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RESEARCH

Powers of graphs & applications to resolutions of powers of monomial ideals

Susan M. Cooper[1], Sabine El Khoury[2], Sara Faridi[3*], Sarah Mayes-Tang[4], Susan Morey[5], Liana M. Şega[6] and Sandra Spiroff[7]

*Correspondence:
Full list of author information is available at the end of the article

This paper is dedicated to Jürgen Herzog, whose interest in powers of ideals has been an inspiration to many, in honor of his 80^{th} birthday

Abstract

This paper is concerned with the question of whether geometric structures such as cell complexes can be used to simultaneously describe the minimal free resolutions of *all* powers of a monomial ideal. We provide a full answer in the case of square-free monomial ideals of projective dimension one by introducing a combinatorial construction of a family of (cubical) cell complexes whose 1-skeletons are powers of a graph that supports the resolution of the ideal.

1 Introduction

The search for topological objects whose chain maps coincide with free resolutions of a given monomial ideal has been a major research area in commutative algebra. Such topological objects are said to *support* a (minimal) free resolution of the ideal. Diana Taylor [20] showed that every ideal in a polynomial ring generated by q monomials has a free resolution supported on a q-simplex.

Our work connects two fruitful directions of research in commutative algebra. On one hand, starting with Taylor's construction there has been a substantial body of work on finding smaller topological structures, such as simplicial or more generally cell complexes, which support free resolutions of a given monomial ideal. We refer to Peeva [19, Chapter III] for the basics of such constructions. On the other hand, the problem of studying the powers of a (not necessarily monomial) ideal I arises naturally in the study of Rees algebras. There are numerous analyses of invariants of the powers I^r, such as depth, regularity, projective dimension, Betti numbers, free resolutions, and more. For a sampling, see Guardo and Van Tuyl [10], Morey [16], Fouli and Morey [9], Engström and Noren [6].

At the intersection of the above-mentioned directions of research lies the following:

Question 1.1 *If Γ is a cell complex supporting a minimal free resolution of a monomial ideal I, can Γ be used to define a family of cell complexes $\{\Gamma^r\}_{r\geq 1}$ such that Γ^r supports a minimal free resolution of I^r for each $r \geq 1$?*

In this paper we provide a positive answer to this question in the case when $\Gamma = G$ is a tree, that is, a graph with no cycles, supporting a minimal free resolution of a square-free monomial ideal I. Given a positive integer r, we build a graph G^r, which can be viewed as a power of G, using an abelianized extension of a graph-theoretic construction called the *box product* (Sect. 3). We then use G^r to construct a polyhedral cell complex $\overline{G^r}$, which is shown to support a minimal free resolution of the ideal I^r.

The polyhedral cell complex $\overline{G^r}$ is built on a skeleton that originates from the graph G^r. More precisely, assuming that G has $q+1$ vertices, we describe an embedding of G^r into $\mathbb{R}^q$ such that all the vertices of G^r have non-negative integer coordinates, and all edges have unit length and are parallel to one of the standard basis vectors in $\mathbb{R}^q$. The graph G^r is no longer a tree, except when $q \leq 1$ or $r = 1$, as multiple cycles are formed among its edges. However, due to our embedding in $\mathbb{R}^q$, the cycles are easily recognizable: They appear in 1-skeletons of cubes of various dimensions. This is detailed in Sect. 3. In Sect. 3, we describe these cubes using an orientation on the edges that ensures that the 1-skeleton of each such cube has a source and a sink, which can be used to identify the cube, and we define the cell complex $\overline{G^r}$ as the collection of these cubes. Proposition 3.23 proves that $\overline{G^r}$ is indeed a polyhedral cell complex.

On the other hand, it is known by Faridi and Hersey [8] that every monomial ideal I of projective dimension one has a minimal free resolution supported on a graph G. When I is square-free, we label the vertices of the cell complex $\overline{G^r}$ described above using the minimal monomial generators of I^r (Sect. 3) and we describe explicitly the differentials of the homogenized cellular chain complex that is supported on $\overline{G^r}$; see (3.26.2). We then show in Proposition 4.7 that this complex is isomorphic to a strand of the Koszul complex resolving the Rees algebra of I. The fact that this chain complex is a minimal free resolution of I^r is a consequence of the fact that the ideal I is of linear type and its Rees algebra is a complete intersection, as shown in Theorems 4.3 and 4.4. In particular, we find explicit formulas for the projective dimension and the Betti numbers of I^r; see Corollaries 4.10 and 4.11.

Our construction of the powers $\overline{G^r}$ points towards the possibility of defining, more generally, the powers of any simplicial (or cell) complex, and providing additional classes where Question 1.1 has a positive answer. This is a topic for ongoing and future work.

The interested reader might be curious about a slightly different, but related version of Question 1.1: if Γ is a q-simplex (which supports a free resolution of *any* ideal generated by q monomials) and r is a positive integer, can we construct a cell complex Γ^r, starting from Γ, which supports a free resolution of I^r, where I is *any* ideal generated by q monomials? This question has been addressed in [4,5].

2 Setup

This section provides the background and notation that will be used throughout the rest of the paper, by building a correspondence between monomial ideals and combinatorial structures that support their resolutions.

Notation 2.1 *If G is a graph with vertices $V(G) = \{x_1, \ldots, x_n\}$, then an undirected edge between vertices x_j and x_i will be denoted $\{x_j, x_i\}$ while a directed edge from x_j to x_i will be written $[x_j, x_i]$. The graphs used in this work will be simple graphs, that is, without loops or multiple edges. Throughout the paper, all graphs will be assumed to be connected unless otherwise indicated.*

2.2. Cell complexes The topological objects in this paper are *polyhedral cell complexes*. See [14,18] for additional resources on these topics.

Definition 2.3 ([15, p. 62]) Let X be a finite collection of convex polytopes in a real vector space $\mathbb{R}^q$. If these convex polytopes, called **faces** of X, satisfy the two properties below, then X is said to be a **polyhedral (or polytopal) cell complex**:

(1) if P is a polytope in X and F is a face of P, then F is in X;
(2) if P and Q are polytopes in X, then $P \cap Q$ is a face of P and a face of Q.

The faces of the polyhedral cell complexes that we will see in this paper will be cubes of varying dimensions. Specifically, our n-cells will always be n-dimensional cubes.

Definition 2.4 An n-**cube** C_n is the Cartesian product of n unit intervals. That is, $C_n = I_1 \times \cdots \times I_n$, for $n \geq 1$, where I_i is a unit interval $[0, 1]$. The **boundary** of the n-cube consists of the $n-1$ cubes formed by replacing one of the unit intervals by one of its two boundary points, which are 0-cubes. Thus, there are $2n$ boundary components, each of which has the form of $I_1 \times \cdots \times \{0\} \times \cdots \times I_n$ or $I_1 \times \cdots \times \{1\} \times \cdots \times I_n$.

An n-cube can be built in $\mathbb{R}^q$ when $q \geq n$ by taking a point $\mathbf{a}$ in $\mathbb{R}^q$ and a collection of n standard unit vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}$. The vertices of the cube are the endpoints of the vectors

$$\mathbf{a} + \sum_{j \in A} \mathbf{e}_{i_j}$$

for all $A \subseteq [n] = \{1, \ldots, n\}$. The edges of the cube, each of which has the form

$$\{\mathbf{a} + \sum_{j \in A} \mathbf{e}_{i_j}, \mathbf{a} + \sum_{j \in A} \mathbf{e}_{i_j} + \mathbf{e}_{i_k}\} \quad \text{for some} \quad k \notin A,$$

inherit a natural direction from that of $\mathbf{e}_{i_k}$. Viewing unit vectors as embedded copies of the unit interval $[0, 1]$, directed from 0 to 1, the directed edges are then

$$[\mathbf{a} + \sum_{j \in A} \mathbf{e}_{i_j}, \mathbf{a} + \sum_{j \in A} \mathbf{e}_{i_j} + \mathbf{e}_{i_k}] \quad \text{for some} \quad k \notin A.$$

A vertex v of a directed graph is a **sink** if every edge that contains v is of the form $[w, v]$ for some vertex w. Similarly, v is a **source** if every edge that contains v is of the form $[v, w]$ for some vertex w. Note that $\mathbf{a}$ is a source of the directed graph formed by the edges of the cube described above and $\mathbf{b} = \mathbf{a} + \sum_{j \in [n]} \mathbf{e}_{i_j}$ is a sink.

Given a polyhedral cell complex Γ let $\Gamma^{(n)}$ be the set of n-cells of Γ. For convenience, $\varnothing$ is considered to be a (-1)-cell, and $\Gamma^{(-1)} = \{\varnothing\}$.

Definition 2.5 For a field k and each $i \geq 0$, let k^i denote an i-dimensional k-vector space. The **oriented chain complex** of Γ is the complex

$$C(\Gamma, \mathsf{k}): \qquad \cdots \to \mathsf{k}^{|\Gamma^{(i)}|} \xrightarrow{\partial_i} \mathsf{k}^{|\Gamma^{(i-1)}|} \to \cdots \to \mathsf{k}^{|\Gamma^{(1)}|} \xrightarrow{\partial_1} \mathsf{k}^{|\Gamma^{(0)}|}$$

with differentials defined as follows: For $c \in \Gamma^{(i)}$ and $c' \in \Gamma^{(i-1)}$, let $\varepsilon(c, c') = 0$ if c' is not a face of c, and otherwise $\varepsilon(c, c') = \pm 1$, chosen by convention, commonly by using an orientation or an incidence function, so that $\partial^2 = 0$. Then for all $i \geq 1$, $c \in \Gamma^{(i)}$, and basis elements $\mathbf{u}_c$, define

$$\partial_i(\mathbf{u}_c) = \sum_{c' \in \Gamma^{(i-1)}} \varepsilon(c, c')\mathbf{u}_{c'}.$$

In particular, if Γ is a polyhedral cell complex (Definition 2.3), the faces of Γ can be oriented (in an arbitrary manner) so that the boundary chain of a face F is $\partial(F) = \sum \varepsilon(F, G)G$, where the sum is taken over all maximal proper faces G of F and

$$\varepsilon(F, G) = \begin{cases} +1 & \text{if the orientation of } F \text{ induces the orientation of } G; \\ -1 & \text{otherwise.} \end{cases}$$

For instance, in Example 3.19, if the 2-cell has a clockwise orientation, then the left and top edges have an orientation that is induced by the clockwise orientation while the bottom and right edges do not.

2.6. Cellular Resolutions Throughout, assume that $R = \mathsf{k}[x_1, \dots, x_n]$ is a polynomial ring over a field k and $I = (m_0, \dots, m_q)$ is an ideal generated by monomials. A **graded free resolution of** I is an exact sequence of free S-modules of the form:

$$\mathbb{F}: \quad 0 \to M_d \xrightarrow{\partial_d} \cdots \to M_i \xrightarrow{\partial_i} M_{i-1} \to \cdots \to M_1 \xrightarrow{\partial_1} M_0 \tag{2.6.1}$$

where $I \cong M_0/\operatorname{im}(\partial_1)$, and each map ∂_i is graded, in the sense that it preserves the degrees of homogeneous elements.

If $\partial_i(M_i) \subseteq (x_1, \dots, x_n)M_{i-1}$ for every $i > 0$, then the free resolution $\mathbb{F}$ is **minimal**. The length of a minimal free resolution of I (which is d in the case of $\mathbb{F}$ above) is another invariant of I called the **projective dimension** and is denoted by $\mathrm{pd}_R(I)$.

One concrete way to calculate a multigraded free resolution is to use chain complexes of topological objects, and in particular of cellular chain complexes. This approach was initiated by Taylor [20], and further developed by Bayer and Sturmfels [1] and many other researchers.

