Received: 25 July 2020 | Revised: 14 September 2020 | Accepted: 12 October 2020

DOI: 10.1002/rnc.5299

RESEARCH ARTICLE

WILEY

Robustness analysis of uncertain time-varying systems using integral quadratic constraints with time-varying multipliers

J. Micah Fry[1,2] | **Dany Abou Jaoude**[3] | **Mazen Farhood**[2]

[1]MIT Lincoln Laboratory, Lexington, Massachusetts

[2]Kevin T. Crofton Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, Virginia

[3]Department of Mechanical Engineering, American University of Beirut, Beirut, Lebanon

Correspondence
Mazen Farhood, Kevin T. Crofton Department of Aerospace and Ocean Engineering, Virginia Tech, 460 Old Turner St., Blacksburg, VA 24061, USA.
Email: farhood@vt.edu

Funding information
Air Force Materiel Command, Grant/Award Number: FA8702-15-D-0001; National Science Foundation (NSF), Division of Civil, Mechanical and Manufacturing Innovation, Grant/Award Number: 1351640; NSF, Division of Computer and Network Systems, Grant/Award Number: 1650465; Office of Naval Research, Grant/Award Number: N00014-18-1-2627

Summary

This article presents a dissipativity approach for robustness analysis using the framework of integral quadratic constraints (IQCs). The derived results apply for linear time-varying nominal systems with uncertain initial conditions. IQC multipliers are used to describe the sets of allowable uncertainty operators, and signal IQC multipliers are used to describe the sets of allowable disturbance signals. The novel concepts of dichotomic nodes and their corresponding factorizations are introduced, which allow for the aforementioned multipliers to be general time-varying operators. The results are illustrated via the robustness analysis of a flight controller for an unmanned aircraft system tasked to perform a Split-S maneuver.

KEYWORDS

integral quadratic constraints, multipliers for signal classes, robust performance, time-varying multipliers, uncertain initial conditions, uncertain time-varying systems

1 | INTRODUCTION

A long legacy of work has demonstrated how integral quadratic constraints (IQCs) can be used to analyze robustness properties of uncertain systems exposed to sets of exogenous disturbances.[1-3] In their seminal work,[4-6] Megretski and Rantzer formulate a homotopy argument to provide sufficient conditions utilizing IQC multipliers in order to determine robustness properties of uncertain systems. While these IQC multipliers may generally be linear, bounded, self-adjoint operators, they are assumed to be elements of $\mathcal{RL}_\infty$ in order to obtain a computational method for checking the hypotheses in the celebrated IQC theorem. This article provides IQC-based theorems which allow IQC multipliers (for either uncertainties or exogenous disturbances) and the nominal system to be linear time-varying (LTV) operators, while still furnishing a computational means of applying the theorems to a broad class of LTV operators.

There are numerous other results that treat time-varying IQC multipliers. Previous results apply IQC analysis to periodic nominal systems G by using periodic IQC multipliers Π.[7] Periodic IQC multipliers are also used to investigate

the stability of periodic solutions for uncertain periodic systems.[8,9] More recently, periodic time-varying multipliers are employed to introduce noncausal scaling for robustness analysis.[10] However, all of these works focus on periodic time-varying systems and naturally link the periodic multipliers back to the frequency domain, typically through one of the various lifting techniques.

The key enabler in these previously cited works is a homotopy argument, which is used to prove that if, among other hypotheses, $[G^* \ I]\Pi[G^* \ I]^* \preccurlyeq -\epsilon I(\epsilon > 0)$, then an associated interconnection is robustly stable. Though G and Π may belong to broad classes of linear bounded operators, checking the previous operator inequality is perhaps the most burdensome task of IQC analysis. Checking this operator inequality becomes tractable by first assuming G and Π are LTI (or lifted into an LTI form), then transforming the operator inequality to an infinite set of inequalities in the frequency domain, which can finally be checked with a finite-dimensional linear matrix inequality (LMI) by virtue of the Kalman-Yakubovich-Popov lemma.[11] The following results remove the practical restriction that the IQC multipliers and nominal system be LTI by relying on a dissipativity,[12,13] instead of a homotopy, argument. We also note that other IQC-based works remove the LTI restrictions by developing theorems using a gap-metric argument and coprime factorizations,[14,15] but such results are concerned with interconnections of linear operators, thereby preventing the incorporation of nonlinearities in the uncertain system.

IQC-based theorems rooted in dissipativity arguments are a recent and growing development;[16] see, for instance, the contributions on uncertain, continuous-time, time-invariant systems,[17-19] and a treatment of the problem in discrete time.[20] Generally speaking, dissipativity arguments define a storage function on the internal state of the system and a supply rate function on the system inputs and outputs in order to reason on system stability properties. A dissipative system is one in which the storage at the end of a trajectory cannot be greater than the storage at the beginning of the trajectory plus the supply accumulated throughout the course of the trajectory, and it has been shown that dissipativity has intrinsic connections to stability properties.[12,13,16] IQC-based dissipativity arguments differ from the homotopy arguments by employing a time-domain, state-space characterization of the nominal system and multipliers, thereby allowing for the extension of the results to more general classes of nominal systems and multipliers. For instance, this approach has been extended to uncertain systems in which the nominal system is an LTV system,[21] a linear parameter-varying system,[22] and an interconnection of LTV subsystems.[23] These extensions are pushed further out by allowing the use of periodic time-varying IQC multipliers in dissipativity results.[24]

This work continues these developments by enabling the use of general time-varying IQC multipliers and providing computationally tractable means of applying IQC analysis when the IQC multipliers and the nominal system are eventually periodic (which generalize finite horizon and periodic systems). To represent the time-varying multipliers, operator machinery[25] is built on and expanded, and the concept of dichotomic nodes is developed. Specifically, as given in Definition 2, a dichotomic node is the sum of a causal node and an anticausal node, where the spaces of dichotomic nodes, causal nodes, and anticausal nodes are defined as the time-varying counterparts/generalizations of $\mathcal{RL}_\infty$, $\mathcal{RH}_\infty$, and $\mathcal{RH}_\infty^\perp$, respectively. In addition, this work demonstrates how signal IQCs may be incorporated in dissipativity-based IQC analysis. Signal IQCs, which characterize the set of exogenous disturbances to which an interconnection is exposed, have already been incorporated in IQC theorems based on the homotopy argument,[3,6,26] but are restricted in those results to be time invariant. Finally, analyzing LTV nominal systems with time-varying IQC multipliers provides an additional means for incorporating uncertain initial conditions in the IQC analysis, thereby complementing existing methods.[26,27] Despite the advantages of reducing conservatism with signal IQCs and characterizing novel uncertainty and disturbance sets with time-varying multipliers, the proposed main theorem only gives a sufficient condition for robust performance. Though other works have investigated the use of IQCs to derive necessary conditions on robustness,[28,29] such a result is not pursued in this work. As such, this theorem is by nature a conservative one, which is useful in generating worst-case analyses, but provides no indication on the magnitude of conservatism contained in the analysis results.

The theoretical results provided in this work apply for general time-varying systems and are expressed in terms of an infinite sequence of LMIs. By adapting previous arguments,[21,30,31] it can be shown that if the nominal system and multipliers are eventually periodic, then the existence of a general time-varying solution sequence to the attendant LMIs is equivalent to the existence of an eventually periodic solution sequence. Such equivalence allows one to search for an eventually periodic solution sequence via a finite-dimensional semidefinite program (SDP) without the concern of introducing conservatism; searching for a general time-varying solution (and attempting to solve the infinite-dimensional optimization problem) will provide no better results. Eventually periodic LTV systems include LTI, periodic LTV, and finite horizon linear systems as special cases.

To summarize the contributions, this article provides novel robustness analysis theorems for uncertain, discrete-time, time-varying systems based on IQC theory. The theoretical results account for the effect of uncertainties in the nominal system's initial conditions and are capable of utilizing time-varying IQC multipliers and time-varying signal IQC multipliers. Time-varying multipliers are primarily beneficial in their capability of characterizing phenomena with properties uniquely described in the time domain. Signal IQCs reduce conservatism because they can incorporate known characteristics of the disturbance signals, and by so doing provide a more accurate technique for analyzing a system's robustness properties.[32,33] Furthermore, we demonstrate how these theoretical results can be applied to eventually periodic systems by solving a finite-dimensional SDP without introducing additional conservatism. Along with these contributions, this article also contains a number of novel time-varying IQC multipliers and signal IQC multipliers that describe time-domain phenomena. The utility of the aforementioned results is made manifest by applying them to a UAS flying a Split-S maneuver and comparing them against robustness tools that can neither incorporate uncertain initial conditions nor utilize signal IQCs or time-varying multipliers.

Preliminary versions of this work build from conference articles, which enable the use of time-varying IQC multipliers[34] and introduce signal IQCs and time-varying performance IQCs into the dissipativity approach to IQC-based analysis, showing how they can be used to compute pointwise output bounds.[35] This article is structured as follows. Section 2 gives the adopted notation. Section 3 develops the necessary operator machinery. Section 4 presents the robustness analysis theorems. Section 5 gives a number of novel time-varying multipliers. Section 6 demonstrates the utility of the results by applying them to a UAS flying a Split-S maneuver. The article concludes with Section 7.

2 | NOTATION

The sets of real vectors of dimension n, real $n\times m$ matrices, and real $n\times n$ symmetric matrices are denoted by $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{S}^n$, respectively. X^* and X^T denote the adjoint and the transpose of X, respectively. For vectors $v\in\mathbb{R}^n$, the Euclidean norm is $\|v\|_2=(v^Tv)^{1/2}$. The symbol I represents the identity operator. Given a positive integer n, I_n denotes the $n\times n$ identity matrix. Given a Hilbert space $\mathcal{H}$ with a corresponding inner product $\langle\cdot,\cdot\rangle$, the operator $X=X^*$ is positive semidefinite ($X\succcurlyeq 0$) if $\langle u,Xu\rangle\geq 0$ for all $u\in\mathcal{H}$. The operator is positive definite ($X\succ 0$) if there exists an $\epsilon>0$ such that $\langle u,Xu\rangle\geq\epsilon\langle u,u\rangle$ for all $u\in\mathcal{H}$. Given operators A and B, their block-diagonal augmentation is denoted by $\mathrm{diag}(A,B)$. Given a positive definite matrix $\Xi\in\mathbb{S}^n$, we define the ellipsoid $\mathcal{E}(\Xi):=\{x\in\mathbb{R}^n|\langle x,\Xi x\rangle\leq 1\}$.

ℓ^n_{2e} denotes the space of $\mathbb{R}^n$-valued semi-infinite sequences $d=(d(0),d(1),\dots)$. The Hilbert space $\ell^n_2\subset\ell^n_{2e}$ consists of square-summable sequences with the inner product and norm defined by $\langle d,d\rangle=\|d\|^2_{\ell_2}:=\sum_{k=0}^{\infty}\|d(k)\|_2^2$. The superscript n is suppressed when the dimension does not pertain to the discussion. Given $d\in\ell_2$, the symbol $\hat{d}$ denotes the corresponding discrete-time Fourier transform. For a bounded operator $M:D\to E$, where D and E are Banach spaces, the D-to-E induced norm is given by $\|M\|_{D\to E}:=\sup_{0\neq d\in D}(\|M(d)\|_E/\|d\|_D)$. If D is a subset of E, then $\|d\|_D=\|d\|_E$.

For any natural number N, we define the sets $N_+=\{N,\ N+1,\ \dots\}$ and $N_-=\{0,\ 1,\ \dots,N-1\}$, where 0_- is the empty set. For any $N\in 0_+$, we define the Hilbert subspaces $\ell_2(N_+)$ and $\ell_2(N_-)$ as the spaces of square-summable sequences with support on N_+ and N_-, respectively. These subspaces are orthogonal complements of each other and satisfy $\ell_2(N_+)\oplus\ell_2(N_-)=\ell_2$. Given any $N\in 0_+$, the operators $\mathbb{P}^+_N$ and $\mathbb{P}^-_N$ are orthogonal projections from ℓ_2 to $\ell_2(N_+)$ and $\ell_2(N_-)$, respectively. Given $N\in 0_+$ and an operator G on ℓ_{2e}, the causal and anticausal Toeplitz operators of G are defined as ${}^NG:=\mathbb{P}^+_NG\mathbb{P}^+_N$ and ${}_NG:=\mathbb{P}^-_NG\mathbb{P}^-_N$, respectively. For any $T\in 0_+$, we use the simplified denotation ${}^T_NG:={}^T({}_NG)={}_N({}^TG)$. An operator G on ℓ_{2e} is said to be causal if ${}^NG=G\mathbb{P}^+_N$ for all $N\in 0_+$, or equivalently, ${}_NG=\mathbb{P}^-_NG$. Given $u\in\ell_2$ and $N\in 0_+$, we define the symbols ${}_Nu:=\mathbb{P}^-_Nu$ and ${}^Nu:=\mathbb{P}^+_Nu$. Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and elements $x_1\in\mathcal{H}_1$ and $x_2\in\mathcal{H}_2$, we interchangeably use the denotations $\{\!\!\{x_1,x_2\}\!\!\}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathcal{H}_1\times\mathcal{H}_2$, where $\mathcal{H}_1\times\mathcal{H}_2$ is the Hilbert space direct product of $\mathcal{H}_1$ and $\mathcal{H}_2$. The direct product of r Hilbert spaces $\mathcal{H}_i$ is denoted by $\prod_{i=1}^r\mathcal{H}_i$.

