}_2(\omega) = \lambda_1 \cdot \big(\kappa \circ \operatorname{Kos}^{\Psi}\big)(\omega \otimes x) + \lambda_2 \cdot \big(\kappa \circ \operatorname{Kos}^{\Psi}\big)(\omega \otimes y) + \lambda_3 \cdot \big(\kappa \circ \operatorname{Kos}^{\Psi}\big)(\omega \otimes z).$$

Recall that

$$\big(\kappa \circ \operatorname{Kos}^{\Psi}\big)(\omega \otimes x) = x \cdot \kappa(y \wedge z \otimes x) - y \cdot \kappa(x \wedge z \otimes x) + z \cdot \kappa(x \wedge y \otimes x)$$

and that

$$\kappa(y \wedge z \otimes x) = \kappa(x \wedge z \otimes y) - \kappa(x \wedge y \otimes z) = \ell_{1,3;2} - \ell_{1,2;3}.$$

Thus,

$$\mathfrak{L}_2(\omega) = \begin{cases} x\lambda_1(\ell_{1,3;2} - \ell_{1,2;3}) & -y\lambda_1\ell_{1,3;1} & +z\lambda_1\ell_{1,2;1} \\ +x\lambda_2\ell_{2,3;2} & -y\lambda_2\ell_{1,3;2} & +z\lambda_2\ell_{1,2;2} \\ +x\lambda_3\ell_{2,3;3} & -y\lambda_3\ell_{1,3;3} & +z\lambda_3\ell_{1,2;3}. \end{cases}$$

When $\boldsymbol{\delta}\mathfrak{L}_2(\omega)$ is written in terms of the basis for $L_{0,2}$, we obtain the matrix on the left side of (6.29). The right most factor in (6.29) is the matrix we have called d_3. The proof is complete as soon as we verify

$$\boldsymbol{\delta}\begin{bmatrix} z\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}-x\begin{bmatrix}0\\ 0\\ \lambda_1\end{bmatrix}\\ -y\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}+x\begin{bmatrix}0\\ \lambda_1\\ 0\end{bmatrix}\\ x\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}\end{bmatrix} = B'\begin{bmatrix}\boldsymbol{\delta}\begin{bmatrix}z^2\\ y^2\\ yz\end{bmatrix}-x\lambda_1\begin{bmatrix}t_{z^2}\\ t_{y^2}\\ t_{yz}\end{bmatrix}\\ x\lambda_2\\ x\lambda_3\end{bmatrix}, \tag{6.29}$$

where B' is the matrix of (6.27) which expresses the basis $\gamma_1, \dots, \gamma_5$ of $(X_{1,2})_{\boldsymbol{\delta}}$ in terms of the basis $\ell_{1,2;1}, \dots, \ell_{2,3;3}$ of $L_{1,2}$ as given in (6.25).

We verify (6.29). The bottom two rows are obvious. The top three rows of the right side of (6.29) is equal to

$$Q\begin{bmatrix} x\lambda_2 t_{yz} + x\lambda_3 t_{z^2}\\ \boldsymbol{\delta} yz - x\lambda_1 t_{yz}\\ \boldsymbol{\delta} z^2 - x\lambda_1 t_{z^2}\end{bmatrix} = S_1 + S_2, \quad \text{with}$$

$$S_1 = xQ\begin{bmatrix}0 & t_{yz} & t_{z^2}\\ -t_{yz} & 0 & 0\\ -t_{z^2} & 0 & 0\end{bmatrix}Q\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad \text{and} \quad S_2 = \boldsymbol{\delta} Q\begin{bmatrix}0\\ yz\\ z^2\end{bmatrix}.$$

Observe that $S_1 = S_1' + S_1''$ with

$$S_1' = xQ\begin{bmatrix}-t_{xz} & 0 & 0\\ -t_{yz} & 0 & 0\\ -t_{z^2} & 0 & 0\end{bmatrix}Q\begin{bmatrix}x\\ y\\ z\end{bmatrix} = x\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\boldsymbol{\delta} & 0 & 0\end{bmatrix}Q\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ -x\boldsymbol{\delta}\lambda_1\end{bmatrix}, \quad \text{and}$$

$$S_1'' = xQ\begin{bmatrix}t_{xz} & t_{yz} & t_{z^2}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}Q\begin{bmatrix}x\\ y\\ z\end{bmatrix} = xQ\begin{bmatrix}\boldsymbol{\delta} z\\ 0\\ 0\end{bmatrix}.$$

Thus,

$$S_1'' + S_2 = z\boldsymbol{\delta}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}$$

and the top three rows of the right side of (6.29) are

$$S_1' + \left(S_1'' + S_2\right) = \begin{bmatrix}0\\ 0\\ -x\boldsymbol{\delta}\lambda_1\end{bmatrix} + z\boldsymbol{\delta}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix},$$

and this is equal to the top three rows of the left side of (6.29). Rows 4, 5, and 6 of (6.29) are treated in the same manner. □

7. Non-empty disjoint sets of orbits

We now turn to Projects 0.4–0.7 when $d = 3$. That is, we identify non-empty disjoint subsets of $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ which are closed under the action of $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$.