Let Γ be a polyhedral cell complex, or more generally any regular CW complex, with vertex set labeled by the monomials $m_0, \dots, m_q$. We label each cell $c \in \Gamma$ by $\mathrm{lcm}(c)$, which is defined as the least common multiple of the labels of its vertices. The **homogenization** of the oriented chain complex $C(\Gamma, \mathsf{k})$ defined in Definition 2.5 is a complex $\mathbb{F} = \mathbb{F}_\Gamma$ as displayed in (2.6.1), such that

$$M_i = \bigoplus_{c \in \Gamma^{(i)}} R(\mathrm{lcm}(c)).$$

We denote the basis element corresponding to the free module $R(\mathrm{lcm}(c))$ in this sum by $\mathbf{u}_c$. The differential of $\mathbb{F}_\Gamma$ is described by

$$\partial_i(\mathbf{u}_c) = \sum_{c' \in \Gamma^{(i-1)}} \varepsilon(c, c')\frac{\mathrm{lcm}(c)}{\mathrm{lcm}(c')}\mathbf{u}_{c'}$$

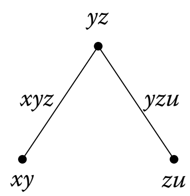
for each $c \in \Gamma^{(i)}$. Recall that $\varepsilon(c, c')$ is nonzero only when c' is a face of c, and in this case $\mathrm{lcm}(c')$ divides $\mathrm{lcm}(c)$.

We say that Γ **supports a resolution** of $I = (m_0, \dots, m_q)$ if the chain complex $\mathbb{F}_\Gamma$ is a resolution of I. In this case, we say $\mathbb{F}_\Gamma$ is a **cellular resolution** of I.

Note that if $\mathbb{F}_\Gamma$ is a resolution of I, then it is minimal if and only if $\mathrm{lcm}(c) \neq \mathrm{lcm}(c')$ for every cell c and every maximal face c' of c, see for example [1] or [19].

2.7. Ideals of projective dimension one In [8, Theorem 27], Faridi and Hersey proved that for a square-free monomial ideal I, having a minimal free resolution supported on a tree is equivalent to having projective dimension one. Here a graph can be viewed as a polyhedral complex where the maximum dimension of a cell is one. Moreover, Faridi and Hersey gave a concrete construction describing how to build the tree given a monomial generating set of an ideal of projective dimension one.

Example 2.8 Let $I = (xy, yz, zu)$ in $R = \mathsf{k}[x, y, z, u]$. Then $\operatorname{pd}_R(I) = 1$, and I has a minimal resolution supported on the labeled graph below.



3 Powers of trees

In this section we show that if G is a graph supporting a minimal free resolution of a square-free monomial ideal I and r is a positive integer, then we can build a polyhedral cell complex $\overline{G^r}$ from the r^{th} power graph G^r (described below), which supports a minimal free resolution of I^r. In this setting, the ideal I has projective dimension one in $R = \mathsf{k}[x_1, \dots, x_n]$, as per [8, Theorem 27].

Our definition of G^r (Definition 3.1) is an abelianized extension of a well-known construction in graph theory called the *box product*. Given two graphs G and H, the **Cartesian product**, or **box product**, of G and H, denoted by $G\square H$, is a new graph whose vertex set is the Cartesian product of the vertices of G and H, and $\{(g, h), (g', h')\}$ is an edge if and only if $\{g, g'\}$ is an edge of G and $h = h'$, or $\{h, h'\}$ is an edge of H and $g = g'$. If G and H have the same vertex set $\{v_0, \dots, v_q\}$, then one can define an abelian version of this product by forming a graph quotient that identifies (v_i, v_j) with (v_j, v_i). Note that no loops are created in this process since if $\{(v_i, v_j), (v_j, v_i)\}$ is an edge of $G\square H$, then $i = j$.

Definition 3.1 (**The (directed) graph** G^r) Let G be a graph on the vertex set $\{v_0, \dots, v_q\}$ and let r be a positive integer. Define

$$\mathcal{N}_r = \{(a_0, \dots, a_q) \in \mathbb{Z}_{\geq 0}^{q+1} \mid a_0 + \cdots + a_q = r\}.$$

Let G^r be the graph with distinct vertices labeled $\mathbf{v}^{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{N}_r$, that is

$$V(G^r) = \{\mathbf{v}^{\mathbf{a}} = v_0^{a_0} v_1^{a_1} \cdots v_q^{a_q} \mid \mathbf{a} = (a_0, \dots, a_q) \in \mathcal{N}_r\},$$

and edge set

$$E(G^r) = \left\{ \{\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}\} \mid \mathbf{v}^{\mathbf{a}} = Wv_j, \mathbf{v}^{\mathbf{b}} = Wv_i \text{ for some } \{v_j, v_i\} \in E(G) \text{ and } W \in V(G^{r-1}) \right\},$$

where if $W = \mathbf{v}^{\mathbf{c}}$ then $Wv_i = \mathbf{v}^{\mathbf{c}+\mathbf{f}_i}$. Here $\mathbf{f}_0, \dots, \mathbf{f}_q$ denotes the standard basis of $\mathbb{R}^{q+1}$; where the indexing starts at 0 for later convenience. More precisely, $\mathbf{f}_i$ denotes the $(i+1)^{st}$ standard basis vector

$$\mathbf{f}_i = (f_0, \dots, f_q), \quad \text{with} \quad f_i = 1 \quad \text{and} \quad f_k = 0 \quad \text{if} \quad k \neq i. \tag{3.1.1}$$

If G is a directed graph, an edge $\{\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}\}$ of G^r as described above inherits its direction from that of $\{v_j, v_i\}$. We denote the directed edge from $\mathbf{v}^{\mathbf{a}}$ to $\mathbf{v}^{\mathbf{b}}$ by $[\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}]$.

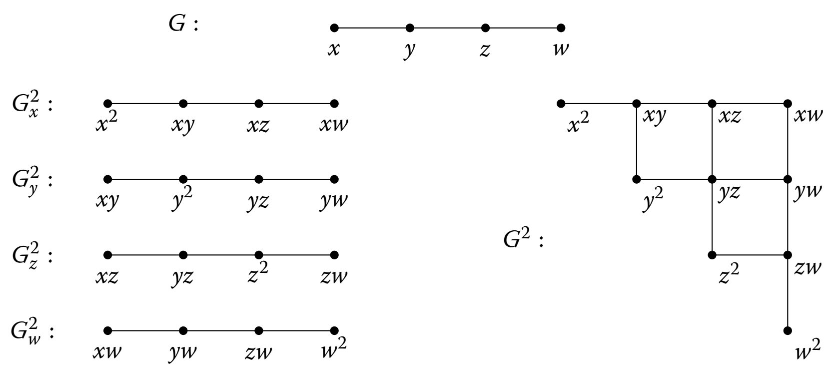
Notice that under this definition, if G is a directed graph then

$$[v_0^{a_0}v_1^{a_1}\cdots v_q^{a_q}, v_0^{b_0}v_1^{b_1}\cdots v_q^{b_q}] \in E(G^r)$$

if and only if there exits an edge $[v_j, v_i]$ of G with

$$a_j + 1 = b_j, a_i - 1 = b_i \quad \text{and} \quad a_k = b_k \quad \text{for} \quad k \neq i, j.$$

Example 3.2 From the path G below, we form the product G^2 by gluing the paths G^2_x, G^2_y, G^2_z, and G^2_w, which are obtained by multiplying the vertices in the path G by $v_0 = x, v_1 = y, v_2 = z$ and $v_3 = w$ respectively. Note that G^2_x and G^2_w, which appear in G^2 along the top and the right, respectively, are glued at xw; G^2_y and G^2_z are glued at yz.



Construction 3.3 (**Labeling and directing a rooted tree**) Let G be a tree with $q + 1$ vertices. Fix a vertex v_0 of G to be the **root** of the tree. Label the remaining vertices so that the vertices along the unique path from v_0 to any vertex are labeled in increasing order, so that if

$$v_0, v_{i_1}, v_{i_2}, \ldots, v_{i_t}$$

are the distinct vertices of a path between v_0 and v_{i_t}, then $i_j < i_k$ whenever $1 \leq j < k \leq t$. With this labeling, a direction on the edges of G is defined by writing every edge $[v_j, v_i]$ as an ordered pair $e_i = [v_j, v_i]$ where $j < i$.
$\boldsymbol{\tau(i)}$: We denote the index j in the directed edge e_i by $\tau(i)$, so that the directed edges of G can be written as

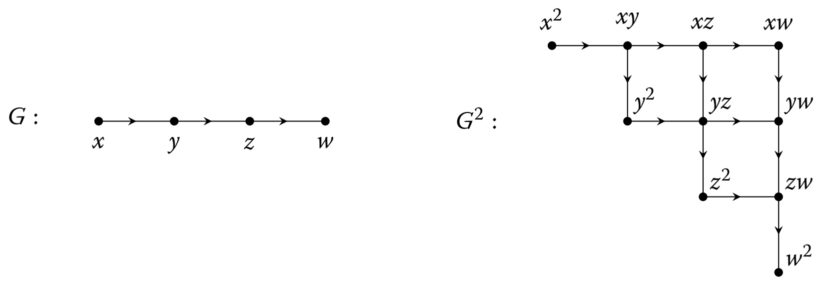
$$e_i = [v_{\tau(i)}, v_i] \quad \text{for} \quad i \in \{1, \ldots, q\}.$$

For the remainder of the paper, all directed graphs will be assumed to have vertices labeled in accordance with Construction 3.3. In particular, v_0 will always denote the root of a directed tree.

Notice that this uniquely labels the q edges of G by $e_1, \ldots e_q$, with each e_i directed toward v_i as seen in Example 3.4 below.

Furthermore, the edges in G^r inherit their direction from the edges of G: if $[v_{\tau(i)}, v_i]$ is a directed edge of G, then the corresponding edge $[\mathbf{v}^{\mathbf{a}}v_{\tau(i)}, \mathbf{v}^{\mathbf{a}}v_i]$ is a directed edge of G^r for every $\mathbf{a} \in \mathcal{N}_{r-1}$.

Example 3.4 We direct the edges of G and G^2 of Example 3.2 by picking $v_0 = x$ as the root.



Lemma 3.5 *Let G be a directed tree on vertices $v_0, \ldots, v_q$, labeled as in Construction 3.3, let $r, q > 0$, and let $\boldsymbol{a} = (a_0, \ldots, a_q)$, $\boldsymbol{b} = (b_0, \ldots, b_q) \in \mathcal{N}_r$. Then $[\boldsymbol{v}^{\boldsymbol{a}}, \boldsymbol{v}^{\boldsymbol{b}}]$ is a directed edge of G^r if and only if for a unique $i \in [q]$, we have*

$$\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{f}_i + \boldsymbol{f}_{\tau(i)}$$

where $\boldsymbol{f}_j$ denotes the $(j+1)^{st}$ standard basis vector in $\mathbb{R}^{q+1}$ as in (3.1.1).

Proof By definition, $[\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}]$ is a directed edge of G^r if and only if for some $i \in [q]$ and $\mathbf{c} = (c_0, \ldots, c_q) \in \mathcal{N}_{r-1}$,

$$\mathbf{v}^{\mathbf{a}} = \mathbf{v}^{\mathbf{c}} \cdot v_{\tau(i)} \quad \text{and} \quad \mathbf{v}^{\mathbf{b}} = \mathbf{v}^{\mathbf{c}} \cdot v_i.$$

This happens if and only if $a_j = b_j = c_j$ when $j \notin \{i, \tau(i)\}$, and since $\tau(i) \neq i$, $b_i = c_i + 1 = a_i + 1$ and $b_{\tau(i)} + 1 = c_{\tau(i)} + 1 = a_{\tau(i)}$. Thus

$$\mathbf{a} = \mathbf{b} - \mathbf{f}_i + \mathbf{f}_{\tau(i)}.$$

The uniqueness of i also follows from the same observation, since the only coordinates in which $\mathbf{a}$ and $\mathbf{b}$ differ are i and $\tau(i)$, and we know $\tau(i) < i$, $b_i = a_i + 1$, and $b_{\tau(i)} + 1 = a_{\tau(i)}$, so the roles of i and $\tau(i)$ cannot be reversed. □

3.6. Embedding G^r in $\mathbb{R}^q$ as 1-skeleta of cubes We now define an explicit embedding of G^r into the Euclidean space $\mathbb{R}^q$ when G is a tree. The embedding chosen is based on the edges of G and designed so that each edge of G^r will be parallel to an axis of $\mathbb{R}^q$.