3 | OPERATOR MACHINERY

This section defines operator machinery adapted from the operator theoretic framework for LTV systems[25,30] in order to present and derive the results of this article. Given a sequence of matrices $A=(A(k))_{k\in 0_+}$, where $A(k)\in\mathbb{R}^{m\times n}$, we define the *memoryless* operator $A:\ell_2\to\ell_2$ by the operation $y=Ax$, where $y(k)=A(k)x(k)$ for all $k\in 0_+$. The operator A has the infinite-dimensional matrix (IDM) representation $A=\mathrm{diag}(A(0),A(1),\dots)$. The slight abuse of notation resulting from using A for a sequence of matrices and a memoryless operator can easily be resolved by context. A is a bounded

operator on ℓ_2 if and only if $\sup_k \|A(k)\| < \infty$. In the following, any given memoryless operator is assumed to be bounded. The operator $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \ell_2^{n_1} \times \ell_2^{n_2} \to \ell_2^{p_1} \times \ell_2^{p_2}$ is a *partitioned memoryless* operator if A, B, C, and D are memoryless operators. We use the denotation

$$\left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\!\right] = \operatorname{diag}\left(\begin{bmatrix} A(0) & B(0) \\ C(0) & D(0) \end{bmatrix}, \begin{bmatrix} A(1) & B(1) \\ C(1) & D(1) \end{bmatrix}, \dots \right)$$

to express the *memoryless realization* of F that maps $\ell_2^{(n_1+n_2)}$ to $\ell_2^{(p_1+p_2)}$. The mapping $[\![\cdot]\!]$ is an isometric homomorphism from the space of partitioned memoryless operators to the space of memoryless operators.[30]

We also introduce the delay operator $Z : \ell_2 \to \ell_2$ defined by $Zv = Z(v(0), v(1), \dots) = (0, v(0), v(1), \dots)$. When the Hilbert space of interest is $\prod_{i=1}^r \ell_2$, the operator $\tilde{Z} : \prod_{i=1}^r \ell_2 \to \prod_{i=1}^r \ell_2$ is defined as $\tilde{Z}\{\!|u_1, \dots, u_r|\!\} = \{\!|Zu_1, \dots, Zu_r|\!\}$. In the sequel, we slightly abuse notation by using the symbol Z to represent both the operator on ℓ_2 and the operator on $\prod_{i=1}^r \ell_2$. The same abuse of notation is applied with the symbols $\mathbb{P}_N^+$ and $\mathbb{P}_N^-$.

For all $k \in 0_+$, consider the difference equations:

$$\begin{aligned} x^+(k+1) &= A(k)x^+(k) + B(k)u(k), & y^+(k) &= C(k)x^+(k) + D(k)u(k) \\ x^-(k) &= A(k)x^-(k+1) + B(k)u(k), & y^-(k) &= C(k)x^-(k+1) + D(k)u(k). \end{aligned}$$

The sequences of matrices $A(k)$, $B(k)$, $C(k)$, and $D(k)$ are assumed to be uniformly bounded and define the memoryless operators A, B, C, and D, respectively. Assuming $x^+(0) = 0$, the equations for x^+ and y^+ may be written in operator form as follows:

$$x^+ = ZAx^+ + ZBu, \qquad y^+ = Cx^+ + Du.$$

The equations for x^- and y^- can also be expressed as in

$$x^- = AZ^*x^- + Bu, \qquad y^- = CZ^*x^- + Du.$$

Noting that $I - ZA$ is causally invertible on ℓ_{2e},[25] we define the operator $G^+ := C(I - ZA)^{-1}ZB + D$ such that $y^+ = G^+u$. G^+ has the following IDM representation:

$$G^+ = \begin{bmatrix} D(0) & 0 & 0 & \dots \\ C(1)B(0) & D(1) & 0 & \dots \\ C(2)A(1)B(0) & C(2)B(1) & D(2) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The operator G^+ is causal, and its IDM representation is block lower-triangular. Since $I - AZ^*$ has an anticausal inverse on ℓ_{2e}, we define $G^- := CZ^*(I - AZ^*)^{-1}B + D$ such that $y^- = G^-u$. The operator G^- is anticausal and admits a block upper-triangular IDM representation. This representation is given by

$$G^- = \begin{bmatrix} D(0) & C(0)B(1) & C(0)A(1)B(2) & \dots \\ 0 & D(1) & C(1)B(2) & \dots \\ 0 & 0 & D(2) & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

For the operators G^+ and G^- to be bounded on ℓ_2, $(I - ZA)^{-1}$ and $(I - AZ^*)^{-1}$ must be bounded on ℓ_2, respectively.

Definition 1. The possibly partitioned memoryless operator A is *causally stable* if there exist positive scalars μ, ν, and ρ and a conformably partitioned memoryless operator $P = P^*$ such that $\rho I \preccurlyeq P \preccurlyeq \mu I$ and $A^*Z^*PZA - P \preccurlyeq -\nu I$. The operator A is *anticausally stable* if there exist positive scalars μ, ν, and ρ and a conformably partitioned memoryless operator $P = P^*$ such that $\rho I \preccurlyeq P \preccurlyeq \mu I$ and $A^*PA - Z^*PZ \preccurlyeq -\nu I$.

Proposition 1. *Let A be a given memoryless operator. $(I-ZA)^{-1}$ is bounded on ℓ_2 if and only if A is causally stable. $(I-AZ^*)^{-1}$ is bounded on ℓ_2 if and only if A is anticausally stable.*

Proposition 1 is a standard result that gives conditions to assert if G^+ and G^- are bounded operators on ℓ_2.[25] Next, we introduce the notions of causal nodes, anticausal nodes, and dichotomic nodes.

Definition 2. Consider the quadruple of memoryless operators (A,B,C,D). If A is causally stable, then (A,B,C,D) defines the *causal node* $G^+ = D + C(I-ZA)^{-1}ZB$, which we denote by $[A,B,C,D]_+$. If A is anticausally stable, then (A,B,C,D) defines the *anticausal node* $G^- = D + CZ^*(I-AZ^*)^{-1}B$, which we denote by $[A,B,C,D]_-$.

Now, consider the octuple of memoryless operators (A,B,C,D,E,U,V,W), where A is *causally stable* and E is anticausally stable. This octuple defines the *dichotomic node*

$$G^d = [A,B,C,D]_+ + [E,U,V,W]_-$$

which we denote by $[A,B,C,D,E,U,V,W]$.

By definition, a causal node is a dichotomic node with $[A,B,C,D]_+ = [A,B,C,D,0,0,0,0]$, and an anticausal node is a dichotomic node with $[A,B,C,D]_- = [0,0,0,0,\ A,B,C,D]$. The previous definitions of causal, anticausal, and dichotomic nodes extend to the cases where the operators A, B, C, D are conformably partitioned memoryless operators.

It can be shown that the adjoint of a causal node is an anticausal node and vice versa. The three types of nodes define bounded operators on $\ell_2(0_+)$. If the nodes were instead defined on the bi-infinite signal space, that is, $\ell_2(\mathbb{Z})$, then it would be possible to view the spaces of causal nodes, anticausal nodes, and dichotomic nodes as the time-varying analogs of $\mathcal{RH}_\infty$, $\mathcal{RH}_\infty^\perp$, and $\mathcal{RL}_\infty$, respectively. The notion of dichotomicity introduced here differs from the one provided by Halanay and Ionescu[25] in that we classify *nodes* as dichotomic. Our approach allows us to deal with a larger class of operators on ℓ_2. For example, consider the transfer function $\hat{L}(z) = 1/z + z$, which can be expressed as the dichotomic node $L = [0,I,I,0,0,I,I,0]$. Though this LTI system has a simple representation given the definition of a dichotomic node adopted herein, the notion of dichotomicity used by Halanay and Ionescu[25] (and not pursued here) cannot be applied to represent such a system.

In IQC-based robustness analysis, the uncertain system consists of a nominal linear system G and an uncertainty operator $\Delta \in \mathbf{\Delta}$, where the set of allowable uncertainties $\mathbf{\Delta}$ is assumed to satisfy an IQC defined by a multiplier Π. In previous results, both G and Π are time-invariant.[4,17,18,20,36] This is later relaxed to allow for LTV nominal systems[21] and periodic time-varying multipliers.[24] The present work leverages the notion of dichotomic nodes defined in the preceding to deal with uncertain systems wherein the nominal system G is LTV and has uncertain initial conditions, and to allow for general time-varying multipliers Π. The remainder of this section gives the necessary definitions and results to deal with the classes of nominal systems and multipliers of interest.

Definition 3. Given the node $G = [A,B,C,D]_+$ and $N \in 0_+$, we define the *zero-input operator* $\mathcal{S}_N^G : \mathbb{R}^n \to \ell_2^n(N_+)$ as $(\mathcal{S}_N^G \xi)(k) = \Phi_A(k,N)\xi$ for $k \geq N$, where

$$\Phi_A(k,N) = \begin{cases} I, & k = N \\ A(k-1)A(k-2)\ \dots\ A(N), & k > N. \end{cases}$$

We also define the *zero-state operator* $\mathcal{L}_N^G : \ell_2^m(N_+) \to \ell_2^n(N_+)$ as $\mathcal{L}_N^G = [A,B,I,0]_+ \mathbb{P}_{\mathbb{N}}^+$.

Definition 4. The causal node $G = [A,B,C,D]_+$ defines an *initial condition (IC) node* $G^{ic} : \ell_2^m \times \mathbb{R}^n \to \ell_2^p$ by $G^{ic}\{\!\{u,\xi\}\!\} = Gu + C\mathcal{S}_0^G\xi$. The IC node G^{ic} is denoted by $[A,B,C,D]_+^{ic} = \begin{bmatrix} G & G_\xi \end{bmatrix}$, where $G_\xi = C\mathcal{S}_0^G$. A *partitioned IC node* is defined analogously.

Lemma 1. *Any self-adjoint dichotomic node Π can be factorized as $\Pi = \Psi^* S \Psi$, where Ψ is a partitioned causal node, and $S = S^*$ is a partitioned memoryless operator.*

Proof. As $\Pi = \Pi^*$ is a dichotomic node, there exist operators A, B, C, D, E, U, V, and W such that $\Pi = \overline{\Psi}^+ + \overline{\Psi}^- = (\overline{\Psi}^+)^* + (\overline{\Psi}^-)^*$, where $\overline{\Psi}^+ = [A,B,C,D]_+$ and $\overline{\Psi}^- = [E,U,V,W]_-$. Subtracting $\overline{\Psi}^-$ and $(\overline{\Psi}^-)^*$ from both sides results in $\overline{\Psi}^+ - (\overline{\Psi}^-)^* = (\overline{\Psi}^+)^* - \overline{\Psi}^-$. As the left-hand side (LHS) of the last equation is a causal node and the right-hand side (RHS) is an anticausal node, both sides are equal to some self-adjoint memoryless operator T. Hence, $\overline{\Psi}^- = (\overline{\Psi}^+)^* - T$ and $\Pi = \overline{\Psi}^+ + (\overline{\Psi}^+)^* - T$. Thus, Π can be factorized as $\Pi = \Psi^* S \Psi$, where

$$\Psi = \begin{bmatrix} (I - ZA)^{-1}ZB \\ I \end{bmatrix}, \qquad S = S^* = \begin{bmatrix} 0 & C^* \\ C & D + D^* - T \end{bmatrix}$$

that is, Ψ is a partitioned causal node and S is a partitioned memoryless operator. ∎

For notational ease, if $\Pi = \Pi^*$ is a dichotomic node, the *factors* (Ψ, S) of Π denote the possibly partitioned causal node Ψ and memoryless operator $S = S^*$ such that $\Pi = \Psi^* S \Psi$. The factors of Π given in the proof of Lemma 1 are not unique: there may be multiple factors (Ψ, S) such that $\Pi = \Psi^* S \Psi$. In Section 4, we find special factors $(\overline{\Psi}, \overline{S})$ for Π that are useful in IQC analysis.

Definition 5. The possibly partitioned causal node $\Psi = [A, B, C, D]_+$, memoryless operator $S = S^*$, time-index $N \geq 0$, initial condition $\xi \in \mathbb{R}^n$, and input $u \in \ell_2(N_+)$ define the *Popov index*

$$J^{\Psi,S}(N, \xi, u) := \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} {}^N Q & {}^N L \\ ({}^N L)^* & {}^N R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle$$
$$\text{subject to : } \; x(k+1) = A(k)x(k) + B(k)u(k) \; \forall k \in N_+, \text{ where } x(N) = \xi, \; x \in \ell_2(N_+) \tag{1}$$

$$Q := C^* S C, \; L := C^* S D, \; R := D^* S D. \tag{2}$$

If Π is partitioned and has two inputs and two outputs, the *lower Popov index* and *upper Popov index* are defined as

$$\underline{J}^{\Psi,S}(N, \xi) = \sup_{\vartheta \in \ell_2(N_+)} \inf_{\varphi \in \ell_2(N_+)} J^{\Psi,S}(N, \xi, \{\!\!\{ \varphi, \vartheta \}\!\!\})$$
$$\overline{J}^{\Psi,S}(N, \xi) = \inf_{\varphi \in \ell_2(N_+)} \sup_{\vartheta \in \ell_2(N_+)} J^{\Psi,S}(N, \xi, \{\!\!\{ \varphi, \vartheta \}\!\!\})$$

respectively. Finally, given a time-index $T \geq N$, the *truncated Popov index* is defined as

$$J_T^{\Psi,S}(N, \xi, u) := \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} {}_N^T Q & {}_N^T L \\ ({}_N^T L)^* & {}_N^T R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle$$
$$\text{subject to : } \; (1) \text{ and } (2).$$

Equation (1) is not written in operator form, as the evolution depends on the initial condition ξ. However, it can be shown that the solution $x \in \ell_2(N_+)$ of (1) satisfies

$$\mathbb{P}_N^+ Z^* x = Ax + Bu. \tag{3}$$

For context, given a dichotomic node Π with factors (Ψ, S), we have $J^{\Psi,S}(N, 0, u) = \langle u, \Psi^* S \Psi u \rangle = \langle u, \Pi u \rangle$ for all $u \in \ell_2(N_+)$. However, the previous inner products are generally not equal to the Popov index if $\xi \neq 0$. Indeed, given the same N, $u \in \ell_2(N_+)$, and $\xi \neq 0$, the Popov indices pertaining to two different factors of Π are different in general. If $\xi = 0$, then the two Popov indices are equal.