Let $\boldsymbol{k}$ be a field and P be the polynomial ring $P = \boldsymbol{k}[x, y, z]$. We first record the action of the group $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$ on the sets $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ and $\mathbb{X}_n(\boldsymbol{k})$ from (0.8). If A is an invertible $(2n+1)\times(2n+1)$ matrix with entries from $\boldsymbol{k}$ and α is an automorphism of the three-dimensional vector space $U = [P]_1$, then $(A, \alpha) \in \mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$ carries the matrix $X = (x_{i,j})$ of $\mathbb{X}_n(\boldsymbol{k})$ to the matrix $A(\alpha(x_{i,j}))A^{-1}$ in $\mathbb{X}_n(\boldsymbol{k})$. If $I \in \mathbb{I}_n^{[3]}(\boldsymbol{k})$, then I is generated by the maximal order Pfaffians of some X in $\mathbb{X}_n(\boldsymbol{k})$ and (A, α) carries I to the ideal generated by the maximal order Pfaffians of (A, α) of X. Notice that the subgroup $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times 1$ of $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$ moves the elements of $\mathbb{X}_n(\boldsymbol{k})$, but acts like the identity on $\mathbb{I}_n^{[3]}(\boldsymbol{k})$.

Throughout the present section we are interested in $n \geq 3$. Indeed, if $n = 2$, then the orbit structure of $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ is not very interesting.

Observation 7.1. *If $\boldsymbol{k}$ is a field of characteristic not equal to 2 which is closed under the taking of square root, then $\mathbb{I}_2^{[3]}(\boldsymbol{k})$ consists of exactly one orbit under the action of $1 \times \mathrm{GL}_3\,\boldsymbol{k}$.*

Proof. Let (ϕ) be the Macaulay Inverse system for some ideal I in $\mathbb{I}_2^{[3]}(\boldsymbol{k})$. Observe that the matrix

$$T_\phi = \begin{bmatrix} \phi(x^2) & \phi(xy) & \phi(xz) \\ \phi(xy) & \phi(y^2) & \phi(yz) \\ \phi(xz) & \phi(yz) & \phi(z^2) \end{bmatrix}$$

represents a non-degenerate symmetric bilinear form on the three vector space U, whose basis is x, y, z. It is well-known, see for example Thm. XIV.3.1 on p. 358 in [25], and easy to see, that one can choose a new basis for U so that the matrix for T_ϕ, in the new basis, is the matrix of Example 1.7. Thus, under a linear change of variables, $\mathrm{ann}(\phi)$ becomes equal to the ideal BE_2. Actually, the standard theorem from linear algebra converts T_ϕ into a diagonal matrix. Our hypothesis about square roots converts the non-degenerate diagonal matrix into an identity matrix and $(x, y, z) \mapsto (\frac{x+\sqrt{-1}y}{\sqrt{2}}, \frac{x-\sqrt{-1}y}{\sqrt{2}}, \sqrt{-1}z)$ converts the identity matrix to

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad \square$$

For each pair of integers n and μ, with $1 \leq n$ and $0 \leq \mu \leq 3$, recall the set

$$\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k}) = \left\{ I \in \mathbb{I}_n^{[3]}(\boldsymbol{k}) \;\middle|\; \begin{array}{l} \exists \text{ linearly independent linear forms } \ell_1, \ldots, \ell_\mu \text{ in } P_1 \\ \text{with } \ell_1^n, \ldots, \ell_\mu^n \text{ in } I \text{ and } \nexists\ \mu + 1 \text{ such forms} \end{array} \right\}$$

of (0.13). It is clear that $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ is the disjoint union of $\bigcup_{\mu=0}^{3} \mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ and each $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ is closed under the action of $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$.

Theorem 7.2. *If $n \geq 3$ and the characteristic of $\boldsymbol{k}$ is zero, then $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ is non-empty for $0 \leq \mu \leq 3$.*

Remark. We do **not** claim that every ideal in $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ may be converted into any other ideal in $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ by using $\mathrm{GL}_{2n+1}\,\boldsymbol{k} \times \mathrm{GL}_3\,\boldsymbol{k}$.