Definition 3.7 Let G be a directed rooted tree on $q + 1$ vertices $v_0, \ldots, v_q$, labeled as in Construction 3.3. Define a $q \times q$ matrix $\Phi = \Phi(G) = (\Phi_{i,j})$ by:

$$\Phi_{i,j} = \begin{cases} 1 & \text{if } e_i \text{ lies on the unique path from } v_0 \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

That is, Φ is the vertex-path incidence matrix whose rows are indexed by edges $e_1, \ldots, e_q$ and whose columns are indexed by the vertices $v_1, \ldots, v_q$ with i, j entry indicating whether or not the e_i is in the path from v_0 to v_j in the graph G. Using this matrix, we can embed the vertices of G^r into $\mathbb{R}^q$.

Example 3.8 Let G be the path graph in Example 2.8 with labels v_0, v_1, v_2 replacing the vertex labels xy, yz, zu, respectively. Since e_1 lies on the unique path from v_0 to v_1 and v_2, but e_2 only lies on the unique path to v_2, the matrix $\Phi(G)$ is

$$\Phi(G) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Definition 3.9 Let G be a directed rooted tree on $q+1$ vertices $v_0, \ldots, v_q$, labeled as in Construction 3.3. Define $\varphi\colon \mathbb{Z}_{\geq 0}^{q+1} \to \mathbb{R}^q$ by

$$\varphi(a_0, a_1, \ldots, a_q) = \Phi(G) \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix}.$$

For convenience, we write $\varphi(v_0^{a_0} v_1^{a_1} \cdots v_q^{a_q}) = \varphi(a_0, \ldots, a_q) = \varphi(\mathbf{a})$.

Lemma 3.10 *The function* $\varphi\big|_{\mathcal{N}_r}$ *is injective.*

Proof First notice that if $(a_0, \ldots, a_q) \in \mathcal{N}_r$, then $a_0 = r - (a_1 + \cdots + a_q)$ and so the value of a_0 is uniquely determined by the vector $(a_1, \ldots, a_q)$.

By Construction 3.3, every vertex v_j appearing on the unique path from v_0 to v_i satisfies $j < i$. Thus, by the labeling of the edges using their terminal vertex, the matrix Φ is an upper triangular matrix. In addition, the diagonal entries are all equal to 1 since by definition, e_i is the final edge in the unique path from v_0 to v_i. Therefore $\Phi(G)$ is a nonsingular matrix, and as a result $\varphi\big|_{\mathcal{N}_r}$ is injective. □

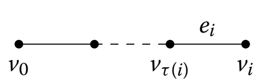
Notice that under the embedding φ, the i^{th} coordinate of the point $\varphi(v_0^{a_0} \cdots v_q^{a_q})$ in $\mathbb{R}^q$ is the total number of times, with multiplicity, that the edge e_i appears on the path from v_0 to any vertex v_k with $a_k \geq 1$ and $k \geq 1$.

Lemma 3.11 *Let G be a tree, labeled as in Construction 3.3, $r, q > 0$, and $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}_r$. If $[\boldsymbol{v^a}, \boldsymbol{v^b}]$ is a directed edge of G^r, and i is as in Lemma 3.5, then*

$$\varphi(\boldsymbol{b}) = \varphi(\boldsymbol{a}) + \mathbf{e}_i,$$

where $\mathbf{e}_i$ denotes the i^{th} standard basis vector in $\mathbb{R}^q$.

Proof Let $\mathbf{a} = (a_0, \ldots, a_q)$ and $\mathbf{b} = (b_0, \ldots, b_q)$. If $[\mathbf{v^a}, \mathbf{v^b}]$ is a directed edge of G^r, then using the unique i from Lemma 3.5, the unique path in G from v_0 to v_i is an extension of the unique path from v_0 to $v_{\tau(i)}$ by the edge e_i.



Therefore, for $i \in [q]$, if $\Phi(G)_i$ denotes the i^{th} column of the matrix,

$$\Phi(G)_i = \begin{cases} \mathbf{e}_i & \text{if} \quad \tau(i) = 0 \\ \Phi(G)_{\tau(i)} + \mathbf{e}_i & \text{if} \quad \tau(i) \geq 1. \end{cases}$$

By Lemma 3.5 we have

$$\begin{aligned} \varphi(\mathbf{b}) =& \Phi(G) \begin{pmatrix} b_1 & \cdots & b_q \end{pmatrix}^T \\ =& \Phi(G) \begin{pmatrix} a_1 & \cdots & a_q \end{pmatrix}^T - \Phi(G)_{\tau(i)} + \Phi(G)_i \\ =& \varphi(\mathbf{a}) + \mathbf{e}_i, \end{aligned}$$

where when $\tau(i) = 0$, we set $\Phi(G)_0 = \mathbf{0}$ for convenience. □

Definition 3.12 (**The directed graph** $\varphi(G^r)$) Let G be a directed rooted tree as in Construction 3.3. We define $\varphi(G^r)$ to be the directed graph embedded in $\mathbb{R}^q$ whose vertices are $\varphi(\mathbf{a})$ where $\mathbf{a} \in \mathcal{N}_r$ and whose edges are $[\varphi(\mathbf{a}), \varphi(\mathbf{b})]$ where $[\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}]$ is an edge of G^r. By Lemma 3.11 we must have $\varphi(\mathbf{b}) = \varphi(\mathbf{a}) + \mathbf{e}_i$ for a unique $i \in [q]$.

To summarize we observe the following correspondence between edges of the directed graphs G^r and $\varphi(G^r)$. If $\mathbf{a}, \mathbf{b} \in \mathcal{N}_r$, then for a unique $j \in [q]$

$$\begin{aligned} \in E(\varphi(G^r)) &\iff \varphi(\mathbf{b}) = \varphi(\mathbf{a}) + \mathbf{e}_i \\ &\iff \mathbf{b} = \mathbf{a} + \mathbf{f}_i - \mathbf{f}_{\tau(i)} \\ &\iff [\mathbf{v}^{\mathbf{a}}, \mathbf{v}^{\mathbf{b}}] \in E(G^r), \end{aligned} \tag{3.12.1}$$

where $\mathbf{e}_i$ and $\mathbf{f}_i$ are unit vectors in $\mathbb{R}^q$ and $\mathbb{R}^{q+1}$ respectively, as in Definition 3.1.

Example 3.13 Let G be the graph in Example 2.8 with matrix $\Phi(G) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ computed in Example 3.8. The vertex set of G^2 is $\{v_0^2, v_0v_1, v_0v_2, v_1^2, v_1v_2, v_2^2\}$ with corresponding exponent vectors

$$\{(2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2)\}.$$

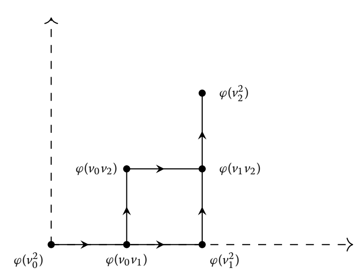
We thus have

$$\varphi(2,0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \varphi(0,1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Similarly we have

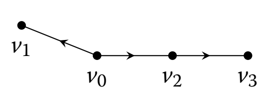
$$\varphi(1,1,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \varphi(1,0,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \varphi(0,2,0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ \varphi(0,0,2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

The graph $\varphi(G^2)$ is above.



Example 3.14 Let G be the directed tree with 4 vertices that starts at the root v_0 and consists of two distinct paths as shown in the figure below. The directed edges of G have the form $[v_j, v_i]$ with $j < i$.

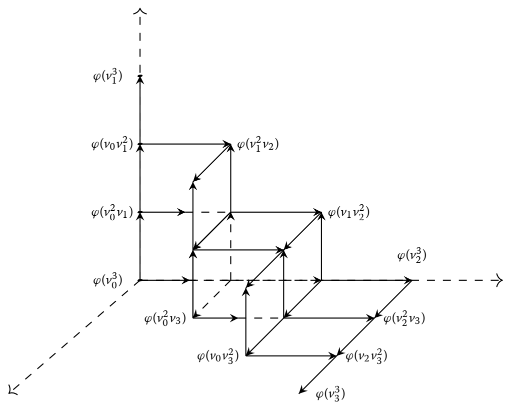
$G:$



The matrix $\Phi(G)$ is

$$\Phi(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The figure below shows the graph $\varphi(G^3)$.



3.15. Building the cubical polyhedral cell complex $\overline{G^r}$ in $\mathbb{R}^q$. Now that we have a directed graph G^r, we focus on building an acyclic polyhedral cell complex which has G^r as its 1-skeleton. Using the concept of a sink (see Definition 2.4), we identify the 1-skeleta of cubes that appear as induced subgraphs of $\varphi(G^r)$. Below, by the **support** of $\mathbf{v}^{\mathbf{a}}$ or of $\mathbf{a} = (a_0, a_1, \dots, a_q) \in \mathbb{Z}_{\geq 0}^{q+1}$ we mean the set

$$\mathrm{Supp}(\mathbf{v}^{\mathbf{a}}) = \mathrm{Supp}(\mathbf{a}) = \{j > 0 \mid a_j \neq 0\} \subseteq [q].$$

Definition 3.16 (**The subgraph** $\mathcal{C}(\mathbf{b}, B)$ **of** $\varphi(G^r)$) If G is a directed rooted tree labeled as in Construction 3.3, $r, q > 0$, $\mathbf{b} \in \mathcal{N}_r$ and $\varnothing \neq B \subseteq \mathrm{Supp}(\mathbf{b})$, then we denote by $\mathcal{C}(\mathbf{b}, B)$ the induced directed subgraph of $\varphi(G^r)$ on the vertex set

$$\{\varphi(\mathbf{b}) - \sum_{i \in B'} \mathbf{e}_i \mid \varnothing \subseteq B' \subseteq B\}.$$

Proposition 3.17 *Let G be a directed rooted tree as in Construction 3.3, $r, q > 0$, $\boldsymbol{b} \in \mathcal{N}_r$ and $\varnothing \neq B \subseteq Supp(\boldsymbol{b})$. Then $\mathcal{C}(\boldsymbol{b}, B)$ is the 1-skeleton of a $|B|$-dimensional cube in $\mathbb{R}^q$ with source $\varphi(\boldsymbol{a})$, where*

$$\boldsymbol{a} = \boldsymbol{b} - \sum_{i \in B} (\boldsymbol{f}_i - \boldsymbol{f}_{\tau(i)}),$$

sink $\varphi(\boldsymbol{b})$, and edges

$$\left[\varphi(\boldsymbol{a}) + \sum_{i \in B'} \boldsymbol{e}_i,\ \varphi(\boldsymbol{a}) + \sum_{i \in B'} \boldsymbol{e}_i + \boldsymbol{e}_k\right] \quad \text{for} \quad \varnothing \subseteq B' \subsetneq B \quad \text{and} \quad k \in B \smallsetminus B'.$$

Proof Let $\mathbf{b} = (b_0, b_1, \dots, b_q)$ and set

$$\mathbf{a} = \mathbf{b} + \sum_{i \in B} \mathbf{f}_{\tau(i)} - \sum_{i \in B} \mathbf{f}_i \tag{3.17.1}$$

as in the statement of the theorem. Since $B \subseteq \mathrm{Supp}(\mathbf{b})$, $\mathbf{a} = (a_0, a_1, \dots, a_q)$ in $\mathcal{N}_r$ and $\varphi(\mathbf{a}) \in \varphi(G^r)$. It follows that $a_{\tau(i)} > 0$ for every $i \in B$, and by Lemmas 3.5 and 3.11, $\varphi(\mathbf{a}) = \varphi(\mathbf{b}) - \sum_{i \in B} \mathbf{e}_i$. Therefore

$$V = \{\varphi(\mathbf{a}) + \sum_{i \in B'} \mathbf{e}_i \mid \varnothing \subseteq B' \subseteq B\}.$$