The solution to Equation (1) can be expressed using the operator $\Upsilon_N^\Psi : \mathbb{R}^n \times \ell_2^m(N_+) \to \ell_2^n(N_+)$, where $\Upsilon_N^\Psi(\xi, u) = \mathcal{S}_N^\Psi \xi + \mathcal{L}_N^\Psi u$. We can thus write the Popov index as follows:

$$\begin{aligned} J^{\Psi,S}(N, \xi, u) &= \left\langle \begin{bmatrix} \Upsilon_N^\Psi(\xi, u) \\ u \end{bmatrix}, \begin{bmatrix} {}^N Q & {}^N L \\ ({}^N L)^* & {}^N R \end{bmatrix} \begin{bmatrix} \Upsilon_N^\Psi(\xi, u) \\ u \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \xi \\ u \end{bmatrix}, \begin{bmatrix} (\mathcal{S}_N^\Psi)^* & 0 \\ (\mathcal{L}_N^\Psi)^* & I \end{bmatrix} \begin{bmatrix} {}^N Q & {}^N L \\ ({}^N L)^* & {}^N R \end{bmatrix} \begin{bmatrix} \mathcal{S}_N^\Psi & \mathcal{L}_N^\Psi \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \xi \\ u \end{bmatrix}, \begin{bmatrix} \mathcal{X}_N^{\Psi,S} & \mathcal{P}_N^{\Psi,S} \\ (\mathcal{P}_N^{\Psi,S})^* & \mathcal{R}_N^{\Psi,S} \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} \right\rangle, \text{ where } \begin{array}{l} \mathcal{X}_N^{\Psi,S} := (\mathcal{S}_N^\Psi)^* {}^N Q \mathcal{S}_N^\Psi, \quad \mathcal{P}_N^{\Psi,S} := (\mathcal{S}_N^\Psi)^* ({}^N Q \mathcal{L}_N^\Psi + {}^N L) \\ \mathcal{R}_N^{\Psi,S} := {}^N R + ({}^N L)^* \mathcal{L}_N^\Psi + (\mathcal{L}_N^\Psi)^* {}^N L + (\mathcal{L}_N^\Psi)^* {}^N Q \mathcal{L}_N^\Psi. \end{array} \end{aligned}$$

It can be shown from the definitions that $\mathcal{R}_N^{\Psi,S} = {}^N\Pi$.

Definition 6. The causal node $\Psi = [A, B, C, D]_+$ and memoryless operator $S = S^*$ define the *Discrete Difference Riccati Equation* (DDRE(Ψ, S)) as follows:

$$X = A^*Z^*XZA + Q - (A^*Z^*XZB + L)(R + B^*Z^*XZB)^{-1}(L^* + B^*Z^*XZA)$$

where $X = X^*$ is a bounded memoryless operator, $R + B^*Z^*XZB$ is boundedly invertible on ℓ_2, and the operators Q, L, and R are defined in (2). Define the *feedback operator* $F := -(R + B^*Z^*XZB)^{-1}(L^* + B^*Z^*XZA)$. Then, a solution X to the DDRE(Ψ, S) is said to be *stabilizing* if $A + BF$ is causally stable.

It is known that if the stabilizing solution X to the DDRE(Ψ, S) exists, it is unique.[25] The DDRE(Ψ, S) can be equivalently expressed as the following sequence of equations for all $k \in 0_+$:

$$X(k) = A(k)^TX(k+1)A(k) + Q(k) - \big(A(k)^TX(k+1)B(k) + L(k)\big)\big(R(k) + B(k)^TX(k+1)B(k)\big)^{-1}\big(L(k)^T + B(k)^TX(k+1)A(k)\big).$$

If all the given operators are time-invariant, the Discrete Algebraic Riccati Equation is retrieved:

$$X = A^TXA + Q - (A^TXB + L)(R + B^TXB)^{-1}(L^T + B^TXA).$$

4 | INTEGRAL QUADRATIC CONSTRAINTS

Consider the uncertain system $(G^{ic}, \mathbf{\Delta})$, where

$$G^{ic} = \left[A^G, \begin{bmatrix} B^{G1} & B^{G2} \end{bmatrix}, \begin{bmatrix} C^{G1} \\ C^{G2} \end{bmatrix}, \begin{bmatrix} D^{G11} & D^{G12} \\ D^{G21} & D^{G22} \end{bmatrix}\right]_+^{ic} = \begin{bmatrix} G_{11} & G_{12} & G_{1\xi} \\ G_{21} & G_{22} & G_{2\xi} \end{bmatrix} \tag{4}$$

is a partitioned IC node, $A^G(k) \in \mathbb{R}^{n_G \times n_G}$ for $k \in 0_+$, and $\mathbf{\Delta}$ is a set of causal operators on ℓ_{2e} bounded on ℓ_2 (see Figure 1). For all $\Delta \in \mathbf{\Delta}$, the interconnection equations are given by

$$\begin{bmatrix} \varphi \\ e \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{1\xi} \\ G_{21} & G_{22} & G_{2\xi} \end{bmatrix} \begin{bmatrix} \vartheta \\ d \\ \xi^G \end{bmatrix}, \qquad \vartheta = \Delta(\varphi). \tag{5}$$

Throughout the article, we assume that there exists a scalar $\mu < \infty$ such that $\sup_{\Delta \in \mathbf{\Delta}} \|\Delta\|_{\ell_2 \to \ell_2} \leq \mu$. This assumption is not overly restrictive, since most uncertainty sets described by IQC multipliers in the literature already satisfy this condition.[4,37] Furthermore, the sets that do not satisfy such a condition may be closely approximated by the subset $\tilde{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} \mid \|\Delta\|_{\ell_2 \to \ell_2} \leq \mu\}$, where μ may be arbitrarily large. For example, the aforementioned assumption is not satisfied for the set of passive nonlinearities, but this set is well-approximated by defining μ to be arbitrarily large and considering only passive nonlinearities whose ℓ_2-induced norm is bounded by μ.

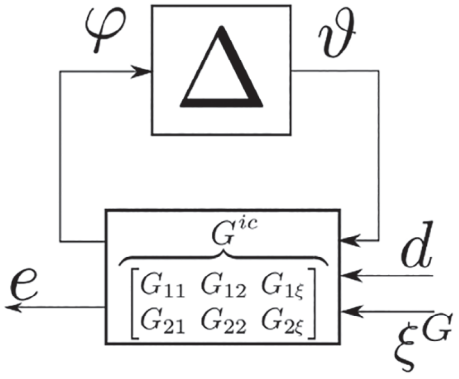


FIGURE 1 The interconnection (G^{ic}, Δ)

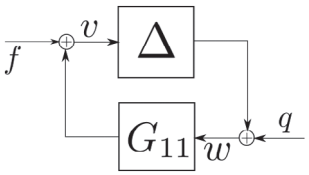


FIGURE 2 Standard feedback interconnection

Definition 7. The uncertain system $(G^{ic}, \mathbf{\Delta})$ defined in (5) is *robustly well-posed* if $(I - G_{11}\Delta)$ has a causal inverse on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$.

The previous definition signifies that, for every $d \in \ell_{2e}$, $\xi^G \in \mathbb{R}^{n_G}$, $\Delta \in \mathbf{\Delta}$, there exist unique signals $\varphi,\ \vartheta,\ e \in \ell_{2e}$ that satisfy (5) and causally depend on d, $G_{1\xi}\xi$, and $G_{2\xi}\xi$.

Definition 8. The uncertain system $(G^{ic}, \mathbf{\Delta})$ defined in (5) is *robustly stable* if it is robustly well-posed and $(I - G_{11}\Delta)^{-1}$ is bounded on ℓ_2 for all $\Delta \in \mathbf{\Delta}$.

The uncertain system $(G^{ic}, \mathbf{\Delta})$ being robustly stable signifies that there exist constants $c_{\vartheta d}$, $c_{\vartheta\xi}$, $c_{\varphi d}$, $c_{\varphi\xi}$, c_{ed}, and $c_{e\xi}$ such that, for all $d \in \ell_2$, $\xi^G \in \mathbb{R}^{n_G}$, and $\Delta \in \mathbf{\Delta}$, we have

$$\|\vartheta\|_{\ell_2} \le c_{\vartheta d}\|d\|_{\ell_2} + c_{\vartheta\xi}\|\xi^G\|_2, \quad \|\varphi\|_{\ell_2} \le c_{\varphi d}\|d\|_{\ell_2} + c_{\varphi\xi}\|\xi^G\|_2, \quad \|e\|_{\ell_2} \le c_{ed}\|d\|_{\ell_2} + c_{e\xi}\|\xi^G\|_2.$$

When considering robust stability, the standard feedback interconnection shown in Figure 2 is typically analyzed.[4,17,18] For each $\Delta \in \mathbf{\Delta}$, this interconnection is described by the equations $v = G_{11}w + f$ and $w = \Delta(v) + q$. To account for nonzero initial conditions $\xi^{\underline{G}}$ in system G_{11}, these equations are modified into

$$v = G_{11}w + G_{1\xi}\xi^{\underline{G}} + f, \qquad w = \Delta(v) + q. \tag{6}$$

For each $\Delta \in \mathbf{\Delta}$, the equations in (6) can be rewritten as

$$\begin{bmatrix} \underline{\varphi} \\ \underline{e} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} & \underline{G}_{1\xi} \\ \underline{G}_{21} & \underline{G}_{22} & \underline{G}_{2\xi} \end{bmatrix}}_{\underline{G}^{ic}} \begin{bmatrix} \underline{\vartheta} \\ \underline{d} \\ \xi^{\underline{G}} \end{bmatrix}, \qquad \underline{\vartheta} = \Delta(\underline{\varphi}) \tag{7}$$

$$\text{where } \underline{G}^{ic} = \begin{bmatrix} G_{11} & \begin{bmatrix} I & G_{11} \end{bmatrix} & G_{1\xi} \\ \begin{bmatrix} G_{11} \\ I \end{bmatrix} & \begin{bmatrix} I & G_{11} \\ 0 & I \end{bmatrix} & \begin{bmatrix} G_{1\xi} \\ 0 \end{bmatrix} \end{bmatrix} = [A^G,\ B^{\underline{G}},\ C^{\underline{G}},\ D^{\underline{G}}]^{ic}_{+}, \quad \underline{G} = \begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} \\ \underline{G}_{21} & \underline{G}_{22} \end{bmatrix} = [A^G,\ B^{\underline{G}},\ C^{\underline{G}},\ D^{\underline{G}}]_{+} \tag{8}$$

$$B^{\underline{G}} = \begin{bmatrix} B^{G1} & \begin{bmatrix} 0 & B^{G1} \end{bmatrix} \end{bmatrix}, \qquad C^{\underline{G}} = \begin{bmatrix} C^{G1} \\ \begin{bmatrix} C^{G1} \\ 0 \end{bmatrix} \end{bmatrix}, \qquad D^{\underline{G}} = \begin{bmatrix} D^{G11} & \begin{bmatrix} I & D^{G11} \end{bmatrix} \\ \begin{bmatrix} D^{G11} \\ I \end{bmatrix} & \begin{bmatrix} 0 & I \\ I & D^{G11} \end{bmatrix} \end{bmatrix} \tag{9}$$

$$\underline{e} = \{\!\{v, w\}\!\}, \quad \underline{d} = \{\!\{f, q\}\!\}.$$

It is not difficult to see that the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ defined in (7) is robustly well-posed (respectively, robustly stable) if and only if the uncertain system $(G^{ic}, \mathbf{\Delta})$ defined in (5) is robustly well-posed (respectively, robustly stable).

Next, we show how to determine if the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ is robustly stable. We begin by defining the notion of a time-varying IQC multiplier.

Definition 9. An *IQC multiplier* $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} = \Pi^*$ is a self-adjoint partitioned dichotomic node.

This definition of an IQC multiplier is a time-varying generalization of the classical IQC multiplier definition. Megretski and Rantzer state that although Π is customarily a Hermitian-valued function in $\mathcal{RL}_\infty$, Π can be any measurable Hermitian-valued function mapping the complex unit circle[1] to $\mathbb{C}^{(m_1+m_2)\times(m_1+m_2)}$.[4] The latter definition permits IQC

[1] the $j\omega$ axis for continuous-time dynamics

multipliers to lie outside $\mathcal{RL}_\infty$, but still does not capture time-varying dynamical systems. By utilizing time-varying IQC multipliers, we can characterize new uncertainty classes with time-varying, time-domain characteristics. Because the set of dichotomic nodes includes the set of time-invariant IQC multipliers (which are typically used), we are also provided with a broader set of multipliers to select from when characterizing uncertainties with time-invariant or frequency-domain properties.

Definition 10. Let Π be an IQC multiplier. The set $\mathbf{\Delta}$ *satisfies the IQC defined by* Π (denoted $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$) if for all $\varphi \in \ell_2$ and $\Delta \in \mathbf{\Delta}$, the following holds:

$$\left\langle \begin{bmatrix} \varphi \\ \Delta(\varphi) \end{bmatrix}, \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ \Delta(\varphi) \end{bmatrix} \right\rangle \geq 0.$$

Given factors (Ψ, S) of Π, where

$$\begin{aligned} \Psi &= \left[A^\Psi, \begin{bmatrix} B^{\Psi 1} & B^{\Psi 2} \end{bmatrix}, C^\Psi, \begin{bmatrix} D^{\Psi 1} & D^{\Psi 2} \end{bmatrix} \right]_+ \\ &= [A^\Psi, B^\Psi, C^\Psi, D^\Psi]_+ \end{aligned} \tag{10}$$

the set $\mathbf{\Delta}$ *satisfies the soft IQC defined by* (Ψ, S) (denoted $\mathbf{\Delta} \in \mathrm{SIQC}(\Psi, S)$) if, for all $\varphi \in \ell_2$ and $\Delta \in \mathbf{\Delta}$,

$$J^{\Psi,S}(0, 0, \{\!|\varphi, \Delta(\varphi)|\!\}) \geq 0.$$

The set $\mathbf{\Delta}$ *satisfies the hard IQC defined by* (Ψ, S) (denoted $\mathbf{\Delta} \in \mathrm{HIQC}(\Psi, S)$) if, for all $\varphi \in \ell_{2e}$, $\Delta \in \mathbf{\Delta}$, and $T \in 0_+$,

$$J_T^{\Psi,S}(0, 0, \{\!|\varphi, \Delta(\varphi)|\!\}) \geq 0.$$

By the previous discussion on Popov indices, $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$ if and only if $\mathbf{\Delta} \in \mathrm{SIQC}(\Psi, S)$ for any factors (Ψ, S) of Π. In addition, if $\mathbf{\Delta} \in \mathrm{HIQC}(\Psi, S)$, then $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$, where (Ψ, S) are factors of Π. However, the converse is not true in general: $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$ does not imply that $\mathbf{\Delta} \in \mathrm{HIQC}(\Psi_1, S_1)$, and $\mathbf{\Delta} \in \mathrm{HIQC}(\Psi_1, S_1)$ does not imply that $\mathbf{\Delta} \in \mathrm{HIQC}(\Psi_2, S_2)$, where (Ψ_1, S_1) and (Ψ_2, S_2) are factors of Π.