Proof of Theorem 7.2. In Definition 7.12 we introduce ideals $I_{n,\mu}(\boldsymbol{k})$ for μ equal to 0, 1, and 2. The ideals $I_{n,\mu}(\boldsymbol{k})$ are shown to be in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ in Proposition 7.13 and to be in $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ in Proposition 7.14. In Proposition 7.24 we exhibit an ideal $J_{n,n-1}$, which is in $\mathbb{I}_{n,3}^{[3]}(\boldsymbol{k})$. $\square$

The ideal $I_{n,2}$ will be defined to be equal to the Buchsbaum–Eisenbud ideal BE_n of Definition 7.3. To prove Theorem 7.2, we first show that when the hypotheses of Theorem 7.2 are in effect, then BE_n is in $\mathbb{I}_{n,2}^{[3]}(\boldsymbol{k})$. It is clear that x^n and y^n are in BE_n. One needs only to show that if ℓ is a linear form, then ℓ^n is in BE_n only if ℓ is a multiple of x or y. We identify homogeneous generators for the ideal BE_n in Proposition 7.6 and the Macaulay inverse system (ϕ_n) for BE_n in Proposition 7.10. These calculations work over any field. We complete the proof that BE_n is in $\mathbb{I}_{n,2}^{[3]}(\boldsymbol{k})$ in Proposition 7.14 by solving the equation $\ell^n(\phi_n) = 0$; this part of the proof is sensitive to the characteristic of $\boldsymbol{k}$. In Definition 7.12 we modify ϕ_n twice producing $\phi_{n,\mu}$ for $\mu = 0$ and $\mu = 1$; we set $\phi_{n,2} = \phi_n$; and we define $I_{n,\mu}$ to be $\mathrm{ann}(\phi_{n,\mu})$. In Proposition 7.13, we prove that the Gorenstein ideals $I_{n,0}$ and $I_{n,1}$ are linearly presented by studying how the determinant of $T_{\phi_{n,\mu}}$ is related to $\det T_{\phi_n}$. In Proposition 7.14 we show that $I_{n,\mu} \in \mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ by solving the equation $\ell^n(\phi_{n,\mu}) = 0$, as ℓ roams over the linear forms of $\boldsymbol{k}[x,y,z]$. We do not know how to modify ϕ_n to produce an element of $\mathbb{I}_{n,3}^{[3]}(\boldsymbol{k})$; however, we can use ideas from the study of the Weak Lefschetz property to see that the Gorenstein ideal $J_{n,n-1} = (x^n, y^n, z^n) : (x+y+z)^{n-1}$ is linearly presented and is generated in degree n (these calculations are very sensitive to the characteristic of $\boldsymbol{k}$); hence is in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$. It is clear that x^n, y^n, and z^n all are in $J_{n,n-1}$; hence, $J_{n,n-1}$ is in $\mathbb{I}_{n,3}^{[3]}(\boldsymbol{k})$.

We first study the alternating matrices introduced by Buchsbaum and Eisenbud in Section 6 of [8]. (Our indexing is slightly different than the indexing of [8]: the matrix that is called H_{2n+1} in [8] is called $H_n(x,y,z)$ in the present paper.)

Definition 7.3. Let x, y, z be elements of a ring R. For each positive integer n, we define the $(2n+1) \times (2n+1)$ alternating matrix $H_n(x,y,z)$. The non-zero entries of $H_n(x,y,z)$ above the main diagonal are

$$\big(H_n(x,y,z)\big)_{i,j} = \begin{cases} x & \text{if } i \text{ is odd and } j = i+1 \\ y & \text{if } i \text{ is even and } j = i+1 \\ z & \text{if } j = 2n+2-i. \end{cases}$$

When P is the ring $\boldsymbol{k}[x,y,z]$, for some field $\boldsymbol{k}$, then we let BE_n denote the ideal of P which is generated by the maximal order Pfaffians of $H_n(x,y,z)$. In other words, BE_n is generated by $\{B_i \mid 1 \le i \le 2n+1\}$ where B_i is the Pfaffian of the $2n \times 2n$ submatrix of $B = H_n(x,y,z)$ which is obtained by deleting row and column i. We call the matrix $H_n(x,y,z)$ a Buchsbaum–Eisenbud matrix and the ideal BE_n a Buchsbaum–Eisenbud ideal.

Example 7.4. The first few Buchsbaum–Eisenbud matrices $H_1 = H_1(x,y,z)$ and $H_2 = H_2(x,y,z)$ are given in (0.9). When $n = 1$, then $B_1 = y$, $B_2 = z$, and $B_3 = x$. When $n = 2$, then $B_1 = y^2$, $B_2 = xz$, $B_3 = xy + z^2$, $B_4 = yz$, and $B_5 = x^2$.