Moreover, for each $B' \subseteq B$,

$$\varphi(\mathbf{a}) + \sum_{i \in B'} \mathbf{e}_i = \varphi\big(\mathbf{a} + \sum_{i \in B'} (\mathbf{f}_i - \mathbf{f}_{\tau(i)})\big) \in \varphi(G^r).$$

By (3.12.1), $[\mathbf{v^c}, \mathbf{v^d}]$ is an edge of G^r if and only if $\mathbf{d} = \mathbf{c} + \mathbf{f}_k - \mathbf{f}_{\tau(k)}$ for some k, or equivalently $\varphi(\mathbf{d}) = \varphi(\mathbf{c}) + \mathbf{e}_k$. Thus the edges of the induced graph on vertex set V are precisely

$$\Big[\varphi(\mathbf{a}) + \sum_{i \in B'} \mathbf{e}_i,\ \varphi(\mathbf{a}) + \sum_{i \in B'} \mathbf{e}_i + \mathbf{e}_k\Big] \quad \text{for} \quad \varnothing \subseteq B' \subsetneq B \quad \text{and} \quad k \in B \setminus B'.$$

By Definition 2.4, these are the edges of a $|B|$-dimensional cube in $\mathbb{R}^q$ with source $\varphi(\mathbf{a})$ and sink $\varphi(\mathbf{b}) = \varphi(\mathbf{a} + \sum_{i \in B} \mathbf{e}_i)$, as desired. □

Consider the graph $\varphi(G^r)$ as a union of 1-skeleta of cubes embedded in $\mathbb{R}^q$, as described in Proposition 3.17. Our next step is to fill each of these cubes to build a polyhedral cell complex. With this goal in mind, we set the following notation.

Notation 3.18 *Let $\overline{\mathcal{C}}(\boldsymbol{b}, B)$ with $B \subseteq Supp(\boldsymbol{b})$ denote the solid cube whose 1- skeleton is $\mathcal{C}(\boldsymbol{b}, B)$ from Proposition 3.17. Let $\overline{G^r}$ be the collection of the solid cubes $\overline{\mathcal{C}}(\boldsymbol{b}, B)$ for each $\boldsymbol{b} \in \mathcal{N}_r$ and $B \subseteq Supp(\boldsymbol{b})$. More precisely*

$$\overline{G^r} = \bigcup_{\substack{\boldsymbol{b} \in \mathcal{N}_r \\ B \subseteq Supp(\boldsymbol{b})}} \{\overline{\mathcal{C}}(\boldsymbol{b}, B)\}. \tag{3.18.1}$$

Note that if $B = \varnothing$, then we set $\overline{\mathcal{C}}(\boldsymbol{b}, B) = \mathcal{C}(\boldsymbol{b}, B)$ to be the 0-dimensional cube $\{\varphi(\boldsymbol{b})\}$.

Example 3.19 If $\varphi(G^r)$ is as in Example 3.13, then $\overline{G^2}$ is shown below. The 2-cell is $\overline{\mathcal{C}}((0, 1, 1), \{1, 2\})$, with

$$\text{source} \quad \varphi(1, 1, 0) = \varphi(v_0 v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and sink} \quad \varphi(0, 1, 1) = \varphi(v_1 v_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

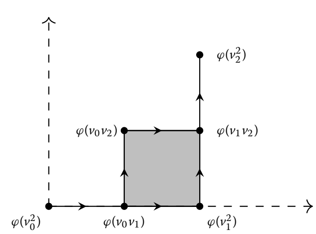
Taking $\mathbf{a} = (1, 1, 0)$, the edges of the 2-cube $\overline{\mathcal{C}}((0, 1, 1), \{1, 2\})$ are

$$\begin{aligned} &\big[\varphi(\mathbf{a}),\ \varphi(\mathbf{a}) + \mathbf{e}_1\big], \big[\varphi(\mathbf{a}),\ \varphi(\mathbf{a}) + \mathbf{e}_2\big], \big[\varphi(\mathbf{a}) + \mathbf{e}_1, \varphi(\mathbf{a}) + \mathbf{e}_1 + \mathbf{e}_2\big], \\ &\big[\varphi(\mathbf{a}) + \mathbf{e}_2,\ \varphi(\mathbf{a}) + \mathbf{e}_2 + \mathbf{e}_1\big], \end{aligned}$$

or equivalently, the edges can be written as

$$[(1, 0), (2, 0)], [(1, 0), (1, 1)], [(2, 0), (2, 1)], [(1, 1), (2, 1)].$$

Similarly, the 1-cell $\overline{\mathcal{C}}((0, 1, 1), \{1\})$ is the top edge of the square, with source $\varphi(1, 0, 1) = \varphi(v_0 v_2) = (1, 1)^T$.



3.20. Constructing $\overline{G^{r+1}}$ from $\overline{G^r}$ when $r \geq q$. Note that, for any $\mathbf{b} \in \mathcal{N}_r$ and for any $B \subseteq \text{Supp}(\mathbf{b})$,

$$\overline{\mathcal{C}}(\mathbf{b}, B)) = \overline{\mathcal{C}}(\mathbf{b} + \mathbf{f}_0, B), \text{ where } \mathbf{f}_0 = (1, 0, \ldots, 0),$$

hence there is an embedding $\overline{G^r} \subseteq \overline{G^{r+1}}$. Since $\mathbf{b} \in \mathcal{N}_r$, we always have $|\text{Supp}(\mathbf{b})| \leq r$. On the other hand when $r \leq q$, there exists at least one $\mathbf{b} \in \mathcal{N}_r$ such that $|\text{Supp}(\mathbf{b})| = r$, hence $\overline{\mathcal{C}}(\mathbf{b}, \mathrm{Supp}(\mathbf{b}))$ is a maximal, r-dimensional cube in $\overline{G^r}$. Thus, each iteration of the construction of $\overline{G^r}$ adds new, higher dimensional cubes, as long as $r \leq q$. We refer the reader to the picture of $\varphi(G^3)$ in Example 3.14 in order to visualize the 3-dimensional cube in $\overline{G^3}$ for that example. On the other hand, the dimension of the cubes cannot exceed the dimension q of the space, so the maximal dimension of a cube stabilizes at q. This does not mean that $\overline{G^r}$ and $\overline{G^{r+1}}$ become equal, but it turns out that, when $r \geq q$, all the cubes of $\overline{G^{r+1}}$ can be described as translations of cubes in $\overline{G^r}$, as discussed below.

We denote by $\overline{\mathcal{C}}(\mathbf{b}, B) + \mathbf{x}$ the translation by the vector $\mathbf{x} \in \mathbb{R}^q$ of a cell $\overline{\mathcal{C}}(\mathbf{b}, B) \in \overline{G^r}$. For any $0 \leq j \leq q$, $\mathbf{b} \in \mathcal{N}_r$, and $B \subseteq \mathrm{Supp}(\mathbf{b})$, observe that

$$\overline{\mathcal{C}}(\mathbf{b}, B) + \varphi(\mathbf{f}_j) = \overline{\mathcal{C}}(\mathbf{b} + \mathbf{f}_j, B). \tag{3.20.1}$$

In particular, translation by the vector $\varphi(\mathbf{f}_j)$ in $\mathbb{R}^q$ induces a cellular map $t_j \colon \overline{G^r} \to \overline{G^{r+1}}$, satisfying

$$t_j(\overline{\mathcal{C}}(\mathbf{b}, B)) = \overline{\mathcal{C}}(\mathbf{b} + \mathbf{f}_j, B) \in \overline{G^{r+1}}.$$

When $r \geq q$, notice that all the cubes in $\overline{G^{r+1}}$ are obtained as translations of cubes in $\overline{G^r}$, namely:

$$\bigcup_{0 \leq j \leq q} t_j(\overline{G^r}) = \overline{G^{r+1}} \quad \text{for all} \quad r \geq q\,. \tag{3.20.2}$$

Indeed, let $\mathbf{a} \in \mathcal{N}_{r+1}$ and $A \subseteq \mathrm{Supp}(\mathbf{a})$. If $r \geq q$, then, since $a_0 + \ldots a_q = r + 1$, either $a_i > 1$ for some $i \in [q]$ or else $a_0 \geq 1$. When $a_0 \geq 1$ and $a_j \leq 1$ for all $j \in [q]$, set $i = 0$. With i as above, one has $\mathbf{a} - \mathbf{f}_i \in \mathcal{N}_r$, $A \subseteq \mathrm{Supp}(\mathbf{a}) = \mathrm{Supp}(\mathbf{a} - \mathbf{f}_i)$, and

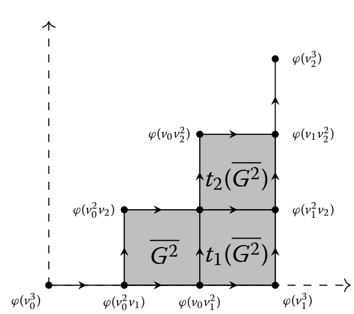
$$\overline{\mathcal{C}}(\mathbf{a}, A) = t_i(\overline{\mathcal{C}}(\mathbf{a} - \mathbf{f}_i, A)) \in t_i(\overline{G^r}),$$

which proves Equation 3.20.2.

Example 3.21 In the context of Example 3.19,

$$\varphi(\mathbf{f}_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \varphi(\mathbf{f}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(\mathbf{f}_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The cell complex $\overline{G^3}$ is shown below, and consists of three overlapping copies of $\overline{G^2}$, obtained by translating $\overline{G^2}$ along the vectors listed above. Note that $t_0(\overline{G^2}) = \overline{G^2}$ and the center most vertex in the picture is $\varphi(v_0 v_1 v_2)$.



Proposition 3.22 *(**The faces of the cube** $\overline{\mathcal{C}}(\boldsymbol{b}, B)$) Let $r, q > 0$, and let G be a directed tree on $q + 1$ vertices labeled as in Construction 3.3, $\boldsymbol{b} \in \mathcal{N}_r$, and $\varnothing \neq B \subseteq \mathrm{Supp}(\boldsymbol{b})$. Then for every face F of $\overline{\mathcal{C}}(\boldsymbol{b}, B)$ there exists a subset $C \subseteq B$ such that*

$$F = \overline{\mathcal{C}}(\boldsymbol{c}, B \setminus C) \quad \textit{where} \quad \boldsymbol{c} = \boldsymbol{b} - \sum_{i \in C'} \left(\boldsymbol{f}_i - \boldsymbol{f}_{\tau(i)}\right) \quad \textit{for some} \quad \varnothing \subseteq C' \subseteq C.$$

In particular, for a $(|B|-1)$*-dimensional face* F *of* $\overline{C}(\boldsymbol{b}, B)$ *there is some* $i \in B$ *such that*

$$F = \overline{C}(\boldsymbol{b}, B \setminus \{i\}) \quad \text{or} \quad F = \overline{C}(\boldsymbol{b} - \boldsymbol{f}_i + \boldsymbol{f}_{\tau(i)}, B \setminus \{i\}).$$

Proof Let $\varphi(\mathbf{a})$ be the source of $\mathcal{C}(\mathbf{b}, B)$ as in Proposition 3.17. Assume that F has dimension $|B| - t$ for $t \geq 1$. We use induction on t to prove the statement. The base case is $t = 1$. By Definition 2.4 for a fixed $i \in B$, the face F can be described as the cube on vertices

$$\Big\{\varphi(\mathbf{a}) + \mathbf{e}_i + \sum_{j \in B'} \mathbf{e}_j \mid B' \subseteq B \setminus \{i\}\Big\} \quad \text{or} \quad \Big\{\varphi(\mathbf{a}) + \sum_{j \in B'} \mathbf{e}_j \mid B' \subseteq B \setminus \{i\}\Big\}.$$