As with the other results based on the dissipativity approach to IQC-based robustness analysis,[18] the results in this article require Π to satisfy an additional condition as defined next.

Definition 11. The IQC multiplier Π is said to be *positive-negative* (respectively, *strictly positive-negative*) *on* $\ell_2(N_+)$ if there exists a scalar $\epsilon \geq 0$ (respectively, $\epsilon > 0$) such that $\langle u, \Pi_{11} u \rangle \geq \epsilon \|u\|^2$ and $\langle v, \Pi_{22} v \rangle \leq -\epsilon \|v\|^2$ for all u and v in $\ell_2(N_+)$. If Π is (strictly) positive-negative on ℓ_2, we simply say it is *(strictly) positive-negative*.

Though the dissipativity approach to IQC-based analysis requires the IQC multipliers to be positive-negative, this requirement is not overly restrictive. Nearly all the IQC multipliers in the literature to date are positive-negative.[4,37] Important exceptions are the full-block IQC multipliers that characterize repeated sector-bounded nonlinearities.[37,38]

4.1 | Robust stability of uncertain systems

To assess the robust stability of the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$, we define the following operators:

$$\begin{aligned} A^{\underline{H}} &= \begin{bmatrix} A^G & 0 \\ B^{\Psi 1} C^{G1} & A^\Psi \end{bmatrix}, \qquad B^{\underline{H}} = \begin{bmatrix} B^{G1} \\ B^{\Psi 1} D^{G11} + B^{\Psi 2} \end{bmatrix}, \qquad B^{\underline{H}2} = \begin{bmatrix} 0 & B^{G1} \\ B^{\Psi 1} & B^{\Psi 1} D^{G11} \end{bmatrix} \\ C^{\underline{H}} &= \begin{bmatrix} D^{\Psi 1} C^{G1} & C^\Psi \end{bmatrix}, \qquad D^{\underline{H}} = D^{\Psi 1} D^{G11} + D^{\Psi 2}, \qquad D^{\underline{H}12} = \begin{bmatrix} D^{\Psi 1} & D^{\Psi 1} D^{G11} \end{bmatrix} \\ C^{\underline{H}2} &= \begin{bmatrix} C^{G1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad D^{\underline{H}21} = \begin{bmatrix} D^{G11} \\ I \end{bmatrix}, \qquad D^{\underline{H}22} = \begin{bmatrix} I & D^{G11} \\ 0 & I \end{bmatrix}. \end{aligned} \tag{11}$$

From the definitions of $\underline{G}$ in (8), Ψ in (10), and the operators in (11), it can be shown that for all $\underline{\vartheta}, \underline{d} \in \ell_{2e}$ (respectively, $\underline{\vartheta}, \underline{d} \in \ell_2$) and $\xi^{\underline{G}} \in \mathbb{R}^{n_G}$, the signals defined by

$$\begin{bmatrix} \underline{\varphi} \\ \underline{e} \end{bmatrix} = \underline{G}^{ic} \left\{ \begin{bmatrix} \begin{bmatrix} \underline{\vartheta} \\ \underline{d} \end{bmatrix}, \xi^{\underline{G}} \end{bmatrix} \right\}, \qquad \underline{r} = \Psi \begin{bmatrix} \underline{\varphi} \\ \underline{\vartheta} \end{bmatrix}$$

$$x^{\underline{G}} = \mathcal{L}_0^{\underline{G}} \begin{bmatrix} \underline{\vartheta} \\ \underline{d} \end{bmatrix} + S_0^{\underline{G}} \xi^{\underline{G}}, \qquad \underline{x}^{\Psi} = \mathcal{L}_0^{\Psi} \begin{bmatrix} \underline{\varphi} \\ \underline{\vartheta} \end{bmatrix}, \qquad x^{\underline{H}} = \begin{bmatrix} x^{\underline{G}} \\ \underline{x}^{\Psi} \end{bmatrix}$$

are in ℓ_{2e} (respectively, ℓ_2) and satisfy

$$\begin{bmatrix} Z^*({}_{T+1}x^{\underline{H}}) \\ {}_T\underline{r} \\ {}_T\underline{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A^{\underline{H}} & B^{\underline{H}} & B^{\underline{H}2} \\ C^{\underline{H}} & D^{\underline{H}} & D^{\underline{H}12} \\ C^{\underline{H}2} & D^{\underline{H}21} & D^{\underline{H}22} \end{bmatrix}}_{\mathcal{H}_{\text{aug}}} \begin{bmatrix} {}_Tx^{\underline{H}} \\ {}_T\underline{\vartheta} \\ {}_T\underline{d} \end{bmatrix}.$$

Theorem 1 is the main result of this section. It provides a sufficient condition to guarantee the robust stability of the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ defined in (7). The theorem statement is followed by a series of intermediate technical lemmas that are needed for its proof.

Theorem 1. *Given the partitioned IC node $\underline{G}^{ic}$ defined in (8), the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ defined in (7) is robustly stable if $(\underline{G}^{ic}, \mathbf{\Delta})$ is robustly well-posed and there exists a positive-negative IQC multiplier Π with factors (Ψ, S) such that*

(a) $\mathbf{\Delta} \in \text{IQC}(\Pi)$*;*
(b) there exist a memoryless operator $P = P^*$ and $\epsilon > 0$ satisfying

$$\begin{bmatrix} A^{\underline{H}} & B^{\underline{H}} \\ C^{\underline{H}} & D^{\underline{H}} \end{bmatrix}^* \begin{bmatrix} Z^*PZ & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A^{\underline{H}} & B^{\underline{H}} \\ C^{\underline{H}} & D^{\underline{H}} \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \preccurlyeq -\epsilon I. \tag{12}$$

Theorem 1 is similar to previous results,[20,21] except that it allows *both* the nominal system and the IQC multiplier to be time-varying. While periodic time-varying IQC multipliers could be incorporated using an earlier result,[24] Theorem 1 allows for general time-varying IQC multipliers.

To prove Theorem 1, we rely on two central ideas. First, we follow the dissipativity approach to IQC-based robust stability analysis to prove Lemma 2. This lemma requires the existence of a positive semidefinite storage function and a factorization of the IQC multiplier Π that yields a hard IQC. Second, we show that, for a positive-negative IQC multiplier Π, if hypotheses (a) and (b) in Theorem 1 hold, then there exists a special factorization for Π such that the stronger (more conservative) counterpart hypotheses in Lemma 2 are satisfied.

Lemma 2. *Given the partitioned IC node $\underline{G}^{ic}$ defined in (8), the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ defined in (7) is robustly stable if $(\underline{G}^{ic}, \mathbf{\Delta})$ is robustly well-posed and there exists a positive-negative IQC multiplier Π with factors (Ψ, S) such that*

(a) $\mathbf{\Delta} \in \text{HIQC}(\Psi, S)$*;*
(b) there exist $\beta > 0$, $\epsilon > 0$, and a memoryless operator $P = P^* \succcurlyeq 0$ such that

$$\mathcal{H}_{\text{aug}}^* \, \text{diag}(Z^*PZ, S, I) \mathcal{H}_{\text{aug}} - \text{diag}(P, 0, \beta^2 I) \preccurlyeq -\epsilon I. \tag{13}$$

Proof. The proof of this lemma makes use of dissipativity arguments.[12,13] Since the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ is robustly well-posed, then for all $\underline{d} \in \ell_2$, $\Delta \in \mathbf{\Delta}$, $\xi^{\underline{G}} \in \mathbb{R}^{n_G}$, there exist unique signals $\underline{\vartheta}$, $\underline{\varphi}$, $\underline{e}$ in ℓ_{2e} that satisfy (7). In addition, define the ℓ_{2e} signals $\underline{r} = \Psi\{\underline{\varphi}, \underline{\vartheta}\}$, $x^{\underline{G}} = \mathcal{L}_0^{\underline{G}}\{\underline{\vartheta}, \underline{d}\} + S_0^{\underline{G}} \xi^{\underline{G}}$, $\underline{x}^{\Psi} = \mathcal{L}_0^{\Psi}\{\underline{\varphi}, \underline{\vartheta}\}$, and $x^{\underline{H}} = \{x^{\underline{G}}, \underline{x}^{\Psi}\}$. By (13), for all $T \in 0_+$ we have

$$\begin{aligned} &\langle \mathcal{H}_{\text{aug}}\{{}_Tx^{\underline{H}}, {}_T\underline{\vartheta}, {}_T\underline{d}\}, \text{diag}(Z^*PZ, S, I)\mathcal{H}_{\text{aug}}\{{}_Tx^{\underline{H}}, {}_T\underline{\vartheta}, {}_T\underline{d}\}\rangle \\ &\qquad - \langle \{{}_Tx^{\underline{H}}, {}_T\underline{d}\}, \text{diag}(P, \beta^2 I)\{{}_Tx^{\underline{H}}, {}_T\underline{d}\}\rangle \leq -\epsilon \left(\|{}_Tx^{\underline{G}}\|_{\ell_2}^2 + \|{}_T\underline{x}^{\Psi}\|_{\ell_2}^2 + \|{}_T\underline{\vartheta}\|_{\ell_2}^2 + \|{}_T\underline{d}\|_{\ell_2}^2 \right) \end{aligned}$$

which simplifies to

$$(\beta^2-\epsilon)\|_T\underline{d}\|_{\ell_2}^2 \geq \epsilon\left(\|_T\underline{\vartheta}\|_{\ell_2}^2+\|_Tx^{\underline{G}}\|_{\ell_2}^2+\|_T\underline{x}^{\Psi}\|_{\ell_2}^2\right)+\langle {}_T\underline{r},S_T\underline{r}\rangle+\left\langle {}_{T+1}x^{\underline{H}},ZZ^*PZZ^*{}_{T+1}x^{\underline{H}}\right\rangle-\left\langle {}_Tx^{\underline{H}},P_Tx^{\underline{H}}\right\rangle+\|_T\underline{e}\|_{\ell_2}^2$$
$$=J_T^{\Psi,S}\left(0,0,\{\!\{\underline{\varphi},\Delta(\underline{\varphi})\}\!\}\right)+\begin{bmatrix}x^{\underline{G}}(T)\\ \underline{x}^{\Psi}(T)\end{bmatrix}^T[\![P]\!](T)\begin{bmatrix}x^{\underline{G}}(T)\\ \underline{x}^{\Psi}(T)\end{bmatrix}-\begin{bmatrix}(\xi^{\underline{G}})\\ 0\end{bmatrix}^T[\![P]\!](0)\begin{bmatrix}(\xi^{\underline{G}})\\ 0\end{bmatrix}+\|_T\underline{e}\|_{\ell_2}^2$$
$$+\epsilon\left(\|_T\underline{\vartheta}\|_{\ell_2}^2+\|_Tx^{\underline{G}}\|_{\ell_2}^2+\|_T\underline{x}^{\Psi}\|_{\ell_2}^2\right).$$

Since $P\succcurlyeq 0$ and $\mathbf{\Delta}\in \mathrm{HIQC}(\Psi,S)$, the previous inequality further simplifies to

$$\epsilon\left(\|_T\underline{\vartheta}\|_{\ell_2}^2+\|_Tx^{\underline{G}}\|_{\ell_2}^2+\|_T\underline{x}^{\Psi}\|_{\ell_2}^2\right)+\|_T\underline{e}\|_{\ell_2}^2\leq(\beta^2-\epsilon)\|_T\underline{d}\|_{\ell_2}^2+\begin{bmatrix}(\xi^{\underline{G}})^T & 0\end{bmatrix}[\![P]\!](0)\begin{bmatrix}(\xi^{\underline{G}})^T & 0\end{bmatrix}^T. \tag{14}$$

Since the LHS of (14) is monotonically nondecreasing with T, and $\lim_{T\to\infty}\|_Td\|_{\ell_2}^2=\|d\|_{\ell_2}^2<\infty$, the limit as $T\to\infty$ of the LHS of (14) exists, whereby we obtain the inequality

$$\epsilon\left(\|\underline{\vartheta}\|_{\ell_2}^2+\|x^{\underline{G}}\|_{\ell_2}^2+\|\underline{x}^{\Psi}\|_{\ell_2}^2\right)+\|\underline{e}\|_{\ell_2}^2\leq(\beta^2-\epsilon)\|\underline{d}\|_{\ell_2}^2+c_P\|\xi^{\underline{G}}\|_2^2 \tag{15}$$

for some $c_P\geq 0$. The previous inequality demonstrates that $\underline{\vartheta}$, $x^{\underline{G}}$, $\underline{x}^{\Psi}$, and $\underline{e}$ are also in ℓ_2. To prove robust stability of $(\underline{G}^{ic},\mathbf{\Delta})$, we must ultimately demonstrate that $(I-G_{11}\Delta)^{-1}$ is bounded on ℓ_2 for all $\Delta\in\mathbf{\Delta}$. This is done by recalling that the signals satisfying (8) also satisfy the simpler system of equations (6) (using the definitions $\underline{e}=\{\!\{v,w\}\!\}$, $\underline{d}=\{\!\{f,q\}\!\}$, $\underline{\varphi}=v$, and $\underline{\vartheta}=w-q$). As $\underline{d}=\{\!\{f,q\}\!\}\in\ell_2$ and $\xi^{\underline{G}}\in\mathbb{R}^{n_G}$ can be selected arbitrarily, we consider the case where both q and $\xi^{\underline{G}}$ are zero, simplifying (6) and (15) such that

$$(I-G_{11}\Delta)(v)=f \tag{16}$$

$$\|v\|_{\ell_2}\leq\|\underline{e}\|_{\ell_2}\leq\beta\|f\|_{\ell_2}\ \text{ for all }\ f\in\ell_2 \tag{17}$$

hold. Since f may be any element in ℓ_2, we see that $I-G_{11}\Delta$ is surjective. Considering the case where $f=0$, we invoke the bound on $\|v\|_{\ell_2}$ from (17) and use (16) to demonstrate that the null space of $I-G_{11}\Delta$ is $\{0\}$, thereby concluding that $I-G_{11}\Delta$ is invertible on ℓ_2. Having the relationship $v=(I-G_{11}\Delta)^{-1}(f)$ and noting (17), it is clear that $\|(I-G_{11}\Delta)^{-1}\|_{\ell_2\to\ell_2}\leq\beta$ for all $\Delta\in\mathbf{\Delta}$, proving robust stability of $(\underline{G}^{ic},\mathbf{\Delta})$. ∎

Though the operator inequality (13) forms an integral part of the proof, Lemma 2 still holds when verifying the simpler operator inequality (12).