In Proposition 7.6 we explicitly identify the maximal order Pfaffians of the Buchsbaum Eisenbud matrices. Our first step is to express the Pfaffians of one of these matrices in terms of the Pfaffians of smaller matrices of the same form. Recall our Pfaffian conventions from Section 1.2

Lemma 7.5. *Let x, y, and z be elements of a ring P and $n \ge 3$ be an integer. If $B = H_n(x,y,z)$, $b = H_{n-1}(y,x,z)$, and $\tilde{b} = H_{n-2}(x,y,z)$, then*

$$\begin{aligned} B_1 &= yb_{2n-1}, & B_2 &= zb_1, \quad B_i = xy\tilde{b}_{i-2} + zb_{i-1} \quad \textit{for } 3 \le i \le 2n-1, \\ B_{2n} &= zb_{2n-1}, \quad \textit{and} & B_{2n+1} &= xb_1. \end{aligned}$$

Proof. Throughout this proof we use the fact that b is the submatrix of B obtained by deleting the first and last rows and columns. Expand the Pfaffians along the last column to obtain

$$B_1 = yB_{1,2n,2n+1} = yb_{2n-1} \quad \text{and} \quad B_{2n} = zB_{1,2n,2n+1} = zb_{2n-1}.$$

Expand the rest of the Pfaffians along the first row: $B_2 = zB_{1,2,2n+1} = zb_1$, $B_{2n+1} = xB_{1,2,2n+1} = xb_1$, and for all i with $3 \le i \le 2n-1$,

$$\begin{aligned} B_i &= xB_{1,2,i} + zB_{1,i,2n+1} = xyB_{1,2,i,2n,2n+1} + zB_{1,i,2n+1} \\ &= xyb_{1,i-1,2n-1} + zb_{i-1} = xy\tilde{b}_{i-2} + zb_{i-1}. \qquad \square \end{aligned}$$

Proposition 7.6. *If x, y, and z are elements of a ring P and n is a positive integer, then the ideal generated by the maximal order Pfaffians of the matrix $H_n(x,y,z)$ is generated by*

$$\{x^i s_{n-i} \mid 1 \le i \le n\} \cup \{s_n\} \cup \{y^i s_{n-i} \mid 1 \le i \le n\},$$

where $s_i = \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{i-j}{j} x^j y^j z^{i-2j}$.

Example 7.7. The first few s_i are $s_0 = 1$, $s_1 = z$, $s_2 = z^2 + xy$, and $s_3 = z^3 + 2xyz$.

Proof of Proposition 7.6. Let $B = H_n(x, y, z)$. We prove the result by showing that the maximal order Pfaffians of B are given by

$$B_i = \begin{cases} x^{n+1-i} s_{i-1} & \text{if } i \text{ is even} \\ y^{n+1-i} s_{i-1} & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad B_{2n+2-i} = \begin{cases} y^{n+1-i} s_{i-1} & \text{if } i \text{ is even} \\ x^{n+1-i} s_{i-1} & \text{if } i \text{ is odd,} \end{cases} \tag{7.8}$$

for $1 \le i \le n+1$. We establish (7.8) by induction on n. If n is 1 or 2, then Examples 7.4 and 7.7 show that (7.8) holds. Henceforth, we assume $3 \le n$ and we apply Lemma 7.5 with $b = H_{n-1}(y, x, z)$ and $\tilde{b} = H_{n-2}(x, y, z)$. Induction gives

$$B_1 = yb_{2n-1} = y \cdot y^{n-1}, \qquad B_2 = zb_1 = z \cdot x^{n-1}, \qquad B_{2n} = zb_{2n-1} = z \cdot y^{n-1}, \quad \text{and}$$
$$B_{2n+1} = xb_1 = x \cdot x^{n-1}.$$

Now suppose $3 \le i \le n+1$. If i is odd, then

$$\begin{aligned} B_i &= xy\tilde{b}_{i-2} + zb_{i-1} \\ &= xy\bigl(y^{n+1-i} s_{i-3}\bigr) + z\bigl(y^{n-i+1} s_{i-2}\bigr) \quad \text{by induction} \\ &= y^{n+1-i}(xys_{i-3} + zs_{i-2}) \end{aligned}$$

and $B_{2n+2-i} = x^{n+1-i}(xys_{i-3} + zs_{i-2})$. If i is even, then

$$B_i = x^{n+1-i}(xys_{i-3} + zs_{i-2}) \quad \text{and} \quad B_{2n+2-i} = y^{n+1-i}(xys_{i-3} + zs_{i-2}).$$

We complete the proof by showing that

$$xys_{\alpha-1} + zs_\alpha = s_{\alpha+1} \tag{7.9}$$

for all α with $1 \le \alpha \le n-1$. Indeed, we see that $xys_{\alpha-1} + zs_\alpha$ is equal to

$$\begin{aligned} & xy \sum_{j=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-1-j}{j} x^j y^j z^{\alpha-1-2j} + z \sum_{j=0}^{\lfloor \frac{\alpha}{2} \rfloor} \binom{\alpha-j}{j} x^j y^j z^{\alpha-2j} \\ &= \sum_{j=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-1-j}{j} x^{j+1} y^{j+1} z^{\alpha-1-2j} + \sum_{j=0}^{\lfloor \frac{\alpha}{2} \rfloor} \binom{\alpha-j}{j} x^j y^j z^{\alpha+1-2j} \\ &= \sum_{j=1}^{\lfloor \frac{\alpha+1}{2} \rfloor} \binom{\alpha-j}{j-1} x^j y^j z^{\alpha+1-2j} + \sum_{j=1}^{\lfloor \frac{\alpha}{2} \rfloor} \binom{\alpha-j}{j} x^j y^j z^{\alpha+1-2j} + z^{\alpha+1}. \end{aligned}$$