Using (3.12.1), it follows immediately that

$$F = \overline{C}(\mathbf{b}, B \setminus \{i\}) \quad \text{or} \quad F = \overline{C}(\mathbf{b} - \mathbf{f}_i + \mathbf{f}_{\tau(i)}, B \setminus \{i\}).$$

If we set $C = \{i\}$, then indeed we have shown that

$$F = \overline{C}(\mathbf{c}, B \setminus C) \quad \text{where} \quad \mathbf{c} = \mathbf{b} - \sum_{i \in C'} \left(\mathbf{f}_i - \mathbf{f}_{\tau(i)}\right) \quad \text{and} \quad C' = \varnothing \quad \text{or} \quad C' = \{i\}.$$

Now suppose $t > 1$, and F is of dimension $|B| - t$. By the definition of a cube (Definition 2.4), F is a the boundary of a $|B| - t + 1 = |B| - (t-1)$-dimensional cube G, which is itself a face of $\overline{C}(\mathbf{b}, B)$. By the induction hypothesis, there exists a subset $D \subseteq B$ such that

$$G = \overline{C}(\mathbf{d}, B \setminus D) \quad \text{where} \quad \mathbf{d} = \mathbf{b} - \sum_{i \in D'} \left(\mathbf{f}_i - \mathbf{f}_{\tau(i)}\right) \quad \text{for some} \quad \varnothing \subseteq D' \subseteq D.$$

By the base case of the induction, we know for some $i \in B \setminus D$,

$$F = \overline{C}(\mathbf{d}, B \setminus (D \cup \{i\})) \quad \text{or} \quad F = \overline{C}(\mathbf{d} - \mathbf{f}_i + \mathbf{f}_{\tau(i)}, B \setminus (D \cup \{i\})).$$

Setting $C = D \cup \{i\}$, we have the desired result where $C' = D'$ or $C' = D' \cup \{i\}$. □

Proposition 3.23 *(*$\overline{G^r}$ ***is a polyhedral cell complex****) Let* $r, q > 0$ *and let* G *be a tree on* $q+1$ *vertices as in Construction 3.3. Then* $\overline{G^r}$ *is a polyhedral cell complex.*

Proof Definition 2.3 (1) follows directly from Proposition 3.22. To verify Definition 2.3 (2), for $\mathbf{b}, \mathbf{d} \in \mathcal{N}_r$ consider

$$F = \overline{C}(\mathbf{b}, B) \cap \overline{C}(\mathbf{d}, D) \quad \text{where} \quad \varnothing \neq B \subseteq \operatorname{Supp}(\mathbf{b}) \quad \text{and} \quad \varnothing \neq D \subseteq \operatorname{Supp}(\mathbf{d}).$$

We need to show

$$F = \overline{C}(\mathbf{c}, C) \quad \text{for some} \quad \mathbf{c} \in \mathcal{N}_r \quad \text{and} \quad C \subseteq \operatorname{Supp}(\mathbf{c}).$$

By Proposition 3.17, the cubes $\overline{C}(\mathbf{b}, B)$ and $\overline{C}(\mathbf{d}, D)$ have vertex sets

$$\{\varphi(\mathbf{b}) - \sum_{j \in A} \mathbf{e}_j \mid A \subseteq B\} \quad \text{and} \quad \{\varphi(\mathbf{d}) - \sum_{j \in A} \mathbf{e}_j \mid A \subseteq D\},$$

respectively.

Claim: if $B_1, B_2 \subseteq B$ are such that

$$u = \varphi(\mathbf{b}) - \sum_{j \in B_1} \mathbf{e}_j \in F \quad \text{and} \quad v = \varphi(\mathbf{b}) - \sum_{j \in B_2} \mathbf{e}_j \in F, \tag{3.23.1}$$

then

$$\varphi(\mathbf{b}) - \sum_{j \in B_1 \cap B_2} \mathbf{e}_j \in F. \tag{3.23.2}$$

To see this, first note that

$$\varphi(\mathbf{b}) - \sum_{j \in B_1 \cap B_2} \mathbf{e}_j \in \overline{\mathcal{C}}(\mathbf{b}, B)$$

by definition. Now since $u, v \in F$, then $u, v \in \overline{\mathcal{C}}(\mathbf{d}, D)$. Hence, for some $D_1, D_2 \subseteq D$

$$u = \varphi(\mathbf{d}) - \sum_{j \in D_1} \mathbf{e}_j \quad \text{and} \quad v = \varphi(\mathbf{d}) - \sum_{j \in D_2} \mathbf{e}_j. \tag{3.23.3}$$

Now (3.23.1) tells us that

$$u - \sum_{j \in B_2 \smallsetminus B_1} \mathbf{e}_j = v - \sum_{j \in B_1 \smallsetminus B_2} \mathbf{e}_j$$

which combined with (3.23.3) implies that

$$\varphi(\mathbf{d}) - \sum_{j \in D_1} \mathbf{e}_j - \sum_{j \in B_2 \smallsetminus B_1} \mathbf{e}_j = \varphi(\mathbf{d}) - \sum_{j \in D_2} \mathbf{e}_j - \sum_{j \in B_1 \smallsetminus B_2} \mathbf{e}_j.$$

This implies that

$$D_1 \cup (B_2 \smallsetminus B_1) = D_2 \cup (B_1 \smallsetminus B_2) \Longrightarrow (B_1 \smallsetminus B_2) \subseteq D_1.$$

Now combining (3.23.1) and (3.23.3) we get

$$\varphi(\mathbf{b}) - \sum_{j \in B_1 \smallsetminus B_2} \mathbf{e}_j - \sum_{j \in B_1 \cap B_2} \mathbf{e}_j = u = \varphi(\mathbf{d}) - \sum_{j \in D_1} \mathbf{e}_j.$$

Moreover, since $(B_1 \smallsetminus B_2) \subseteq D_1$, it follows that

$$\varphi(\mathbf{b}) - \sum_{j \in B_1 \cap B_2} \mathbf{e}_j = \varphi(\mathbf{d}) - \sum_{j \in D_1 \smallsetminus (B_1 \smallsetminus B_2)} \mathbf{e}_j \in \overline{\mathcal{C}}(\mathbf{b}, D) \cap \overline{\mathcal{C}}(\mathbf{d}, D).$$

This establishes (3.23.2) as claimed. It follows immediately that there exist unique minimal subsets B_0 of B and D_0 of D such that

$$\varphi(\mathbf{b}) - \sum_{j \in B_0} \mathbf{e}_j = \varphi(\mathbf{d}) - \sum_{j \in D_0} \mathbf{e}_j \in F.$$

Set

$$\mathbf{c} = \mathbf{b} - \sum_{j \in B_0} (\mathbf{f}_j - \mathbf{f}_{\tau(j)}) \in \mathcal{N}_r \quad \text{and} \quad C = (B \smallsetminus B_0) \cap (D \smallsetminus D_0) \subseteq \operatorname{Supp}(\mathbf{c})$$

so that

$$\varphi(\mathbf{c}) = \varphi(\mathbf{b}) - \sum_{j \in B_0} \mathbf{e}_j \in F,$$

and moreover

$$\varphi(\mathbf{c}) - \sum_{j \in C'} \mathbf{e}_j \in F \quad \text{for all} \quad C' \subseteq C.$$

In other words,

$$\overline{\mathcal{C}}(\mathbf{c}, C) \subseteq F. \tag{3.23.4}$$

To see the reverse inclusion to (3.23.4), take $w \in F$ so that

$$w = \varphi(\mathbf{b}) - \sum_{j \in B'} \mathbf{e}_j = \varphi(\mathbf{d}) - \sum_{j \in D'} \mathbf{e}_j$$

for some

$$B_0 \subseteq B' \subseteq B \quad \text{and} \quad D_0 \subseteq D' \subseteq D.$$

So we can write

$$w = \varphi(\mathbf{b}) - \sum_{j \in B_0} \mathbf{e}_j - \sum_{j \in B' \smallsetminus B_0} \mathbf{e}_j = \varphi(\mathbf{c}) - \sum_{j \in B' \smallsetminus B_0} \mathbf{e}_j. \tag{3.23.5}$$

Similarly

$$w = \varphi(\mathbf{c}) - \sum_{j \in D' \smallsetminus D_0} \mathbf{e}_j. \tag{3.23.6}$$

Equations (3.23.5) and (3.23.6) indicate that

$$\sum_{j \in B' \smallsetminus B_0} \mathbf{e}_j = \sum_{j \in D' \smallsetminus D_0} \mathbf{e}_j \Longrightarrow B' \smallsetminus B_0 = D' \smallsetminus D_0 \subseteq C.$$

Hence $w \in \overline{\mathcal{C}}(\mathbf{c}, C)$. We conclude that $F = \overline{\mathcal{C}}(\mathbf{c}, C)$ as desired. □

3.24. Labeling $\overline{G^r}$ with monomials. If I is a monomial ideal generated by $m_0, \ldots, m_q$, the ideal I^r is generated by monomials of the form

$$\mathbf{m}^{\mathbf{a}} = m_0^{a_0} \cdots m_q^{a_q}$$

where $\mathbf{m} = \{m_0, \ldots, m_q\}$ and $\mathbf{a} = (a_0, a_1, \ldots, a_q) \in \mathcal{N}_r$.

When I is square-free of projective dimension one, [3, Proposition 4.1] shows that

$$\mathbf{m}^{\mathbf{a}} = \mathbf{m}^{\mathbf{b}} \iff \mathbf{a} = \mathbf{b}.$$

In particular, the set $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}_r\}$ is a minimal generating set for the ideal I^r in this case. Thus for the remainder of the paper we will assume the following setting unless otherwise noted.

Setting 3.25 Let I be a square-free monomial ideal of projective dimension one in a polynomial ring $R = \mathsf{k}[x_1, \ldots, x_n]$. By [8, Theorem 27], there is an ordering of the minimal monomial generators $m_0, m_1, \ldots, m_q$ of I that produces a tree G, labeled as in Construction 3.3, such that G supports a minimal free resolution of I when each vertex v_i is labeled by m_i. We fix G to be such a tree, with this ordering and labeling.

In this setting, we label each vertex $\mathbf{v}^{\mathbf{a}}$ of $\overline{G^r}$ with the monomial $\mathbf{m}^{\mathbf{a}}$. Our goal is to show that the polyhedral cell complex $\overline{G^r}$ with this labeling supports a minimal free resolution of I^r.

With this monomial labeling, we homogenize the cellular (cubical) chain complex of $\overline{G^r}$. The cell complex $\overline{G^r}$ gives rise to the oriented chain complex $C(\overline{G^r}, \mathsf{k})$, as described in Definition 2.5. In order to define the signs of the maps, let

$$B = \{j_1, \ldots, j_i\} \quad \text{where} \quad j_1 < j_2 < \cdots < j_i.$$

Using Proposition 3.22 the differential ∂_i is described by:

$$\partial_i(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = \sum_{1 \le k \le i} (-1)^{k+1} \mathbf{u}_{\mathcal{C}(\mathbf{b},B \smallsetminus \{j_k\})} + \sum_{1 \le k \le i} (-1)^{k} \mathbf{u}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)},B \smallsetminus \{j_k\})}. \tag{3.25.1}$$

To homogenize these maps, denote the lcm of the monomial labels of the vertices of a cube $\overline{\mathcal{C}}(\mathbf{b}, B)$ by $\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}$. The chain complex in (3.25.1) then homogenizes as described in Sect. 2 to a $\mathbb{Z}^n$-graded cellular complex

$$\mathbb{F}_{\overline{G^r}}: \qquad \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \tag{3.25.2}$$

where F_i is the free graded R-module with basis elements $\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}$ with $\overline{\mathcal{C}}(\mathbf{b},B) \in \overline{G^r}$ and where $\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}$ is considered to be in degree equal to the exponent vector of $\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}$. For each $i > 0$ the differential ∂_i of $\mathbb{F}_{\overline{G^r}}$ is described by:

$$\begin{aligned}\partial_i(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = &\sum_{1\le k\le i} (-1)^{k+1} \frac{\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}}{\mathbf{m}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})}}\, \mathbf{u}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})} + \\ &+ \sum_{1\le k\le i} (-1)^{k} \frac{\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}}{\mathbf{m}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)},B\smallsetminus\{j_k\})}}\, \mathbf{u}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)},B\smallsetminus\{j_k\})}.\end{aligned} \tag{3.25.3}$$

We now focus on the two monomial coefficients appearing in (3.25.3).