Lemma 3. *The existence of $\beta>0$, $\epsilon>0$, and memoryless operators $P\succcurlyeq 0$ and $S=S^*$ such that (13) holds is equivalent to the existence of $\epsilon>0$ and memoryless operators $P\succcurlyeq 0$ and $S=S^*$ such that (12) holds.*

Proof. First, suppose that there exist $\epsilon>0$, $\beta>0$ and memoryless operators $P\succcurlyeq 0$ and $S=S^*$ such that (13) holds. Pre- and post-multiplying (13) by $\begin{bmatrix}I & 0 & 0\\ 0 & I & 0\end{bmatrix}$ and its adjoint, respectively, result in

$$\begin{bmatrix}A^{\underline{H}} & B^{\underline{H}}\\ C^{\underline{H}} & D^{\underline{H}}\\ C^{\underline{H}2} & D^{\underline{H}21}\end{bmatrix}^*\begin{bmatrix}Z^*PZ & 0 & 0\\ 0 & S & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}A^{\underline{H}} & B^{\underline{H}}\\ C^{\underline{H}} & D^{\underline{H}}\\ C^{\underline{H}2} & D^{\underline{H}21}\end{bmatrix}-\mathrm{diag}(P,0)\preccurlyeq-\epsilon I \tag{18}$$

which implies that (12) holds.

Second, suppose that there exist $\epsilon>0$ and memoryless operators $P\succcurlyeq 0$ and $S=S^*$ such that (12) holds. Due to the boundedness of the operators in (18), $P_\alpha=\alpha P$ and $S_\alpha=\alpha S$ satisfy (18) for a sufficiently large $\alpha>0$ (replacing P and S in

(18) with P_α and S_α, respectively). (13) can be equivalently stated as $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \preccurlyeq -\epsilon I$, where Y_{11} is equal to the LHS of (18),

$$Y_{12} = \begin{bmatrix} A^{\underline{H}} & B^{\underline{H}} \\ C^{\underline{H}} & D^{\underline{H}} \\ C^{\underline{H}2} & D^{\underline{H}21} \end{bmatrix}^* \begin{bmatrix} Z^*PZ & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} B^{\underline{H}2} \\ D^{\underline{H}12} \\ D^{\underline{H}22} \end{bmatrix}, \text{ and } Y_{22} = \begin{bmatrix} B^{\underline{H}2} \\ D^{\underline{H}12} \\ D^{\underline{H}22} \end{bmatrix}^* \begin{bmatrix} Z^*PZ & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} B^{\underline{H}2} \\ D^{\underline{H}12} \\ D^{\underline{H}22} \end{bmatrix} - \beta^2 I.$$

By the Schur complement formula, $Y \prec 0$ if and only $Y_{11} \prec 0$ and $Y_{22} - Y_{12}^* Y_{11}^{-1} Y_{12} \prec 0$. We have already shown that $Y_{11} \preccurlyeq -\epsilon I$. Exploiting the boundedness of Y_{11}^{-1}, Y_{12}, and each operator defining Y_{22}, there exists a sufficiently large $\beta > 0$ such that $Y_{22} - Y_{12}^* Y_{11}^{-1} Y_{12} \prec 0$. ∎

Hypotheses (a) and (b) in Lemma 2 are more restrictive than their counterparts in Theorem 1. These hypotheses can be relaxed when using positive-negative IQC multipliers.

Lemma 4. *Given an IQC multiplier* Π *with factors* (Ψ, S)*, if there exists a memoryless operator* $P = P^*$ *such that (12) holds, then for all* $N \in 0_+$*,* $\xi^G \in \mathbb{R}^{n_G}$*, and* $\xi \in \mathbb{R}^n$*,*

$$(\xi^{\underline{H}})^T [\![P]\!](N) \xi^{\underline{H}} \geq \underline{J}^{\Psi,S}(N, \xi), \quad \text{where } \xi^{\underline{H}} = \begin{bmatrix} (\xi^G)^T & \xi^T \end{bmatrix}^T.$$

Proof. For any $\vartheta \in \ell_2(N_+)$, $\xi^G \in \mathbb{R}^{n_G}$, and $\xi \in \mathbb{R}^n$, define

$$\begin{aligned} x^G &= \mathcal{L}_N^{G_{11}} \vartheta + \mathcal{S}_N^{G_{11}} \xi^G, & x^\Psi &= \mathcal{L}_N^\Psi \{\!\{\varphi, \vartheta\}\!\} + \mathcal{S}_N^\Psi \xi \\ \varphi &= G_{11}\vartheta + C^{G_1} \mathcal{S}_N^{G_{11}} \xi^G = C^{G_1} x^G + D^{G11} \vartheta, & r &= \Psi \{\!\{\varphi, \vartheta\}\!\} + C^\Psi \mathcal{S}_N^\Psi \xi = C^\Psi x^\Psi + \begin{bmatrix} D^{\Psi 1} & D^{\Psi 2} \end{bmatrix} \{\!\{\varphi, \vartheta\}\!\}. \end{aligned}$$

Since G_{11} and Ψ are causal nodes, then x^G, x^Ψ, φ, and r are in $\ell_2(N_+)$. Let $x^{\underline{H}} = \{\!\{x^G, x^\Psi\}\!\}$. Then, by recalling the operators defined in (11) and the form of Equation (3), it follows that $\mathbb{P}_N^+ Z^* x^{\underline{H}} = A^{\underline{H}} x^{\underline{H}} + B^{\underline{H}} \vartheta$ and $r = C^{\underline{H}} x^{\underline{H}} + D^{\underline{H}} \vartheta$. Thus, for all $\vartheta \in \ell_2(N_+)$, (12) implies that

$$\begin{aligned} -\epsilon \left(\|x^G\|_{\ell_2}^2 + \|x^\Psi\|_{\ell_2}^2 + \|\vartheta\|_{\ell_2}^2 \right) &\geq \left\langle \mathbb{P}_N^+ Z^* x^{\underline{H}}, Z^* P Z \mathbb{P}_N^+ Z^* x^{\underline{H}} \right\rangle - \left\langle x^{\underline{H}}, P x^{\underline{H}} \right\rangle + J^{\Psi,S}(N, \xi, \{\!\{\varphi, \vartheta\}\!\}) \\ &= -(\xi^{\underline{H}})^T [\![P]\!](N) \xi^{\underline{H}} + J^{\Psi,S}(N, \xi, \{\!\{\varphi, \vartheta\}\!\}) \end{aligned}$$

and so we can conclude that

$$(\xi^{\underline{H}})^T [\![P]\!](N) \xi^{\underline{H}} \geq \sup_{\vartheta \in \ell_2(N_+)} J^{\Psi,S}(N, \xi, \{\!\{\varphi, \vartheta\}\!\}) \geq \sup_{\vartheta \in \ell_2(N_+)} \inf_{\varphi \in \ell_2(N_+)} J^{\Psi,S}(N, \xi, \{\!\{\varphi, \vartheta\}\!\}) = \underline{J}^{\Psi,S}(N, \xi).$$

∎

Lemma 5. *Given an IQC multiplier* Π *with factors* (Ψ, S)*, if* $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$*, then*

$$J_N^{\Psi,S}(0, 0, \{\!\{\varphi, \Delta(\varphi)\}\!\}) \geq -\overline{J}^{\Psi,S}(N, x(N)) \tag{19}$$

for all $\varphi \in \ell_{2e}$*,* $\Delta \in \mathbf{\Delta}$*,* $N \in 0_+$*, where* $x = \mathcal{L}_0^\Psi \{\!\{\varphi, \Delta(\varphi)\}\!\}$*.*

Proof. Since $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$, then, for all $\varphi \in \ell_2$, $N \in 0_+$, and $\Delta \in \mathbf{\Delta}$, we have

$$0 \leq J^{\Psi,S}(0, 0, \{\!\{\varphi, \Delta(\varphi)\}\!\}) = J_N^{\Psi,S}(0, 0, \{\!\{{}_N\varphi, {}_N(\Delta({}_N\varphi))\}\!\}) + J^{\Psi,S}(N, x(N), \{\!\{{}^N\varphi, {}^N(\Delta(\varphi))\}\!\})$$

where the equality follows from the causality of Ψ and Δ. Therefore, for all $\varphi \in \ell_2$, $N \in 0_+$, and $\Delta \in \mathbf{\Delta}$, we have

$$J_N^{\Psi,S}(0, 0, \{\!\{{}_N\varphi, {}_N(\Delta({}_N\varphi))\}\!\}) \geq -J^{\Psi,S}(N, x(N), \{\!\{{}^N\varphi, {}^N(\Delta(\varphi))\}\!\}).$$

The term ${}^N(\Delta(\varphi))$ is the only term on the RHS of the previous inequality that depends on the "past," that is, time-steps $k<N$. To remove this dependence, the RHS of the previous inequality is replaced by a corresponding lower bound. Namely, the following inequality is obtained, which holds for all $\varphi\in\ell_2$, $N\in 0_+$, and $\Delta\in\mathbf{\Delta}$:

$$J_N^{\Psi,S}(0,0,\{\!\|{}_N\varphi,{}_N(\Delta({}_N\varphi))\|\!\}) \geq \inf_{{}^N\vartheta\in\ell_2(N_+)} -J^{\Psi,S}(N,x(N),\{\!\|{}^N\varphi,{}^N\vartheta\|\!\}).$$

Since this inequality holds for all $\varphi\in\ell_2$ and its LHS is independent of ${}^N\varphi$, taking the supremum over ${}^N\varphi\in\ell_2(N_+)$ on the RHS gives

$$J_N^{\Psi,S}(0,0,\{\!\|{}_N\varphi,{}_N(\Delta({}_N\varphi))\|\!\}) \geq \sup_{{}^N\varphi\in\ell_2(N_+)}\inf_{{}^N\vartheta\in\ell_2(N_+)} -J^{\Psi,S}(N,x(N),\{\!\|{}^N\varphi,{}^N\vartheta\|\!\}) = -\overline{J}^{\Psi,S}(N,x(N)).$$

Finally, by causality of Ψ and all $\Delta\in\mathbf{\Delta}$, the LHS of the previous inequality can be rewritten as $J_N^{\Psi,S}(0,0,\{\!\|\varphi,(\Delta(\varphi))\|\!\})$.

Up to this point we have proven that (19) holds for all $\Delta\in\mathbf{\Delta}$, $N\in 0_+$, and $\varphi\in\ell_2$. However, (19) must hold for all signals φ in the larger set ℓ_{2e}. To establish this, we show that, for all $\varphi\in\ell_{2e}$, $N\in 0_+$, and $\Delta\in\mathbf{\Delta}$, there exists a $\tilde{\varphi}\in\ell_2$ such that

$$J_N^{\Psi,S}(0,0,\{\!\|\varphi,\Delta(\varphi)\|\!\}) = J_N^{\Psi,S}(0,0,\{\!\|\tilde{\varphi},\Delta(\tilde{\varphi})\|\!\}) \text{ and} \tag{20}$$

$$\overline{J}^{\Psi,S}(N,x(N)) = \overline{J}^{\Psi,S}(N,\tilde{x}(N)) \tag{21}$$

where $x=\mathcal{L}_0^{\Psi}\{\!\|\varphi,\Delta(\varphi)\|\!\}$ and $\tilde{x}=\mathcal{L}_0^{\Psi}\{\!\|\tilde{\varphi},\Delta(\tilde{\varphi})\|\!\}$. One choice of $\tilde{\varphi}$ is given by $\tilde{\varphi}={}_N\tilde{\varphi}={}_N\varphi\in\ell_2$. We prove (20) by noting that

$$\begin{aligned}
J_N^{\Psi,S}(0,0,\{\!\|\varphi,\Delta(\varphi)\|\!\}) &= J_N^{\Psi,S}(0,0,\{\!\|{}_N\varphi,{}_N(\Delta(\varphi))\|\!\}) && \text{(by causality of } \Psi)\\
&= J_N^{\Psi,S}(0,0,\{\!\|{}_N\varphi,{}_N(\Delta({}_N\varphi))\|\!\}) && \text{(by causality of } \Delta)\\
&= J_N^{\Psi,S}(0,0,\{\!\|{}_N\tilde{\varphi},{}_N(\Delta(\tilde{\varphi}))\|\!\}) && \text{(by choice of } \tilde{\varphi})\\
&= J_N^{\Psi,S}(0,0,\{\!\|\tilde{\varphi},(\Delta(\tilde{\varphi}))\|\!\}) && \text{(by causality of } \Psi).
\end{aligned}$$

Similar arguments invoking causality of Ψ and Δ establish that $x(N)=\tilde{x}(N)$, thereby proving (21). ∎

The counterpart of Lemma 5 for time-invariant IQC multipliers[18] gives $J_N^{\Psi,S}(0,0,\{\!\|\varphi,\Delta(\varphi)\|\!\})\geq -\overline{J}^{\Psi,S}(0,x(N))$. Here, since Π may be time-varying, the lower bound in (19) is given by the upper Popov index *starting at N* rather than 0.