Notice that $\lfloor \frac{\alpha+1}{2} \rfloor = \lfloor \frac{\alpha}{2} \rfloor + \chi$, where

$$\chi = \begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd;} \end{cases}$$

and therefore, $xys_{\alpha-1}+zs_\alpha$ is equal to

$$\chi x^{\lfloor\frac{\alpha+1}{2}\rfloor}y^{\lfloor\frac{\alpha+1}{2}\rfloor}+\sum_{j=1}^{\lfloor\frac{\alpha}{2}\rfloor}\binom{\alpha-j}{j-1}x^jy^jz^{\alpha+1-2j}+\sum_{j=1}^{\lfloor\frac{\alpha}{2}\rfloor}\binom{\alpha-j}{j}x^jy^jz^{\alpha+1-2j}+z^{\alpha+1}$$

$$=\chi x^{\lfloor\frac{\alpha+1}{2}\rfloor}y^{\lfloor\frac{\alpha+1}{2}\rfloor}+\sum_{j=1}^{\lfloor\frac{\alpha}{2}\rfloor}\left[\binom{\alpha-j}{j-1}+\binom{\alpha-j}{j}\right]x^jy^jz^{\alpha+1-2j}+z^{\alpha+1}$$

$$=\chi x^{\lfloor\frac{\alpha+1}{2}\rfloor}y^{\lfloor\frac{\alpha+1}{2}\rfloor}+\sum_{j=1}^{\lfloor\frac{\alpha}{2}\rfloor}\binom{\alpha+1-j}{j}x^jy^jz^{\alpha+1-2j}+z^{\alpha+1}$$

$$=\sum_{j=0}^{\lfloor\frac{\alpha+1}{2}\rfloor}\binom{\alpha+1-j}{j}x^jy^jz^{\alpha+1-2j}=s_{\alpha+1}.$$

We have established (7.9); therefore the proof is complete. □

Proposition 7.10. *Let n be a positive integer, $\boldsymbol{k}$ be a field, U be a vector space of dimension three over $\boldsymbol{k}$ with basis x,y,z, P be the polynomial ring $P=\operatorname{Sym}_\bullet^{\boldsymbol{k}}(U)=\boldsymbol{k}[x,y,z]$, and BE_n be the Buchsbaum–Eisenbud ideal. Then the Macaulay inverse system for BE_n is the P-submodule of $D_\bullet^{\boldsymbol{k}}(U^*)$ which is generated by*

$$\phi_n=\sum_{i=0}^{n-1}(-1)^ic_ix^{*(n-1-i)}y^{*(n-1-i)}z^{*(2i)}\in D_{2n-2}^{\boldsymbol{k}}(U^*),$$

where c_i is the ith Catalan number $c_i=\frac{1}{i+1}\binom{2i}{i}$.

Note. The first few ϕ's are

$$\phi_1=1,\qquad \phi_2=x^*y^*-z^{*(2)},\quad\text{and}\quad \phi_3=x^{*(2)}y^{*(2)}-x^*y^*z^{*(2)}+2z^{*(4)}.$$

Proof. The ideal BE_n is presented by the matrix $H_n(x,y,z)$, which has homogeneous linear entries. It follows that the socle degree of BE_n is $2n-2$; and therefore Macaulay's Theorem guarantees that $\operatorname{ann}\mathrm{BE}_n$ is a homogeneous cyclic P-submodule of $D_\bullet(U^*)$ generated by an element of $D_{2n-2}(U^*)$. As a consequence, any non-zero element of degree $2n-2$ in $\operatorname{ann}\mathrm{BE}_n$ is a generator of $\operatorname{ann}\mathrm{BE}_n$. To prove the result, it suffices to show that $\phi_n\in\operatorname{ann}\mathrm{BE}_n$. In light of Proposition 7.6, it suffices to show that $x^{n-i}s_i(\phi_n)$ and $y^{n-i}s_i(\phi_n)$ are zero for $0\le i\le n$. The expressions s_i and ϕ_n are both symmetric in x and y; consequently, it suffices to show that $x^{n-i}s_i(\phi_n)=0$ for $0\le i\le n$. We compute

$$x^{n-i}(\phi_n)=x^{n-i}\left(\sum_{k=0}^{n-1}(-1)^kc_kx^{*(n-1-k)}y^{*(n-1-k)}z^{*(2k)}\right)$$

$$= \sum_{k=0}^{i-1} (-1)^k c_k x^{*(i-1-k)} y^{*(n-1-k)} z^{*(2k)}.$$

It follows that

$$s_i\big(x^{n-i}\phi_n\big) = \sum \binom{i-j}{j} c_k (-1)^k x^{*(i-1-k-j)} y^{*(n-1-k-j)} z^{*(2k-i+2j)},$$

where the sum is taken over all pairs (k, j) which satisfy:

$$0 \le j \le \left\lfloor \frac{i}{2} \right\rfloor, \qquad 0 \le k \le i-1, \qquad 0 \le i-1-k-j, \quad \text{and} \quad 0 \le 2k-i+2j.$$