Lemma 3.26 *Using Sect. 3.25, let $\boldsymbol{b} \in \mathcal{N}_r$, $B \subseteq Supp(\boldsymbol{b})$ and $i \in B$. The following equalities then hold:*

(1) $\dfrac{\boldsymbol{m}_{\mathcal{C}(\boldsymbol{b},B)}}{\boldsymbol{m}_{\mathcal{C}(\boldsymbol{b},B\smallsetminus\{i\})}} = \dfrac{lcm(m_i, m_{\tau(i)})}{m_i}$;

(2) $\dfrac{\boldsymbol{m}_{\mathcal{C}(\boldsymbol{b},B)}}{\boldsymbol{m}_{\mathcal{C}(\boldsymbol{b}-\boldsymbol{f}_i+\boldsymbol{f}_{\tau(i)},B\smallsetminus\{i\})}} = \dfrac{lcm(m_i, m_{\tau(i)})}{m_{\tau(i)}}$.

Proof By Proposition 3.17 and (3.12.1) the vertices of $\mathcal{C}(\mathbf{b},B)$ are the images under φ of

$$\mathbf{b} - \sum_{j\in B'} \mathbf{f}_j + \sum_{j\in B'} \mathbf{f}_{\tau(j)} \quad \text{for all } B' \text{ such that } \varnothing \subseteq B' \subseteq B.$$

Notice that for each j, the monomial label associated to

$$\mathbf{b} - \mathbf{f}_j + \mathbf{f}_{\tau(j)} \quad \text{is} \quad \frac{\mathbf{m}^{\mathbf{b}} m_{\tau(j)}}{m_j},$$

so for each $B' \subseteq B$, the label associated to

$$\mathbf{b} - \sum_{j\in B'} \mathbf{f}_j + \sum_{j\in B'} \mathbf{f}_{\tau(j)} \quad \text{is} \quad \mathbf{m}^{\mathbf{b}} \prod_{j\in B'} \frac{m_{\tau(j)}}{m_j}.$$

As a result, considering all $2^{|B|}$ vertices of $\mathcal{C}(\mathbf{b},B)$, we have

$$\begin{aligned}\mathbf{m}_{\mathcal{C}(\mathbf{b},B)} &= \operatorname{lcm}\left\{\mathbf{m}^{\mathbf{b}} \prod_{j\in B'} \frac{m_{\tau(j)}}{m_j} \;\middle|\; \varnothing \subseteq B' \subseteq B\right\} \\ &= \operatorname{lcm}\left(\mathbf{m}^{\mathbf{b}}, \left\{\frac{\mathbf{m}^{\mathbf{b}} \cdot m_{\tau(j_1)} \cdots m_{\tau(j_t)}}{m_{j_1} \cdots m_{j_t}}\right\}_{\{j_1,\ldots,j_t\}\subseteq B}\right) \\ &= \mathbf{m}^{\mathbf{b}} \prod_{j\in B} \left(\prod_{x|m_{\tau(j)},\, x\nmid m_j} x\right) \quad \text{where} \quad x \in \{x_1,\ldots,x_n\}.\end{aligned} \tag{3.26.1}$$

Similarly,

$$\mathbf{m}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{i\})} = \mathbf{m}^{\mathbf{b}} \prod_{j\in B\setminus\{i\}} \left(\prod_{x|m_{\tau(j)},\, x\nmid m_j} x\right).$$

Thus the quotient is

$$\frac{\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}}{\mathbf{m}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{i\})}} = \prod_{x|m_{\tau(i)}, x\nmid m_i} x = \frac{m_{\tau(i)}}{\gcd(m_i, m_{\tau(i)})} = \frac{\operatorname{lcm}(m_i, m_{\tau(i)})}{m_i}$$

and (1) follows. To see equality (2), note that by (3.26.1)

$$\frac{\mathbf{m}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_i+\mathbf{f}_{\tau(i)},B\smallsetminus\{i\})}}{\mathbf{m}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{i\})}}=\frac{m_{\tau(i)}}{m_i}$$

and thus using equality (1),

$$\frac{\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}}{\mathbf{m}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_i+\mathbf{f}_{\tau(i)},B\smallsetminus\{i\})}}=\frac{m_i}{m_{\tau(i)}}\frac{\mathbf{m}_{\mathcal{C}(\mathbf{b},B)}}{\mathbf{m}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{i\})}}=\frac{\operatorname{lcm}(m_i,m_{\tau(i)})}{m_{\tau(i)}}.$$

□

As a result of Proposition 3.26, the homogenized differentials described in (3.25.3) can be written as

$$\begin{aligned}\partial_i(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)})=&\sum_{1\le k\le i}(-1)^{k+1}\frac{\operatorname{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{j_k}}\,\mathbf{u}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})}\\&+\sum_{1\le k\le i}(-1)^{k}\frac{\operatorname{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{\tau(j_k)}}\,\mathbf{u}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)},B\smallsetminus\{j_k\})}.\end{aligned}\tag{3.26.2}$$

Example 3.27 Let G be the path graph in Examples 2.8 and 3.13. In the cell complex $\overline{G^2}$ drawn in Example 3.19, let $c=\overline{\mathcal{C}}((0,1,1),\{1,2\})$ be the shaded square. The cells of dimension 1 are the 6 line segments representing the edges, with the horizontal segments on the first line below and the vertical ones on the second.

$$\begin{array}{lll}c_1=\overline{\mathcal{C}}((1,1,0),\{1\}) & c_2=\overline{\mathcal{C}}((0,2,0),\{1\}) & c_3=\overline{\mathcal{C}}((0,1,1),\{1\})\\ c_4=\overline{\mathcal{C}}((0,1,1),\{2\}) & c_5=\overline{\mathcal{C}}((1,0,1),\{2\}) & c_6=\overline{\mathcal{C}}((0,0,2),\{2\}).\end{array}$$

The 0-dimensional cells are shown below, where the first entry is $\varphi(2,0,0)=(0,0)$.

$$\begin{array}{lll}c'_1=\overline{\mathcal{C}}((2,0,0),\varnothing) & c'_2=\overline{\mathcal{C}}((1,1,0),\varnothing) & c'_3=\overline{\mathcal{C}}((0,2,0),\varnothing)\\ c'_4=\overline{\mathcal{C}}((0,1,1),\varnothing) & c'_5=\overline{\mathcal{C}}((1,0,1),\varnothing) & c'_6=\overline{\mathcal{C}}((0,0,2),\varnothing).\end{array}$$

With this notation, the differential of the complex $C(\overline{G^2},\mathsf{k})$ is given as follows:

$$\begin{array}{ll}\partial(\mathbf{u}_c)=\mathbf{u}_{c_2}+\mathbf{u}_{c_4}-\mathbf{u}_{c_3}-\mathbf{u}_{c_5} & \partial(\mathbf{u}_{c_1})=\mathbf{u}_{c'_2}-\mathbf{u}_{c'_1}\\ \partial(\mathbf{u}_{c_2})=\mathbf{u}_{c'_3}-\mathbf{u}_{c'_2} & \partial(\mathbf{u}_{c_3})=\mathbf{u}_{c'_4}-\mathbf{u}_{c'_5}\\ \partial(\mathbf{u}_{c_4})=\mathbf{u}_{c'_4}-\mathbf{u}_{c'_3} & \partial(\mathbf{u}_{c_5})=\mathbf{u}_{c'_5}-\mathbf{u}_{c'_2}\\ \partial(\mathbf{u}_{c_6})=\mathbf{u}_{c'_6}-\mathbf{u}_{c'_4} & \end{array}$$

The complex $C(\overline{G^2},\mathsf{k})$ is thus as follows:

$$0\to\mathsf{k}\xrightarrow{\begin{bmatrix}0\\1\\-1\\1\\-1\\0\end{bmatrix}}\mathsf{k}^6\xrightarrow{\begin{bmatrix}-1&0&0&0&0&0\\1&-1&0&0&-1&0\\0&1&0&-1&0&0\\0&0&1&1&0&-1\\0&0&-1&0&1&0\\0&0&0&0&0&1\end{bmatrix}}\mathsf{k}^6.$$

Now we homogenize this complex using the monomial generators of I, namely $m_0 = xy$, $m_1 = yz$, and $m_2 = zu$, which yields:

$$\mathbb{F}_{\overline{G^2}}:\quad 0 \to R(-4) \xrightarrow{\begin{bmatrix} 0 \\ u \\ -y \\ x \\ -z \\ 0 \end{bmatrix}} R(-3)^6 \xrightarrow{\begin{bmatrix} -z & 0 & 0 & 0 & 0 & 0 \\ x & -z & 0 & 0 & -u & 0 \\ 0 & x & 0 & -u & 0 & 0 \\ 0 & 0 & x & y & 0 & -u \\ 0 & 0 & -z & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & y \end{bmatrix}} R(-2)^6$$

and we get a multigraded resolution as follows:

$$0 \to R(xy^2z^2u) \xrightarrow{\begin{bmatrix} 0 \\ u \\ -y \\ x \\ -z \\ 0 \end{bmatrix}} \begin{array}{c} R(x^2y^2z) \\ \oplus R(xy^2z^2) \\ \oplus R(xyz^2u) \\ \oplus R(y^2z^2u) \\ \oplus R(xy^2zu) \\ \oplus R(yz^2u^2) \end{array} \xrightarrow{\begin{bmatrix} -z & 0 & 0 & 0 & 0 & 0 \\ x & -z & 0 & 0 & -u & 0 \\ 0 & x & 0 & -u & 0 & 0 \\ 0 & 0 & x & y & 0 & -u \\ 0 & 0 & -z & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & y \end{bmatrix}} \begin{array}{c} R(x^2y^2) \\ \oplus R(xy^2z) \\ \oplus R(y^2z^2) \\ \oplus R(yz^2u) \\ \oplus R(xyzu) \\ \oplus R(z^2u^2) \end{array}.$$

Remark 3.28 Let $r \geq 1$. In Sect. 3, we introduced the translations $t_i\colon \overline{G^r} \to \overline{G^{r+1}}$ and we noted they are cellular maps. At the level of the associated homogenized chain complexes, these cellular maps induce chain maps $\widetilde{t_i}\colon \mathbb{F}_{\overline{G^r}} \to \mathbb{F}_{\overline{G^{r+1}}}$ described by

$$\widetilde{t_i}(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = \mathbf{u}_{\mathcal{C}(\mathbf{b}+\mathbf{f}_i,B)},$$

for all $\overline{\mathcal{C}}(\mathbf{b}, B) \in \overline{G^r}$ and all i with $0 \leq i \leq q$. One can check that these maps are indeed chain maps, using the description of the differentials in Example 3.26.2. As will be shown in Sect. 4, under the hypotheses of Sect. 3.25 $F_{\overline{G^r}}$ is a minimal free resolution of I^r and $F_{\overline{G^{r+1}}}$ is a minimal free resolution of I^{r+1}.