The next lemma is key to using Lemmas 4 and 5 to derive Theorem 1 from Lemma 2.

Lemma 6. *For every strictly positive-negative IQC multiplier* Π *with factors* (Ψ,S)*,*

(1) there exists a stabilizing solution X to the DDRE(Ψ,S)*;*

(2) $\underline{J}^{\Psi,S}(N,\xi)=\overline{J}^{\Psi,S}(N,\xi)=\xi^TX(N)\xi$ *for all* $N\in 0_+$ *and* $\xi\in\mathbb{R}^n$*;*

(3) there exist factors $(\overline{\Psi},\overline{S})$ *for* Π *such that* $\overline{\Psi}=[A^{\Psi},B^{\Psi},C^{\overline{\Psi}},D^{\overline{\Psi}}]_+$*,* $\overline{\Psi}^{-1}$ *is a bounded causal node, and the stabilizing solution to the* DDRE$(\overline{\Psi},\overline{S})$ *is* $\overline{X}=0$*.*

Proof. (1): By the definition of a strictly positive-negative multiplier, there exists a scalar $\epsilon>0$ such that $\langle u,\Pi_{11}u\rangle\geq\epsilon\|u\|^2$ and $\langle v,\Pi_{22}v\rangle\leq-\epsilon\|v\|^2$ for all u and v in ℓ_2. This implies that $\langle\overline{u},\Pi_{11}\overline{u}\rangle\geq\epsilon\|\overline{u}\|^2$ and $\langle\overline{v},\Pi_{22}\overline{v}\rangle\leq-\epsilon\|\overline{v}\|^2$ for all $\overline{u}$ and $\overline{v}$ in $\ell_2(N_+)$, which in turn implies that ${}^N\Pi$ is strictly positive-negative on $\ell_2(N_+)$ for all $N\in 0_+$. Because ${}^N\Pi_{22}$ is negative definite, it is bijective on $\ell_2(N_+)$, and by the bounded inverse theorem,[39] it has a bounded inverse on $\ell_2(N_+)$. Since ${}^N\Pi_{11}$ is positive definite, then ${}^N\Pi^{\times}={}^N\Pi_{11}-{}^N\Pi_{12}\,({}^N\Pi_{22})^{-1}\,{}^N\Pi_{12}^*$ is positive definite and has a bounded inverse on $\ell_2(N_+)$. Moreover,

$${}^N\Pi=\begin{bmatrix} I & {}^N\Pi_{12}\,{}^N\Pi_{22}^{-1}\\ 0 & I\end{bmatrix}\begin{bmatrix}{}^N\Pi^{\times} & 0\\ 0 & {}^N\Pi_{22}\end{bmatrix}\begin{bmatrix} I & 0\\ {}^N\Pi_{22}^{-1}\,{}^N\Pi_{12}^* & I\end{bmatrix}$$

has a bounded inverse on $\ell_2(N_+)$ that can be easily computed. Invoking Theorem 3.2.2 from Halanay and Ionescu,[25] if ${}^N\Pi$ has a bounded inverse on $\ell_2(N_+)$ for all $N\in 0_+$, then there exists a stabilizing solution to the DDRE(Ψ,S).

(2): Define $u_N^\star = \{\!\!\{\varphi_N^\star, \vartheta_N^\star\}\!\!\} = -(\mathcal{R}_N^{\Psi,S})^{-1}(\mathcal{P}_N^{\Psi,S})^*\xi$ and $x_N^\star = \mathcal{L}_N^\Psi u_N^\star + \mathcal{S}_N^\Psi \xi$. Since $(\mathcal{R}_N^{\Psi,S})^{-1} = ({}^N\Pi)^{-1}$ is bounded, then $\varphi_N^\star$ and $\vartheta_N^\star$ are in $\ell_2(N_+)$, and the quantity $J^{\Psi,S}(N,\xi,u_N^\star)$ is well-defined. Hence,

$$J^{\Psi,S}(N,\xi,u_N^\star) = \left\langle \begin{bmatrix} \xi \\ u_N^\star \end{bmatrix}, \begin{bmatrix} \mathcal{X}_N^{\Psi,S} & \mathcal{P}_N^{\Psi,S} \\ (\mathcal{P}_N^{\Psi,S})^* & \mathcal{R}_N^{\Psi,S} \end{bmatrix} \begin{bmatrix} \xi \\ u_N^\star \end{bmatrix} \right\rangle = \left\langle \xi, \left(\mathcal{X}_N^{\Psi,S} - \mathcal{P}_N^{\Psi,S}(\mathcal{R}_N^{\Psi,S})^{-1}(\mathcal{P}_N^{\Psi,S})^* \right)\xi \right\rangle.$$

Following similar arguments as in the proof of [25, Theorem 3.2.2], $\mathcal{X}_N^{\Psi,S} - \mathcal{P}_N^{\Psi,S}(\mathcal{R}_N^{\Psi,S})^{-1}(\mathcal{P}_N^{\Psi,S})^* = X(N)$. Now, for any $\varphi_N, \vartheta_N \in \ell_2(N_+)$, define $u_N = \{\!\!\{\varphi_N, \vartheta_N\}\!\!\}$, $x_N = \mathcal{L}_N^\Psi u_N + \mathcal{S}_N^\Psi \xi$, $\tilde{u}_N = u_N - u_N^\star$, and $\tilde{x}_N = x_N - x_N^\star$, and observe that

$$\begin{aligned}
J^{\Psi,S}(N,\xi,u_N) &= \left\langle \begin{bmatrix} x_N^\star + \tilde{x}_N \\ u_N^\star + \tilde{u}_N \end{bmatrix}, \begin{bmatrix} {}^NQ & {}^NL \\ ({}^NL)^* & {}^NR \end{bmatrix} \begin{bmatrix} x_N^\star + \tilde{x}_N \\ u_N^\star + \tilde{u}_N \end{bmatrix} \right\rangle \\
&= J^{\Psi,S}(N,\xi,u_N^\star) + J^{\Psi,S}(N,0,\tilde{u}_N) + 2\left\langle \begin{bmatrix} \tilde{x}_N \\ \tilde{u}_N \end{bmatrix}, \begin{bmatrix} {}^NQ & {}^NL \\ ({}^NL)^* & {}^NR \end{bmatrix} \begin{bmatrix} x_N^\star \\ u_N^\star \end{bmatrix} \right\rangle \\
&= J^{\Psi,S}(N,\xi,u_N^\star) + J^{\Psi,S}(N,0,\tilde{u}_N) + 2\left\langle \begin{bmatrix} 0 \\ \tilde{u}_N \end{bmatrix}, \begin{bmatrix} \mathcal{S}_N^\Psi & \mathcal{L}_N^\Psi \\ 0 & I \end{bmatrix}^* \begin{bmatrix} {}^NQ & {}^NL \\ ({}^NL)^* & {}^NR \end{bmatrix} \begin{bmatrix} \mathcal{S}_N^\Psi & \mathcal{L}_N^\Psi \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ u_N^\star \end{bmatrix} \right\rangle \\
&= J^{\Psi,S}(N,\xi,u_N^\star) + J^{\Psi,S}(N,0,\tilde{u}_N) + 2\left\langle \begin{bmatrix} 0 \\ \tilde{u}_N \end{bmatrix}, \begin{bmatrix} \mathcal{X}_N^{\Psi,S} & \mathcal{P}_N^{\Psi,S} \\ (\mathcal{P}_N^{\Psi,S})^* & \mathcal{R}_N^{\Psi,S} \end{bmatrix} \begin{bmatrix} \xi \\ u_N^\star \end{bmatrix} \right\rangle \\
&= J^{\Psi,S}(N,\xi,u_N^\star) + J^{\Psi,S}(N,0,\tilde{u}_N) + 2\left\langle \begin{bmatrix} 0 \\ \tilde{u}_N \end{bmatrix}, \begin{bmatrix} X(N) \\ 0 \end{bmatrix} \xi \right\rangle \\
&= J^{\Psi,S}(N,\xi,u_N^\star) + J^{\Psi,S}(N,0,\tilde{u}_N).
\end{aligned}$$

Then, for all φ_N and ϑ_N in $\ell_2(N_+)$, we have

$$\begin{aligned}
J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N,\vartheta_N^\star\}\!\!\}) - J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N^\star,\vartheta_N^\star\}\!\!\}) &= J^{\Psi,S}(N,0,\{\!\!\{\tilde{\varphi}_N,0\}\!\!\}) \geq 0 \\
J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N^\star,\vartheta_N\}\!\!\}) - J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N^\star,\vartheta_N^\star\}\!\!\}) &= J^{\Psi,S}(N,0,\{\!\!\{0,\tilde{\vartheta}_N\}\!\!\}) \leq 0
\end{aligned}$$

since ${}^N\Pi_{11}$ is positive definite, ${}^N\Pi_{22}$ is negative definite, and $J^{\Psi,S}(N,0,u_N) = \langle u_N, \Pi u_N\rangle$ for all $u_N \in \ell_2(N_+)$. Thus, for all φ_N and ϑ_N in $\ell_2(N_+)$, the following holds:

$$J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N^\star,\vartheta_N\}\!\!\}) \leq J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N^\star,\vartheta_N^\star\}\!\!\}) \leq J^{\Psi,S}(N,\xi,\{\!\!\{\varphi_N,\vartheta_N^\star\}\!\!\}).$$

In other words, $(\varphi_N^\star, \vartheta_N^\star)$ is a saddle-point solution, whose existence implies that the lower and upper Popov indices are equal,[40] that is, $\overline{J}^{\Psi,S}(N,\xi) = \underline{J}^{\Psi,S}(N,\xi) = \xi^T X(N)\xi$.

(3): The existence of factors $(\overline{\Psi}, \overline{S})$ for Π such that $\overline{\Psi} = [A^\Psi, B^\Psi, C^{\overline{\Psi}}, D^{\overline{\Psi}}]_+$ is proven in [25, Proposition 4.4.3]. Namely, if we partition $R + (B^\Psi)^* Z^* XZB^\Psi = \begin{bmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{12}^* & \mathcal{Y}_{22} \end{bmatrix}$ and define $\mathcal{Y}_{22}^\times = \mathcal{Y}_{22} - \mathcal{Y}_{12}^* \mathcal{Y}_{11}^{-1}\mathcal{Y}_{12}$, then the desired $\overline{S}$, $D^{\overline{\Psi}}$, and $C^{\overline{\Psi}}$ can be defined as follows:

$$\overline{S} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad D^{\overline{\Psi}} = \begin{bmatrix} \mathcal{Y}_{11}^{\frac{1}{2}} & \mathcal{Y}_{11}^{-\frac{1}{2}}\mathcal{Y}_{12} \\ 0 & (-\mathcal{Y}_{22}^\times)^{\frac{1}{2}} \end{bmatrix}$$
$$C^{\overline{\Psi}} = \overline{S}\left((D^{\overline{\Psi}})^{-1}\right)^* \left((B^\Psi)^* Z^* XZA^\Psi + (D^\Psi)^* SC^\Psi\right).$$

To prove $(\overline{\Psi}, \overline{S})$ define factors for Π, we show that

$$\langle u, \Pi u\rangle = \langle u, \Psi^* S\Psi u\rangle = \left\langle u, \overline{\Psi}^* \overline{S}\, \overline{\Psi} u \right\rangle \quad \text{for all} \quad u \in \ell_2. \tag{22}$$

This is done by first defining $Q := (C^{\Psi})^* S C^{\Psi}$, $L := (C^{\Psi})^* S D^{\Psi}$, and $R := (D^{\Psi})^* S D^{\Psi}$ and invoking the DDRE(Ψ, S) to demonstrate that

$$\begin{bmatrix} \overline{Q} & \overline{L} \\ \overline{L}^* & \overline{R} \end{bmatrix} := \begin{bmatrix} (C^{\overline{\Psi}})^* \\ (D^{\overline{\Psi}})^* \end{bmatrix} \overline{S} \begin{bmatrix} C^{\overline{\Psi}} & D^{\overline{\Psi}} \end{bmatrix} = \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} + \begin{bmatrix} (A^{\Psi})^* Z^* X Z A^{\Psi} - X & (A^{\Psi})^* Z^* X Z B^{\Psi} \\ (B^{\Psi})^* Z^* X Z A^{\Psi} & (B^{\Psi})^* Z^* X Z B^{\Psi} \end{bmatrix}.$$

Proving (22) amounts to showing that, for all $u \in \ell_2$ and $x = (I - Z A^{\Psi})^{-1} Z B^{\Psi} u \in \ell_2$,

$$\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} \overline{Q} & \overline{L} \\ \overline{L}^* & \overline{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle.$$

The equality of interest indeed holds since

$$\begin{aligned} \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} (A^{\Psi})^* Z^* X Z A^{\Psi} - X & (A^{\Psi})^* Z^* X Z B^{\Psi} \\ (B^{\Psi})^* Z^* X Z A^{\Psi} & (B^{\Psi})^* Z^* X Z B^{\Psi} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \left(\begin{bmatrix} (Z A^{\Psi})^* \\ (Z B^{\Psi})^* \end{bmatrix} X \begin{bmatrix} Z A^{\Psi} & Z B^{\Psi} \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} Z A^{\Psi} & Z B^{\Psi} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, X \begin{bmatrix} Z A^{\Psi} & Z B^{\Psi} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle - \langle x, X x \rangle = 0. \end{aligned}$$

To prove that $\overline{\Psi}^{-1}$ is a causal node (boundedness is implied by the definition of a causal node), we use the fact that, for a causal node $\mathcal{G} = [A, B, C, D]_+$, if D has a bounded inverse and $A - BD^{-1}C$ is causally stable, then $\mathcal{G}^{-1}$ is a causal node defined by $\mathcal{G}^{-1} = [A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1}]_+$. By construction, the operator $D^{\overline{\Psi}}$ has a bounded inverse and $A^{\Psi} - B^{\Psi}(D^{\overline{\Psi}})^{-1} C^{\overline{\Psi}} = A^{\Psi} + B^{\Psi} F$ is causally stable, where F is the feedback gain associated with the stabilizing solution X to the DDRE(Ψ, S). Therefore, we define $\overline{\Psi}^{-1}$ by the causal node $[A^{\Psi} - B^{\Psi}(D^{\overline{\Psi}})^{-1} C^{\overline{\Psi}}, B^{\Psi}(D^{\overline{\Psi}})^{-1}, -(D^{\overline{\Psi}})^{-1} C^{\overline{\Psi}}, (D^{\overline{\Psi}})^{-1}]_+$.