Replace k with $i-1-\ell-j$ to obtain

$$s_i\big(x^{n-i}\phi_n\big) = \sum_{\ell=0}^{\lfloor \frac{i-2}{2} \rfloor} (-1)^{i-1-\ell} \left[\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-j}{j} (-1)^j c_{i-1-\ell-j} \right] x^{*(\ell)} y^{*(\ell+n-i)} z^{*(i-2-2\ell)}.$$

The sum inside the brackets is zero due to Bennett's identity [3]:

$$\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j c_{m-n-j} \binom{m-j}{j} = 0 \tag{7.11}$$

when m and n are positive integers with $2n \le m$. The proof of (7.11) that is given in [3] is based on generating functions. The power series expansion for $F(z) = (1 - \sqrt{1-4z})/(2z)$ is $\sum_{i=0}^{\infty} c_i z^i$ and the coefficient $x^m z^{m-n}$ in the power series expansion of $\frac{F(z)}{1-(x-zx^2)}$ is equal to the left side of (7.11). □

Definition 7.12. Let n be a positive integer, $\boldsymbol{k}$ be a field, U be a vector space of dimension three over $\boldsymbol{k}$ with basis x, y, z, P be the polynomial ring $P = \operatorname{Sym}_\bullet^{\boldsymbol{k}}(U) = \boldsymbol{k}[x, y, z]$, and μ be one of the integers 0, 1, or 2. Define $\phi_{n,u}$ is the element

$$\phi_{n,\mu} = \sum_{i=0}^{n-1} (-1)^i c_i x^{*(n-1-i)} y^{*(n-1-i)} z^{*(2i)} + \chi(\mu \le 1) x^{*(2n-2)} + 2\chi(\mu = 0) y^{*(2n-2)}$$

of $D_{2n-2}^{\boldsymbol{k}} U^*$ and define $I_{n,\mu}$ to be the ideal $I_{n,\mu} = \operatorname{ann}(\phi_{n,\mu})$ of P.

Remark. The symbol χ is defined in (1.2). In particular,

$$\phi_{n,2} = \phi_n, \qquad \phi_{n,1} = \phi_n + x^{*(2n-2)}, \quad \text{and} \quad \phi_{n,0} = \phi_n + x^{*(2n-2)} + 2y^{*(2n-2)},$$

for ϕ_n as defined in Proposition 7.10. It is a consequence of Proposition 7.10, that $\mathrm{BE}_n = I_{n,2}$.

Proposition 7.13. *If* $\boldsymbol{k}$ *is a field,* n *is a positive integer, and* μ *is equal to* 0, 1, *or* 2, *then the ideal* $I_{n,\mu}$ *of Definition 7.12 is in the set* $\mathbb{I}_n^{[3]}(\boldsymbol{k})$ *of (0.8).*

Proof. Recall, from Definition 1.5, that $T_{\phi_{n,\mu}}$ is the $N \times N$ matrix $(\phi_{n,\mu}(m_i m_j))$, where $N = \binom{n+1}{2}$ and $\{m_i\}$ is a basis for $\operatorname{Sym}_{n-1}^{\boldsymbol{k}} U$. Let $m_1 = x^{n-1}$ and $m_2 = y^{n-1}$. Observe that the matrix $T_{\phi_{n,\mu}}$ has the form

$$T_{\phi_{n,\mu}} = \begin{pmatrix} M_\mu & 0 \\ 0 & M' \end{pmatrix},$$

where M_μ is the 2×2 matrix with entries $(\phi_n(m_i m_j))$ and $(\phi_{n,\mu}(m_i m_j))$, respectively, with $1 \le i, j \le 2$. A quick calculation yields that

$$M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad M_0 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The construction of the ideal $\mathrm{BE}_n = I_{n,2}$ puts this ideal in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$; and therefore, according to Proposition 1.8, $\det T_{\phi_{n,2}} \ne 0$; hence, $\det M' \ne 0$. It is clear that both matrices M_0 and M_1 have non-zero determinant and therefore $\det T_{\phi_{n,\mu}} \ne 0$, for μ equal to 0 and 1. It follows from Proposition 1.8 that $I_{n,0}$ and $I_{n,1}$ are both in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$. □