Note that the chain maps $\widetilde{t_i}$ can also be described as the chain maps induced by the map $I^r \to I^{r+1}$ given by multiplication by m_i. Additionally, if we consider the map $\psi\colon \bigoplus_{i=0}^q I^r \to I^{r+1}$ whose i^{th} component is given by multiplication by m_i for each i, then the induced chain maps

$$\bigoplus_{i=0}^{q} \widetilde{t_i}\colon \bigoplus_{i=0}^{q} \mathbb{F}_{\overline{G^r}} \to \mathbb{F}_{\overline{G^{r+1}}}$$

are surjective when $r \geq q$, in view of Equation 3.20.2. Hence, the induced maps

$$\operatorname{Tor}_j^R(\psi, \Bbbk)\colon \operatorname{Tor}_j^R(\bigoplus_{i=0}^{q} I^r, \Bbbk) \to \operatorname{Tor}_j^R(I^{r+1}, \Bbbk)$$

are surjective for all $j \geq 0$ and all $r \geq q$.

4 Minimal cellular resolutions of I^r

In this section, we construct a cellular resolution of I^r, where I is a square-free monomial ideal of projective dimension one, showing that Question 1.1 has a positive answer for this class of ideals. At this point, using the graph G as in Sect. 3.25, we have constructed $\overline{G^r}$ as a polyhedral cell complex labeled with the monomials that generate I^r. What remains in this section is to prove that the labeled chain complex of $\overline{G^r}$ gives a minimal free resolution of I^r. To complete this task we will compare the labeled chain complex of $\overline{G^r}$ to another chain complex, which is in turn isomorphic to the minimal free resolution of I^r.

The *Rees algebra* of an ideal I is a well-studied object that encodes all powers of I into a single construction. Having a family of minimal resolutions of I^r for all $r > 0$ that are constructed from a common base, namely G, naturally leads one to wonder if these resolutions are related to a resolution of the Rees algebra. Indeed, the other chain complex mentioned in the previous paragraph turns out to stem from the Rees algebra. We will show that for square-free monomial ideals of projective dimension one, the Rees algebra can be presented as a quotient of a polynomial ring by a complete intersection ideal. A strand of the Koszul resolution of the complete intersection is the chain complex that will be isomorphic to the chain complex of $\overline{G^r}$.

4.1. Rees algebras and ideals of linear type We first recall background information related to the Rees algebra and show that square-free monomial ideals of projective dimension one are of linear type.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. The **Rees algebra** of I, denoted $R[It]$, is a subalgebra of the polynomial ring $R[t]$ consisting of polynomials for which the coefficient of t^s is in I^s for all s. That is,

$$R[It] = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots.$$

When $R = \mathsf{k}[x_1, \ldots, x_n]$ is a polynomial ring, the definition carries through with $\mathfrak{m} = (x_1, \ldots, x_n)$ the homogeneous maximal ideal of R. A common way to gain insight into $R[It]$ is to embed it in a larger polynomial ring and then study its defining ideal. Suppose I is a monomial ideal of projective dimension one minimally generated by square-free monomials $m_0, \ldots, m_q$, and let $S = R[T_0, \ldots, T_q]$. The map $\psi : S \to R[It]$ defined by $\psi(T_i) = m_it$ gives a surjective R-algebra homomorphism. Hence, if $J = \ker\psi$, then

$$R[It] = S/J = R[T_0, \ldots, T_q]/J.$$

The map ψ is graded so its kernel $J = J_1 + J_2 + \cdots$ is a homogeneous ideal, commonly referred to as the **defining ideal** of $R[It]$. The graded component of J of degree 1, namely J_1, is generated by elements of the form:

$$\{a_0T_0 + a_1T_1 + \cdots + a_qT_q \mid a_0m_0 + \ldots + a_qm_q = 0\}$$

which correspond to the generators of the first syzygy module of I. In the case where $I = (m_0, \ldots, m_q)$ is square-free of projective dimension one, we know that ([8, proof of Theorem 8]) the first syzygy module can be generated by the homogenized generators of the chain complex of the graph supporting the resolution of I, namely

$$g_k = \frac{\operatorname{lcm}(m_k, m_{\tau(k)})}{m_k}T_k - \frac{\operatorname{lcm}(m_k, m_{\tau(k)})}{m_{\tau(k)}}T_{\tau(k)} \quad \text{for} \quad k \in [q]. \tag{4.1.1}$$

Therefore, $J_1 = (g_1, \ldots, g_q)$.

We now show that I is of **linear type**, that is, J is generated by its degree one elements.

Lemma 4.2 *Let I be an ideal of projective dimension one minimally generated by $q+1$ square-free monomials in the polynomial ring $R = \mathsf{k}[x_1, \ldots, x_n]$ over a field* k*. Then $q+1 \leq n$.*

Proof If $q = 0$ the result follows trivially, so suppose $q \geq 1$. From the concrete construction of the tree G that supports a minimal free resolution of I it follows [3, (4.0.1)] that for every $i \in [q]$ there exists a variable $x_{a_i} \in \{x_1, \ldots, x_n\}$ such that

$$x_{a_i} \nmid m_i \quad \text{and} \quad x_{a_i} \mid m_j \quad \text{for} \quad 0 \leq j < i. \tag{4.2.1}$$

This implies that $q \leq n$. On the other hand, by (4.2.1) $x_{a_1} \cdots x_{a_q} \mid m_0$. If for any j, $x \mid m_j$ implies $x = x_{a_i}$ for some i, then since I is square-free, $m_j \mid m_0$, a contradiction. Thus we must have at least $q+1$ variables; i.e. $q+1 \leq n$. □

Theorem 4.3 *Let I be a square-free monomial ideal of projective dimension one in the polynomial ring $R = \mathsf{k}[x_1, \ldots, x_n]$ over a field* k*. Then I is of linear type.*

Proof Let $\mu(I)$ denote the minimal number of generators of I. If we show that $\mu(I_p) \leq \operatorname{depth} R_p$ for all prime ideals p of R containing I, then by a result of Tchernev [21, Theorem 5.1] I is of linear type.

Let p be a prime ideal of R containing I, and suppose p' is the ideal generated by all the variables in p, i.e. $p' = (x_i \mid x_i \in p) \subseteq p$, and suppose that p' is generated by n' variables. It is not difficult to see that (e.g. see [7, proof of Lemma 1]) I_p and $I_{p'}$ have the same (square-free) monomial generating set, and so

$$\mu(I_p) = \mu(I_{p'}). \tag{4.3.1}$$

These generators form a square-free monomial ideal in the polynomial ring R' generated by the n' variables generating p', and $\operatorname{pd}_{R'}(I_{p'}) \leq 1$, and so from Lemma 4.2 it follows that

$$\mu(I_{p'}) \leq n' = \operatorname{height} p' \leq \operatorname{height} p = \operatorname{depth} R_p. \tag{4.3.2}$$

Equations (4.3.1) and (4.3.2) together imply $\mu(I_p) \leq \operatorname{depth} R_p$ for all primes $I \subseteq p$, and we are done. □

Theorem 4.3 tells us that

$$R[It] = S/J_1 = S/(g_1, \ldots, g_q).$$

Next we show that the generators $g_1, \ldots, g_k$ listed in (4.1.1) form a regular sequence, and make use of the associated Koszul complex $\mathbb{K}$ to extract a chain complex that leads to a minimal free resolution of I^r, where $r > 0$.

Theorem 4.4 *Let I be a square-free monomial ideal of projective dimension one in a polynomial ring R over a field. Then*

$$R[It] \cong R[T_0, \ldots, T_q]/(g_1, \ldots, g_q),$$

where $g_1, \ldots, g_q$ are as in (4.1.1) and form a regular sequence.

Proof Recall that the Rees algebra $R[It]$ has dimension equal to $\dim R + 1 = n + 1$. In particular,

$$\operatorname{height}(J_1) = \dim R[T_0, \ldots, T_q] - \dim R[It] = n + q + 1 - (n+1) = q\,.$$

This yields the desired conclusion. □

Theorem 4.4 shows that the generators of J_1 form a regular sequence in $S = R[T_0, \dots, T_q]$ and as a result, the Koszul complex of $(g_1, \dots, g_q)$ over the ring S is a minimal free resolution of $S/(g_1, \dots, g_q)$ (see [2] for basic facts about exterior algebras and Koszul complexes). Since $S/(g_1, \dots, g_q) = R[It]$, we gain information about the free resolutions of powers of I by obtaining information on the graded strands of the Koszul complex of $S/(g_1, \dots, g_q)$. To make that information easier to access, we use the fact that exterior algebras commute with base extensions (see, e.g., [11, §6.13]). In other words, instead of starting with a rank q free module over S, we let F be a rank q free module over R with basis $e_1, \dots e_q$.

The augmented Koszul complex on the elements $g_1, \dots, g_q$ is

$$\begin{aligned} 0 \to \Lambda^q F(-q) \otimes S \to & \Lambda^{q-1} F(-q+1) \otimes S \to \cdots \\ & \cdots \to \Lambda^1 F(-1) \otimes S \to S \to S/J_1 \to 0. \end{aligned}$$

In this complex, the degree shifts refer to the T-grading on J, which is determined by an element's total degree in $T_0, \dots, T_q$. We write $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ where S_i is the ith graded component of S relative to T. We then take the linear strand of the Koszul complex above with T-degree equal to r, we obtain the complex

$$\begin{aligned} \mathbb{K}^r: \qquad 0 \to \Lambda^r F(-r) \otimes S_0 \to & \Lambda^{r-1} F(-r+1) \otimes S_1 \to \dots \\ & \cdots \to \Lambda^1 F(-1) \otimes S_{r-1} \to S_r \to I^r \to 0. \end{aligned}$$

We note that $\Lambda^i F = 0$ for all $i > q$. The differential of this complex can be described by

$$\partial_i^{\mathbb{K}^r}(e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes w) = \sum_{k=1}^{i} (-1)^{k-1} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i} \otimes g_{j_k} w.$$

where $w \in S_{r-i}$.

Example 4.5 Let I be the ideal in Example 2.8, with $R = \mathsf{k}[x, y, z, u]$. Then $S = R[T_0, T_1, T_2]$ and the elements g_1, g_2 defined in (4.1.1) are:

$$\begin{aligned} g_1 &= \frac{\operatorname{lcm}(yz, xy)}{yz} T_1 - \frac{\operatorname{lcm}(yz, xy)}{xy} T_0 = xT_1 - zT_0, \\ g_2 &= \frac{\operatorname{lcm}(zu, yz)}{zu} T_2 - \frac{\operatorname{lcm}(zu, yz)}{yz} T_1 = yT_2 - uT_1. \end{aligned}$$

If F is a free R-module with basis e_1, e_2, the complex $\mathbb{K}^2$ is then the complex of free R-modules

$$0 \to \Lambda^2 F(-2) \otimes S_0 \xrightarrow{\partial_2} F(-1) \otimes S_1 \xrightarrow{\partial_1} S_2 \to 0,$$

where:

- a basis for $\Lambda^2 F(-2) \otimes S_0$ consists of the element $e_1 \wedge e_2 \otimes 1$;
- a basis of $F(-1) \otimes S_1$ consists of the six elements $e_k \otimes T_i$, with $i \in \{0, 1, 2\}$ and $k \in \{1, 2\}$;
- a basis of S_2 consists of the six elements $T_i T_j$ with $i, j \in \{0, 1, 2\}$, $i \leq j$.

The differential ∂_2 acts as follows:

$$\begin{aligned}\partial_2(e_1 \wedge e_2 \otimes 1) &= e_2 \otimes g_1 - e_1 \otimes g_2 \\ &= x(e_2 \otimes T_1) - z(e_2 \otimes T_0) - y(e_1 \otimes T_2) + u(e_1 \otimes T_1)\end{aligned}$$

and the differential ∂_1 is described by

$$\partial_1(e_1 \otimes T_i) = g_1 T_i = -zT_0T_i + xT_1T_i,$$
$$\partial_1(e_2 \otimes T_i) = g_2 T_i = -uT_1T_i + yT_2T_i$$

for $i \in \{0, 1, 2\}$. At this point, the reader can observe that $\mathbb{K}^2$ is exactly the homogenized complex $\mathbb{F}_{\overline{G^2}}$ described in example 3.27. This is true in general, as we shall prove next.