That $\overline{X} = 0$ is a solution to the DDRE$(\overline{\Psi}, \overline{S})$ is proved by substituting $\overline{X} = 0$ into the DDRE$(\overline{\Psi}, \overline{S})$ and verifying that $0 = \overline{Q} - \overline{L}\,\overline{R}^{-1}\overline{L}^*$, where $\overline{R} = R + (B^{\Psi})^* Z^* X Z B^{\Psi}$ is known to have a bounded inverse. The equality of interest holds by the definition of $\overline{Q}$, $\overline{L}$, and $\overline{R}$ since X is the stabilizing solution to the DDRE(Ψ, S). Finally, $\overline{X} = 0$ is the stabilizing solution to the DDRE$(\overline{\Psi}, \overline{S})$ because the associated feedback operator $\overline{F} = -\overline{R}^{-1}\overline{L}^* = F$, where F is the feedback operator associated with the stabilizing solution X of the DDRE(Ψ, S). ∎

Proof of Theorem 1. Following the approach in Seiler's work,[18] we first consider the case where Π is a strictly positive-negative multiplier and has factors (Ψ, S). By the hypotheses of Theorem 1, $\mathbf{\Delta} \in \text{IQC}(\Pi)$ and there exists a memoryless operator $P = P^*$ such that (12) holds. By Lemma 6, Π has factors $(\overline{\Psi}, \overline{S})$ such that $\underline{J}^{\overline{\Psi},\overline{S}}(N, \xi) = \underline{J}^{\overline{\Psi},\overline{S}}(N, \xi) = \xi^T \overline{X}(N) \xi = 0$ for all $N \in 0_+$ and $\xi \in \mathbb{R}^n$, where $\overline{X} = 0$ is the stabilizing solution to the DDRE$(\overline{\Psi}, \overline{S})$. By Lemma 5, $J_T^{\overline{\Psi},\overline{S}}(0, 0, \{\!\{\varphi, \Delta(\varphi)\}\!\}) \geq 0$ for all $T \in 0_+$, $\varphi \in \ell_{2e}$, and $\Delta \in \mathbf{\Delta}$, that is, $\mathbf{\Delta} \in \text{HIQC}(\overline{\Psi}, \overline{S})$. By lengthy manipulations, it can be shown that (12) is satisfied if and only if

$$\begin{bmatrix} A^{\underline{\overline{H}}} & B^{\underline{\overline{H}}} \\ C^{\underline{\overline{H}}} & D^{\underline{\overline{H}}} \end{bmatrix}^* \begin{bmatrix} Z^* \overline{P} Z & 0 \\ 0 & \overline{S} \end{bmatrix} \begin{bmatrix} A^{\underline{\overline{H}}} & B^{\underline{\overline{H}}} \\ C^{\underline{\overline{H}}} & D^{\underline{\overline{H}}} \end{bmatrix} - \begin{bmatrix} \overline{P} & 0 \\ 0 & 0 \end{bmatrix} \preccurlyeq -\epsilon I$$

where $\overline{P} = P - \begin{bmatrix} 0 & I \end{bmatrix}^* X \begin{bmatrix} 0 & I \end{bmatrix}$, X is the stabilizing solution of the DDRE(Ψ, S), and the operators $A^{\underline{\overline{H}}}$, $B^{\underline{\overline{H}}}$, $C^{\underline{\overline{H}}}$, and $D^{\underline{\overline{H}}}$ are defined as in (11) by replacing all instances of $\underline{H}$ and Ψ with $\underline{\overline{H}}$ and $\overline{\Psi}$, respectively.[2] Moreover, by Lemma 4, $[\![\overline{P}]\!](N) \succcurlyeq 0$ for all $N \in 0_+$.

From the above discussion and by calling on Lemma 3, one can see that the hypotheses of Lemma 2 are satisfied when using the factors $(\overline{\Psi}, \overline{S})$ for Π, which provides the conclusion of Theorem 1 for the case of strictly positive-negative IQC multipliers.

[2]By Lemma 6, $A^{\overline{\Psi}} = A^{\Psi}$ and $B^{\overline{\Psi}} = B^{\Psi}$, and so $C^{\underline{\overline{H}}}$ and $D^{\underline{\overline{H}}}$ are the only operators that differ from $C^{\underline{H}}$ and $D^{\underline{H}}$, respectively.

Suppose now that Π is nonstrictly positive-negative. Since there exists $\mu > 0$ such that $\sup_{\Delta\in\mathbf{\Delta}}\|\Delta\|_{\ell_2\to\ell_2} \le \mu$, then $\mathbf{\Delta} \in \mathrm{IQC}(\Pi^\kappa)$, where $\Pi^\kappa = \kappa\,\mathrm{diag}(\mu^2 I, -I) + \Pi$ is a strictly positive-negative multiplier for all $\kappa > 0$. Because the LHS of (12) is strictly negative definite, we can choose κ small enough such that (12) still holds with the factors of Π^κ. The remainder of the proof then follows similarly to the first part by considering the strictly positive-negative multiplier Π^κ. ∎

4.2 | Robust performance of uncertain systems

In this section, we provide a robust performance theorem that allows the use of signal IQC multipliers to describe the set $\mathcal{D}$ of allowable disturbances. Typical robust performance theorems assume that the disturbance d is an ℓ_2 signal. However, oftentimes more information is known about the disturbance signals, such as their characteristics in the frequency-domain or time-domain. These characteristics can be incorporated by using signal IQCs to constrain the disturbance signals d and provide less conservative measures on the robust performance level of the uncertain system.

Definition 12. *A signal IQC multiplier* $\Phi = \Phi^*$ is defined as a self-adjoint dichotomic node. The signal set $\mathcal{D} \subseteq \ell_2$ *satisfies the signal IQC defined by* Φ ($\mathcal{D} \in \mathrm{SigIQC}(\Phi)$) if

$$\langle d, \Phi d\rangle \ge 0 \ \text{ for all } \ d \in \mathcal{D}.$$

Given factors (Θ, U) of Φ, where $\Theta = [A^\Theta, B^\Theta, C^\Theta, D^\Theta]_+$, the set $\mathcal{D} \in \mathrm{SigIQC}(\Phi)$ if $J^{\Theta,U}(0,0,d) \ge 0$ for all $d \in \mathcal{D}$.

Section 4.1 was concerned with determining robust stability for $(\underline{G}^{ic}, \mathbf{\Delta})$ defined in (7), which is equivalent to robust stability of $(G^{ic}, \mathbf{\Delta})$ defined in (5). In this section, we focus on computing a robust performance level for the uncertain system $(G^{ic}, \mathbf{\Delta})$ defined in (5) directly.

Definition 13. The uncertain system $(G^{ic}, \mathbf{\Delta})$ defined in (5) has a *robust* $(\mathcal{E}(\Xi), \mathcal{D})$*-to-*$\ell_2$ *gain performance level of* (λ, γ) if it is robustly stable and $\|e\|_{\ell_2}^2 \le \gamma^2\|d\|_{\ell_2}^2 + \lambda^2$ for all $d \in \mathcal{D}$, $\xi^G \in \mathcal{E}(\Xi)$, and $\Delta \in \mathbf{\Delta}$. The term "robust $(\mathcal{E}(\Xi), \mathcal{D})$-to-$\ell_2$ gain performance level" is sometimes referred to as "robust performance level" for simplicity.

Theorem 2 provides conditions for determining a robust performance level for $(G^{ic}, \mathbf{\Delta})$. To state the theorem, we define the following partitioned memoryless operators:

$$A^H = \begin{bmatrix} A^G & 0 & 0 \\ B^{\Psi 1}C^{G1} & A^\Psi & 0 \\ 0 & 0 & A^\Theta \end{bmatrix}, \quad B^H = \begin{bmatrix} B^{G1} \\ B^{\Psi 1}D^{G11} + B^{\Psi 2} \\ 0 \end{bmatrix}, \quad B^{H2} = \begin{bmatrix} B^{G2} \\ B^{\Psi 1}D^{G12} \\ B^\Theta \end{bmatrix}$$

$$C^H = \begin{bmatrix} D^{\Psi 1}C^{G1} & C^\Psi & 0 \end{bmatrix}, \quad D^H = D^{\Psi 1}D^{G11} + D^{\Psi 2}, \quad D^{H12} = D^{\Psi 1}D^{G12}$$

$$C^{H2} = \begin{bmatrix} C^{G2} & 0 & 0 \end{bmatrix}, \quad D^{H21} = D^{G21}, \quad D^{H22} = D^{G22}$$

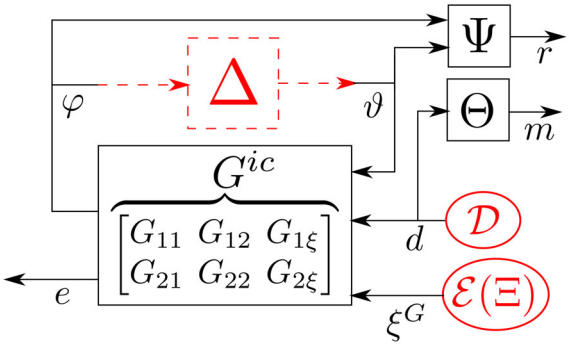
$$C^{H3} = \begin{bmatrix} 0 & 0 & C^\Theta \end{bmatrix}, \quad D^{H31} = 0, \quad D^{H32} = D^\Theta.$$

Consider Figure 3 that shows the augmented system used in robust performance analysis. For any $\xi^G \in \mathbb{R}^{n_G}$ and signals d and ϑ in ℓ_2, the signals $x^G, x^\Psi, x^\Theta, \varphi, e, r$, and m defined by

$$\begin{bmatrix} \varphi \\ e \end{bmatrix} = G^{ic}\left\{\begin{bmatrix} \vartheta \\ d \end{bmatrix}, \xi^G\right\}, \quad r = \Psi\begin{bmatrix} \varphi \\ \vartheta \end{bmatrix}, \quad m = \Theta d$$

$$x^G = \mathcal{L}_0^G\begin{bmatrix} \vartheta \\ d \end{bmatrix} + S_0^G\xi^G, \quad x^\Psi = \mathcal{L}_0^\Psi\begin{bmatrix} \varphi \\ \vartheta \end{bmatrix}, \quad x^\Theta = \mathcal{L}_0^\Theta d \tag{23}$$

FIGURE 3 Augmented system used in robust performance analysis [Colour figure can be viewed at wileyonlinelibrary.com]



are in ℓ_2 and satisfy

$$\begin{bmatrix} Z^*x^H \\ r \\ e \\ m \end{bmatrix} = \underbrace{\begin{bmatrix} A^H & B^H & B^{H2} \\ C^H & D^H & D^{H12} \\ C^{H2} & D^{H21} & D^{H22} \\ C^{H3} & D^{H31} & D^{H32} \end{bmatrix}}_{\mathcal{H}_{\text{perf}}} \begin{bmatrix} x^H \\ \vartheta \\ d \end{bmatrix}$$

where $x^H = \{\![x^G, x^\Psi, x^\Theta]\!\}$.

Theorem 2. *Given the partitioned IC node G^{ic} defined in (4), the uncertain system $(G^{ic}, \mathbf{\Delta})$ defined by (5) has a robust $(\mathcal{E}(\Xi), \mathcal{D})$-to-$\ell_2$ gain performance level of (λ, γ) if*

(a) $(G^{ic}, \mathbf{\Delta})$ is robustly well-posed;
(b) there exists a positive-negative IQC multiplier Π with factors (Ψ, S) such that $\mathbf{\Delta} \in \text{IQC}(\Pi)$;
(c) there exists a signal IQC multiplier Φ with factors (Θ, U) such that $\mathcal{D} \in \text{SigIQC}(\Phi)$;
(d) there exist $\epsilon > 0$ and a memoryless operator $P = P^*$ such that

$$\mathcal{H}_{\text{perf}}^* \operatorname{diag}(Z^*PZ, S, I, U)\mathcal{H}_{\text{perf}} - \operatorname{diag}(P, 0, \gamma^2 I) \preccurlyeq -\epsilon I \tag{24}$$

$$\text{and} \quad \begin{bmatrix} I_{n_G} & 0 \end{bmatrix} [\![P]\!](0) \begin{bmatrix} I_{n_G} & 0 \end{bmatrix}^T \preccurlyeq \lambda^2 \Xi. \tag{25}$$

Proof. We first prove that $(G^{ic}, \mathbf{\Delta})$ is robustly stable. Define the operator

$$W = \begin{bmatrix} V & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad \text{where } V = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

Pre- and post-multiplying (24) by W and W^*, respectively, we obtain

$$\begin{bmatrix} \tilde{A}^H & \tilde{B}^H \\ C^{\underline{H}} & D^{\underline{H}} \\ \tilde{C}^{H2} & D^{H21} \end{bmatrix}^* \begin{bmatrix} Z^*PZ & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}^H & \tilde{B}^H \\ C^{\underline{H}} & D^{\underline{H}} \\ \tilde{C}^{H2} & D^{H21} \end{bmatrix} - \operatorname{diag}(VPV^*, 0) \preccurlyeq -\epsilon I$$

where $\tilde{A}^H = \begin{bmatrix} A^{\underline{H}} \\ 0 \end{bmatrix}$, $\tilde{B}^H = \begin{bmatrix} B^{\underline{H}} \\ 0 \end{bmatrix}$, and $\tilde{C}^{H2} = \begin{bmatrix} C^{G2} & 0 \end{bmatrix}$. Partition P as follows:

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} = \left[\begin{array}{cc:c} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ \hdashline P_{13}^* & P_{23}^* & P_{33} \end{array}\right]$$

where $P_1 : \ell_2^{n_G} \times \ell_2^n \to \ell_2^{n_G} \times \ell_2^n$ is conformable with $A^{\underline{H}}$. Given this partitioning of P, it is not difficult to see that the previous inequality implies that (12) holds (replacing all instances of P with P_1). Hence, by Theorem 1, the uncertain system $(\underline{G}^{ic}, \mathbf{\Delta})$ is robustly stable, and equivalently, $(G^{ic}, \mathbf{\Delta})$ is robustly stable.