Proposition 7.14. *Let* $\boldsymbol{k}$ *be a field of characteristic zero,* $n \ge 3$ *be a positive integer,* μ *equal* 0, 1, *or* 2, *and* $I_{n,\mu} \subseteq P$ *be the ideal of Definition 7.12. If* $\ell = \alpha x + \beta y + \gamma z$ *is an arbitrary linear form in* P, *with* α, β, *and* γ *in* $\boldsymbol{k}$, *then*

$$\ell^n \in I_{n,\mu} \iff \begin{cases} \alpha = \gamma = 0 \quad \textit{or} \quad \beta = \gamma = 0 & \textit{when } \mu = 2 \\ \alpha = \gamma = 0 & \textit{when } \mu = 1 \\ \alpha = \beta = \gamma = 0 & \textit{when } \mu = 0. \end{cases} \tag{7.15}$$

In particular, the ideal $I_{n,\mu}$ *is in the set* $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$ *of (0.13).*

Example. The hypothesis $n \ge 3$ in Proposition 7.14 is necessary. Indeed, if $\boldsymbol{k}$ is the field $\mathbb{Q}[\sqrt{2}]$, then x^2, y^2, and $(x + y + \sqrt{2}z)^2$ all are in the ideal BE_2.

Proof of Proposition 7.14. We proved in Proposition 7.13 that $I_{n,\mu}$ is in $\mathbb{I}_n^{[3]}(\boldsymbol{k})$, which is the disjoint union $\bigcup_{i=0}^{3} \mathbb{I}_{n,i}^{[3]}(\boldsymbol{k})$. To prove that $I_{n,\mu}$ is in $\mathbb{I}_{n,\mu}^{[3]}(\boldsymbol{k})$, it suffices to establish (7.15). The direction ($\Leftarrow$) of (7.15) is obvious. We prove ($\Rightarrow$). Fix ℓ with $\ell^n \in I_{n,\mu}$.

We first assume that $4 \le n$. One calculates

$$\ell^n = \sum_{a+b+c=n} \binom{n}{a, b, c} (\alpha x)^a (\beta y)^b (\gamma z)^c \quad \text{and}$$

$$\ell^n(\phi_{n,\mu}) = \begin{cases} \sum_{a+b+c=n} \sum_{i=0}^{n-1} (-1)^i c_i \binom{n}{a,b,c} \alpha^a \beta^b \gamma^c x^{*(n-1-i-a)} y^{*(n-1-i-b)} z^{*(2i-c)} \\ \quad + \chi(\mu \le 1)\alpha^n x^{*(n-2)} + 2\chi(\mu = 0)\beta^n y^{*(n-2)}. \end{cases}$$

Recall that the binomial coefficient $\binom{n}{a,b,c}$ is zero if any of the parameters a, b, or c is negative. Each coefficient of $\ell^n(\phi_{n,\mu})$ is zero; so, in particular:

the coefficient of	in $\ell^n(\phi_{n,\mu})$ is	
$x^{*(n-2)}$	$0 = \beta^{n-2}\big(n\alpha\beta - \binom{n}{2}\gamma^2\big) + \chi(\mu \le 1)\alpha^n$	(7.16)
$y^{*(n-2)}$	$0 = \alpha^{n-2}\big(n\alpha\beta - \binom{n}{2}\gamma^2\big) + 2\chi(\mu = 0)\beta^n$	(7.17)
$z^{*(n-2)}$	$0 = \sum_{i=0}^{n-1}(-1)^i c_i \binom{n}{n-1-i,n-1-i,2i+2-n} \alpha^{n-1-i}\beta^{n-1-i}\gamma^{2i+2-n}$	
	$= (-1)^{n-1}c_{n-1}\gamma^n + \alpha\beta\kappa,$ for some integer κ in $\boldsymbol{k}$	(7.18)
$x^{*(n-3)}z^*$	$0 = \beta^{n-3}\gamma\big(2\binom{n}{3}\gamma^2 - \binom{n}{1,n-2,1}\alpha\beta\big)$	(7.19)
$x^*y^{*(n-3)}$	$0 = \alpha^{n-4}\big(2\binom{n}{4}\gamma^4 - \binom{n}{n-3,1,2}\alpha\beta\gamma^2 + \binom{n}{2}\alpha^2\beta^2\big)$	(7.20)

The hypothesis $4 \le n$ ensures that the 5 listed elements of $D_{n-2}^{\boldsymbol{k}}U^*$ are distinct.

We first show that

$$\alpha\beta\gamma = 0. \tag{7.21}$$

Indeed, if all three constants are non-zero, then we may combine (7.19) and (7.20) to see that $(\alpha\beta, \gamma^2)$ is a point in the intersection

$$0 = 2\binom{n}{3}Y - \binom{n}{1, n-2, 1}X \quad \text{and} \quad 0 = 2\binom{n}{4}Y^2 - \binom{n}{n-3, 1, 2}XY + \binom{n}{2}X^2.$$

However the only intersection point is $(0, 0)$. Thus, in every case, at least one of the constants α, β, or γ must be zero and (7.21) is established.