4.6. Isomorphism of complexes. We now compare $\mathbb{K}^r$ to the $\mathbb{Z}^n$-graded cellular complex $\mathbb{F}_{\overline{G^r}}$ established in (3.25.2) with differentials given in Equation (3.25.3). We prove that the complexes are isomorphic, which gives us that the labeled chain complex $\overline{G^r}$ supports the minimal free resolution of I^r.

Proposition 4.7 *Under Setting 3.25, the chain complexes $\mathbb{F}_{\overline{G^r}}$ and $\mathbb{K}^r$ are isomorphic.*

Proof To describe an isomorphism $\rho\colon \mathbb{F}_{\overline{G^r}} \to \mathbb{K}^r$, we need to specify how it acts on the basis elements of the free modules in $\mathbb{F}_{\overline{G^r}}$. Let $\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}$ be a basis element of F_i, as in Equation 3.25.2, with $B = \{j_1, \ldots, j_i\}$ and $j_1 < j_2 < \cdots < j_i$. Since $\mathbf{b} \in \mathcal{N}_r$, $b_0 + \cdots + b_q = r$. Let $\mathbf{T}^{\mathbf{b}} = T_0^{b_0} \cdots T_q^{b_q} \in S_r$. Define $\mathbf{b}' = \mathbf{b} - \sum_{i\in B} \mathbf{f}_i$. Then $\mathbf{T}^{\mathbf{b}'} \in S_{r-i}$. We define

$$\rho(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes \mathbf{T}^{\mathbf{b}'} \in \Lambda^i F \otimes S_{r-i}.$$

We also note that

$$\rho(\mathbf{u}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})}) = e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \ldots \wedge e_{j_i} \otimes \mathbf{T}^{\mathbf{b}'} T_{j_k} \in \Lambda^{i-1} F \otimes S_{r-i+1}$$

and hence

$$\rho(\mathbf{u}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})}) = e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \ldots \wedge e_{j_i} \otimes \mathbf{T}^{\mathbf{b}'+\mathbf{f}_{j_k}}.$$

The map ρ is clearly bijective. We need to verify now that ρ is indeed a homomorphism of complexes (i.e. commutes with the differential). Setting $B = \{j_1, \ldots, j_i\}$ in Equation (3.26.2), we get:

$$\begin{aligned}\partial_i^{\mathbb{F}_{\overline{G^r}}}(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = &\sum_{1\le k\le i} (-1)^{k+1} \frac{\operatorname{lcm}(m_{j_k}, m_{\tau(j_k)})}{m_{j_k}} \mathbf{u}_{\mathcal{C}(\mathbf{b},B\smallsetminus\{j_k\})} \\ &+ \sum_{1\le k\le i} (-1)^{k} \frac{\operatorname{lcm}(m_{j_k}, m_{\tau(j_k)})}{m_{\tau(j_k)}} \mathbf{u}_{\mathcal{C}(\mathbf{b}-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)},B\smallsetminus\{j_k\})}.\end{aligned}$$

We then have

$$\rho\left(\partial_i^{\mathbb{F}_{\overline{G^r}}}(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)})\right) = \sum_{1\le k\le i}(-1)^{k+1}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_k}}\wedge\cdots\wedge e_{j_i}\otimes\frac{\mathrm{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{j_k}}\mathbf{T}^{\mathbf{b}'+\mathbf{f}_{j_k}}$$
$$+\sum_{1\le k\le i}(-1)^{k}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_k}}\wedge\cdots\wedge e_{j_i}\otimes\frac{\mathrm{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{\tau(j_k)}}\mathbf{T}^{\mathbf{b}'-\mathbf{f}_{j_k}+\mathbf{f}_{\tau(j_k)}+\mathbf{f}_{j_k}}.$$

On the other hand, we have:

$$\begin{aligned}
&\partial_i^{\mathbb{K}^r}(\rho(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)}) = \partial_i^{\mathbb{K}^r}(e_{j_1}\wedge\cdots\wedge e_{j_i}\otimes\mathbf{T}^{\mathbf{b}'})\\
&=\sum_{1\le k\le i}(-1)^{k-1}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_k}}\wedge\cdots\wedge e_{j_i}\otimes\left(\frac{\mathrm{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{j_k}}T_{j_k}-\frac{\mathrm{lcm}(m_{j_k},m_{\tau(j_k)})}{m_{\tau(j_k)}}T_{\tau(j_k)}\right)\mathbf{T}^{\mathbf{b}'}\\
&=\rho\left(\partial_i^{\mathbb{F}_{\overline{G^r}}}(\mathbf{u}_{\mathcal{C}(\mathbf{b},B)})\right).
\end{aligned}$$

□

4.8. Main results So far, we have seen in Theorem 4.4 that $g_1,\dots,g_q$ is a regular sequence and the Koszul complex on these elements is acyclic. Furthermore, since the number of generators of J_1 is less than the dimension of R, the strand $\mathbb{K}^r$ of $\mathbb{K}$ is a minimal free resolution of the r^{th} graded piece of $R[It]\cong S/J_1$, which is I^r. From this, we obtain the result below.

Theorem 4.9 *Let I be a square-free monomial ideal of projective dimension one, and $r>0$. Then I^r has a cellular minimal free resolution supported on the polyhedral cell complex $\overline{G^r}$.*

Proof This follows immediately from Proposition 4.7. □

Corollary 4.10 *(**The projective dimension of** I^r **and** I^r/I^{r+1}) If I is generated by $q+1$ square-free monomials in the polynomial ring R, I has projective dimension one, and r is a positive integer, then*

$$pd_R I^r = \begin{cases} q & \text{if } \ r\ge q\\ r & \text{if } \ r<q\end{cases}, \quad \text{and} \quad pd_R I^r/I^{r+1} = \begin{cases} q+1 & \text{if } \ r\ge q-1,\\ r+2 & \text{if } \ r<q-1.\end{cases}.$$

Proof The projective dimension of I^r can be read from the complex $\mathbb{K}^r$ (which is isomorphic to $\mathbb{F}_{\overline{G^r}}$), as we note that $\Lambda^i F=0$ if and only if $i>q$.

To obtain the formula for $\mathrm{pd}_R I^r/I^{r+1}$ one uses the fact that the map

$$\mathrm{Tor}_i^R(I^{r+1},\mathsf{k})\to\mathrm{Tor}_i^R(I^r,\mathsf{k})$$

induced by the inclusion $I^{r+1}\subseteq I^r$ is zero for all $i\ge 1$; this follows from a result of Maleki [13, Proposition 3.5]. The map above can be seen to be zero when $i=0$ as well. For any $i\ge 1$ there is thus a short exact sequence

$$0\to\mathrm{Tor}_i^R(I^r,\mathsf{k})\to\mathrm{Tor}_i^R(I^r/I^{r+1},\mathsf{k})\to\mathrm{Tor}_{i-1}^R(I^{r+1},\mathsf{k})\to 0$$

that gives $\mathrm{pd}_R I^r/I^{r+1}=\max\{\mathrm{pd}_R I^r, 1+\mathrm{pd}_R I^{r+1}\}=1+\mathrm{pd}_R I^{r+1}$. □

Given the structure of this resolution, we are able to say more. We can find the precise Betti numbers for each power of I.

Corollary 4.11 *(The Betti numbers of I^r) If I is generated by $q+1$ square-free monomials in the polynomial ring R and I has projective dimension one, then the t^{th} Betti number of I^r is $\binom{q}{t}\cdot\binom{q+r-t}{r-t}$ if $t \le r$ and 0 otherwise. In particular, the Betti numbers of I^r do not depend on the characteristic of the base field.*

Proof If $\mathbf{b} \in G^r$ then each distinct $B \subseteq \mathrm{Supp}(\mathbf{b})$ with $|B| = t$ determines a cell of size t embedded as in Sect. 3. For each t, there are $\binom{q}{t}$ distinct sets B of size t. For each such B, there are $\binom{q+r-t}{r-t}$ vertices of G^r whose support contains B. □

Example 4.12 In our running Example 3.13, the path has three vertices, and $I = (xy, yz, zu)$. By Corollary 4.10, $\mathrm{pd}_S I^r = q = 2$ for all $r \ge 2$. Furthermore, applying Corollary 4.11 for $r = 3$, we obtain $\beta_0(I^3) = \binom{5}{3} = 10$, $\beta_1(I^3) = 2\cdot\binom{4}{2} = 12$, and $\beta_2(I^3) = \binom{3}{1} = 3$.

4.13. An application to the fiber cone. While the result below also follows directly from Theorem 4.4, we present an alternate proof that illustrates additional properties of the ideals I. Before stating the result, we briefly recall the relevant definitions and background. Additional information can be found in [22,23].

An ideal $J \subseteq I$ is a **reduction** [17] of I if there exists an integer r such that $JI^r = I^{r+1}$. Reductions can be viewed as approximations of an ideal I that share asymptotic behavior but have fewer generators. The **analytic spread** of I, denoted by $\ell(I)$, is the minimal number of generators of a minimal reduction of I.

An interesting class of ideals are those that are their own minimal reductions. Since the dimension of the **fiber cone of** I

$$\mathcal{F}(I) = \frac{R[It]}{\mathfrak{m}R[It]} = R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots$$

is the analytic spread, the fiber cone is often used to detect this property.

Corollary 4.14 *If I is a square-free monomial ideal of projective dimension one minimally generated by $(m_0, \ldots, m_q)$, then I has no proper non-trivial reductions. Moreover, the fiber cone is isomorphic to a polynomial ring in q variables.*

Proof Consider the standard presentation map

$$\phi : R[T_0, \ldots, T_q] \to R[It] \text{ induced by } \phi(T_i) = m_i t.$$

Since I is a monomial ideal, $J = \ker(\phi)$ is a binomial ideal. In general, $J(0) = (J+\mathfrak{m})/\mathfrak{m}R[It]$ is the defining ideal of $\mathcal{F}(I)$. That is,

$$\mathcal{F}(I) \cong \mathsf{k}[T_0, \ldots, T_q]/J(0).$$

Thus the generators of $J(0)$ have the form $T_0^{a_0}\cdots T_q^{a_q} - T_0^{b_0}\cdots T_q^{b_q}$ where $\phi(\mathbf{T}^{\mathbf{a}}) = \phi(\mathbf{T}^{\mathbf{b}})$. By definition of ϕ, this implies $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{m}^{\mathbf{a}} = \mathbf{m}^{\mathbf{b}}$, which by [3, Proposition 4.1] means that $\mathbf{a} = \mathbf{b}$ and therefore $J(0) = (0)$. Thus

$$\mathcal{F}(I) \cong \mathsf{k}[T_0, \ldots, T_q].$$

It is well known (see, for example, [22]) that $\ell(I) = \dim \mathcal{F}(I)$ and $\ell(I) \le \mu(I)$, where $\mu(I)$ is the minimal number of generators of I. Hence $\ell(I) = \dim \mathcal{F}(I) = q + 1 = \mu(I)$, which implies I has no proper non-trivial reductions. □

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Data availability statement
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Author details
[1]Department of Mathematics, University of Manitoba, 520 Machray Hall, 186 Dysart Road, Winnipeg, MB R3T 2N2, Canada, [2]Department of Mathematics, American University of Beirut, Bliss Hall 315, P.O. Box 11-0236, Beirut 1107-2020, Lebanon, [3]Department of Mathematics & Statistics, Dalhousie University, 6316 Coburg Rd., PO BOX 15000, Halifax, NS B3H 4R2, Canada, [4]Department of Mathematics, University of Toronto, 40 St. George Street, Room 6290, Toronto, ON M5S 2E4, Canada, [5]Department of Mathematics, Texas State University, 601 University Dr., San Marcos, TX 78666, USA, [6]Department of Mathematics and Statistics, University of Missouri-Kansas City, Kansas City, MO 64110, USA. e-mail: spiroff@olemiss.edu, URL: http://math.olemiss.edu/sandra-spiroff/, [7]Department of Mathematics, University of Mississippi, Hume Hall 335, P.O. Box 1848, University, MS 38677, USA.

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