Next we show that

$$\gamma = 0. \tag{7.22}$$

Indeed, (7.21) ensures that at least one of the constants are zero. Furthermore, we may apply (7.18) to see that if $\alpha = 0$ or $\beta = 0$, then γ is also zero. We conclude that (7.22) holds.

Now that (7.22) holds, we apply (7.20) again to see that

$$\alpha\beta = 0. \tag{7.23}$$

The proof is complete if $\mu = 2$.

We now focus on $\mu = 1$. We have shown that $0 = \gamma = \alpha\beta$. If $\beta = 0$, then apply (7.16) to conclude $\alpha = 0$. Thus α must be zero if $\mu = 1$ and the proof is complete in this case.

Finally, we assume that $\mu = 0$. We have shown that (7.22) and (7.23) hold. If $\alpha = 0$, then (7.17) yields that β is also zero. If $\beta = 0$, then (7.16) yields that α is also zero. Thus all three constants are zero and the proof is complete in this case.

Now we treat the case $n = 3$. The argument is similar. Each coefficient of $\ell^3(\phi_{3,\mu})$ is zero; so, in particular:

the coefficient of	in $\ell^3(\phi_{3,\mu})$ is	
$x^{*(n-2)}$	$0 = 3\alpha\beta^2 - 3\beta\gamma^2 + \chi(\mu \le 1)\alpha^3$	(7.16′)
$y^{*(n-2)}$	$0 = 3\alpha^2\beta - 3\alpha\gamma^2 + 2\chi(\mu = 0)\beta^3$	(7.17′)
$z^{*(n-2)}$	$0 = -6\alpha\beta\gamma + 2\gamma^3.$	(7.18′)

As before, we first establish (7.21). If μ is 1 or 2, then an easy argument yields that every simultaneous solution of (7.17′) and (7.18′) is also a solution of (7.21). If $\mu = 0$, then one can show that every simultaneous solution of (7.16′), (7.17′) and (7.18′) is also a solution of (7.21). One now uses (7.18′) to show that (7.22) holds.

If $1 \le \mu$, then (7.23) follows from (7.17′), which now is $0 = 3\alpha^2\beta$ since $\gamma = 0$. If $\mu = 0$, then one can use (7.16′) and (7.17′), which now are $0 = 3\alpha\beta^2 + \alpha^3$ and $0 = 3\alpha^2\beta + 2\beta^3$ to conclude (7.23). Thus, (7.23) holds in all cases. The proof is complete when $\mu = 2$.

To complete the proof when $\mu = 1$, we use (7.16′), together with (7.22) and (7.23), to see that $\alpha = \gamma = 0$. To complete the proof when $\mu = 0$, we use (7.16′) and (7.17′) together with (7.22) and (7.23), to see that $\alpha = \beta = \gamma = 0$. □

We could not modify the generator of the Macaulay inverse system ϕ_n of the Buchsbaum–Eisenbud ideal BE_n to produce an element of $\mathbb{I}^{[3]}_{n,3}(\boldsymbol{k})$; however the ideal

$$J_{n,n-1} = \left(x^n, y^n, z^n\right) : (x + y + z)^{n-1},$$

which arises in the study of the Weak Lefschetz Property (see, for example, Observation 0.15) is in $\mathbb{I}^{[3]}_{n,3}(\boldsymbol{k})$ when $\boldsymbol{k}$ has characteristic zero.

Proposition 7.24. *Let n be a positive integer, $\boldsymbol{k}$ be a field, P be the polynomial ring $\boldsymbol{k}[x, y, z]$, and $J_{n,n-1}$ be the ideal $(x^n, y^n, z^n) : (x + y + z)^{n-1}$ of P. If the characteristic of $\boldsymbol{k}$ is zero, then $J_{n,n-1}$ is in $\mathbb{I}^{[3]}_{n,3}(\boldsymbol{k})$.*

Proof. It is clear that $J_{n,n-1}$ contains x^n, y^n, and z^n. We apply Proposition 1.8 to show that $J_{n,n-1}$ is in $\mathbb{I}^{[3]}_{n}(\boldsymbol{k})$. It suffices to show that $[P]_{2n-1} \subseteq J_{n,n-1}$ and $[J_{n,n-1}]_{n-1}$ is equal to 0. It is clear that $[P]_{3n-2} \subseteq (x^n, y^n, z^n)$; and therefore $[P]_{2n-1} \subseteq J_{n,n-1}$. Furthermore, Theorem 5 in [30] guarantees that the minimal generator degree of $\frac{J_{n,n-1}}{(x^n,y^n,z^n)}$ is at least n; and therefore, $[J_{n,n-1}]_{n-1} = 0$. □

Example. A quick calculation shows that

$$J_{1,0} = (x, y, z) \quad \text{and} \quad J_{2,1} = \bigl(x^2, y^2, z^2, z(x-y), y(x-z)\bigr).$$

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