Since $(G^{ic}, \mathbf{\Delta})$ is robustly stable, then for any $d \in \ell_2$ and $\xi^G \in \mathbb{R}^{n_G}$, there exist ϑ, φ, and e in ℓ_2 that satisfy (5). In particular, we restrict $d \in \mathcal{D} \subseteq \ell_2$ and $\xi^G \in \mathcal{E}(\Xi)$. Because G^{ic} is a partitioned IC node and Ψ and Θ are causal nodes, it then follows that the signals x^G, r, x^Ψ, m, and x^Θ defined in (23) are also in ℓ_2. Invoking (23) and (24), we have

$$\left\langle Z^* x^H, Z^* P Z Z^* x^H \right\rangle - \left\langle x^H, P x^H \right\rangle + \langle r, Sr \rangle + \langle m, Um \rangle + \|e\|_{\ell_2}^2 + \epsilon(\|\vartheta\|_{\ell_2}^2 + \|x^G\|_{\ell_2}^2 + \|x^\Psi\|_{\ell_2}^2 + \|x^\Theta\|_{\ell_2}^2) \leq (\gamma^2 - \epsilon)\|d\|_{\ell_2}^2.$$

Noting that $x^\Psi(0)$ and $x^\Theta(0)$ are zero, the previous inequality implies

$$J^{\Psi,S}(0, 0, \{\!\{\varphi, \vartheta\}\!\}) + J^{\Theta,U}(0, 0, d) + \|e\|_{\ell_2}^2 \leq (\gamma^2 - \epsilon)\|d\|_{\ell_2}^2 + (\xi^G)^T P_{11}(0) \xi^G. \tag{26}$$

Since $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$ and $\mathcal{D} \in \mathrm{SigIQC}(\Phi)$, then $J^{\Psi,S}(0, 0, \{\!\{\varphi, \vartheta\}\!\}) \geq 0$ and $J^{\Theta,U}(0, 0, d) \geq 0$ for all $\vartheta = \Delta(\varphi)$, $\varphi \in \ell_2$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathcal{D}$. Moreover, since $\xi^G \in \mathcal{E}(\Xi)$, then (25) implies that $(\xi^G)^T P_{11}(0) \xi^G \leq \lambda^2$. In conclusion, $\|e\|_{\ell_2}^2 \leq \gamma^2 \|d\|_{\ell_2}^2 + \lambda^2$. ■

As observed in a recent work,[27] from (26) and the fact that $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$, it follows that if $d \equiv 0$, then $\|e\|_{\ell_2}^2 \leq (\xi^G)^T P_{11}(0) \xi^G$ for all $\xi^G \in \mathbb{R}^{n_G}$, that is, $P_{11}(0) \succcurlyeq 0$.

The volume of the ellipsoid $\mathcal{E}(\Xi)$ is proportional to $\sqrt{\det(\Xi^{-1})}$; hence, we can make $\mathcal{E}(\Xi)$ arbitrarily small by choosing the eigenvalues of Ξ to be arbitrarily large. Therefore, by dropping the constraint (25), Theorem 2 gives an upper bound γ on the robust $\mathcal{D}$-to-ℓ_2 gain performance level of the uncertain system $(G, \mathbf{\Delta})$, wherein the nominal system G has zero initial conditions.

Suppose that some components of $\xi^G \in \mathbb{R}^{n_G}$ are known to be zeros, whereas the remaining components are uncertain. Specifically, assume that $\xi^G = E \xi^G_{\mathrm{red}}$ for an appropriately defined matrix E, where $\xi^G_{\mathrm{red}} \in \mathbb{R}^{n_{G,\mathrm{red}}}$, $n_{G,\mathrm{red}} < n_G$, and $\xi^G_{\mathrm{red}} \in \mathcal{E}(\Xi_{\mathrm{red}})$. The rows of E that correspond to the zero components of ξ^G have zero elements. In this case, the condition (25) is modified from $P_{11}(0) \preccurlyeq \lambda^2 \Xi$ to $E^T P_{11}(0) E \preccurlyeq \lambda^2 \Xi_{\mathrm{red}}$, and we speak of a robust $(\mathcal{E}(\Xi_{\mathrm{red}}), \mathcal{D})$-to-$\ell_2$ gain performance level of (λ, γ) for the uncertain system $(G^{ic}, \mathbf{\Delta})$.

While Theorem 2 can be used when G^{ic} is an arbitrary IC node and Π and Φ are arbitrary dichotomic nodes, if G^{ic} and the factors (Ψ, S) of Π and (Θ, U) of Φ are (h, q)-eventually periodic, we can compute a robust performance level for $(G^{ic}, \mathbf{\Delta})$ by solving a finite-dimensional SDP as given in Corollary 1. Indeed, in the case of (h, q)-eventually periodic G^{ic} and factors for Π and Φ, by adapting arguments from prior results,[21,30,31] it can be shown that (24) and (25) admit a solution P if and only if they admit an (h, q)-eventually periodic solution.

Definition 14. A partitioned memoryless operator P is (h, q)-*eventually periodic* for some $h \in 0_+$ and $q \in 1_+$ if $[\![P]\!](h + iq + k) = [\![P]\!](h + k)$ for all i and k in 0_+. A dichotomic node $G^d = [A, B, C, D, E, U, V, W]$ is (h, q)-*eventually periodic* if the operators A, B, C, D, E, U, V, and W that define it are (h, q)-eventually periodic. The partitioned IC node $G^{ic} = [A, B, C, D]^{ic}_+$ is (h, q)-*eventually periodic* if the causal node $G = [A, B, C, D]_+$ that defines it is (h, q)-eventually periodic.

Corollary 1. *Given an (h_G, q_G)-eventually periodic partitioned IC node G^{ic} defined by (4), the uncertain system $(G^{ic}, \mathbf{\Delta})$ defined by (5) has a robust $(\mathcal{E}(\Xi), \mathcal{D})$-to-$\ell_2$ gain performance level of (λ, γ) if*

(a) $(G^{ic}, \mathbf{\Delta})$ is robustly well-posed;
(b) there exists a positive-negative IQC multiplier Π with (h_Π, q_Π)-eventually periodic factors (Ψ, S) such that $\mathbf{\Delta} \in \mathrm{IQC}(\Pi)$;
(c) there exists a signal IQC multiplier Φ with (h_Φ, q_Φ)-eventually periodic factors (Θ, U) such that $\mathcal{D} \in \mathrm{SigIQC}(\Phi)$;
(d) there exist an $\epsilon > 0$ and a finite sequence of symmetric matrices $(P(k))_{k \in [0, h+q]}$ with $P(h+q) = P(h)$ such that

$$[\![\mathcal{H}_{\mathrm{perf}}]\!](k)^T \mathrm{diag}(P(k+1), [\![S]\!](k), I, [\![U]\!](k)) [\![\mathcal{H}_{\mathrm{perf}}]\!](k) - \mathrm{diag}(P(k), 0, \gamma^2 I) \preccurlyeq -\epsilon I \tag{27}$$

$$\begin{bmatrix} I_{n_G} & 0 \end{bmatrix} P(0) \begin{bmatrix} I_{n_G} & 0 \end{bmatrix}^T \preccurlyeq \lambda^2 \Xi \tag{28}$$

for all $k = 0, 1, \ldots, h + q - 1$, where $h = \max(h_G, h_\Pi, h_\Phi)$ and q is the least common multiple of q_G, q_Π, and q_Φ.

In practice, using Corollary 1 to compute a robust performance level for $(G^{ic}, \mathbf{\Delta})$ amounts to defining a priori the (h, q)-eventually periodic causal nodes Ψ and Θ and solving an SDP for γ^2, λ^2, and (h, q)-eventually periodic sequences of matrices $P(k)$, $[\![S]\!](k)$, and $[\![U]\!](k)$ such that (27) and (28) hold. When the initial condition of G is zero, we minimize γ^2 to find the smallest upper bound on the $\mathcal{D}$-to-ℓ_2 gain performance level of $(G, \mathbf{\Delta})$. However, by incorporating the effect of the uncertain initial condition and attempting to minimize λ^2 in addition to minimizing γ^2, the SDP becomes a vector

optimization problem. Thus, the scalarized objective function to be minimized is of the form $\gamma^2 + \kappa\lambda^2$ for some $\kappa > 0$. The value of κ, which gives different weights to the components in the objective function, can be varied to trace the optimal trade-off curve.

5 | TIME-VARYING IQC MULTIPLIERS

This section presents a number of time-varying IQC multipliers and signal IQC multipliers that either extend their standard time-invariant counterparts or introduce novel IQC characterizations for uncertainty sets and signal sets with time-varying, time-domain properties.

5.1 | Time-varying, arbitrarily fast scalar uncertainties

Let $\mathbf{\Delta}$ be the set of memoryless operators Δ such that $\Delta(k) = \delta(k)I$, $|\delta(k)| \le \alpha(k)$, and $\alpha(k) > 0$ for all $k \in 0_+$. Then, $\mathbf{\Delta} \in \text{IQC}(\Pi)$ for all memoryless $\Pi = \Pi^*$ such that

$$[\![\Pi]\!](k) = \begin{bmatrix} \alpha^2(k)X(k) & Y(k) \\ Y(k)^T & -X(k) \end{bmatrix}, \quad \begin{matrix} X(k) = X(k)^T \succcurlyeq 0 \\ Y(k) = -Y(k)^T. \end{matrix}$$

5.2 | Time-varying, rate-bounded scalar uncertainties

Let $\mathbf{\Delta}$ be the set of memoryless operators Δ such that $\Delta(k) = \delta(k)I$, $|\delta(k)| \le \alpha(k)$, $|\delta(k+1) - \delta(k)| \le \beta(k)$, and $\alpha(k), \beta(k) > 0$ for all $k \in 0_+$. Robustness analysis for $(G^{ic}, \mathbf{\Delta})$ can be performed by constructing an extended nominal system G_e^{ic} and extended uncertainty set $\mathbf{\Delta}_e$ and conducting robustness analysis on $(G_e^{ic}, \mathbf{\Delta}_e)$.[41] In this approach, $\mathbf{\Delta}_e \in \text{IQC}(\Pi)$, where $\Pi = \Psi^* S \Psi$ and

$$\Psi = \left[\begin{bmatrix} A^\Psi & 0 \\ 0 & A^\Psi \end{bmatrix}, \begin{bmatrix} B^\Psi & 0 & 0 \\ 0 & B^\Psi & I \end{bmatrix}, \begin{bmatrix} C^\Psi & 0 \\ A^\Psi & 0 \\ 0 & C^\Psi \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} D^\Psi & 0 & 0 \\ B^\Psi & 0 & 0 \\ 0 & D^\Psi & 0 \\ 0 & 0 & I \end{bmatrix}\right]_+, \quad [\![S]\!](k) = \begin{bmatrix} \alpha^2(k)X(k) & 0 & Y(k) & 0 \\ 0 & \beta^2(k)\overline{X}(k) & 0 & \overline{Y}(k) \\ Y(k)^T & 0 & -X(k) & 0 \\ 0 & \overline{Y}(k)^T & 0 & -\overline{X}(k) \end{bmatrix}$$

$$\Delta_e = \begin{bmatrix} \Delta \\ (Z^*\Delta Z - \Delta)[A^\Psi, B^\Psi, A^\Psi, B^\Psi]_+ \end{bmatrix}, \qquad \begin{matrix} X(k) \succcurlyeq 0, & Y(k) = -Y(k)^T \\ \overline{X}(k) \succcurlyeq 0, & \overline{Y}(k) = -\overline{Y}(k)^T. \end{matrix}$$

These multipliers can characterize uncertainties that transition to/from time-varying and time-invariant parameters. Such characterization is especially useful for analyzing systems undergoing transition maneuvers.

5.3 | Time-varying, sector-bounded uncertainties

Let $\mathbf{\Delta}$ be the set of memoryless nonlinear operators $\Delta : \mathbb{R} \to \mathbb{R}$ defined by $(\Delta(\varphi))(k) = \Delta(k)(\varphi(k))$ such that $(\Delta(k)(x) - \alpha(k)x)(\beta(k)x - \Delta(k)(x)) \ge 0$ for all $x \in \mathbb{R}$ and $k \in 0_+$. This condition assumes $\beta(k) \ge 0 \ge \alpha(k)$. Then, $\mathbf{\Delta} \in \text{IQC}(\Pi)$, where the memoryless operator $\Pi = \Pi^*$ is defined by

$$[\![\Pi]\!](k) = \begin{bmatrix} -2\alpha(k)\beta(k) & \alpha(k) + \beta(k) \\ \alpha(k) + \beta(k) & -2 \end{bmatrix}.$$

5.4 | Disturbances with energy restricted to time-intervals

Consider the signal set $\mathcal{D} = \{d \in \ell_2 \mid \|d(k)\|_2 = 0 \text{ for all } k \in \mathbb{I} \subset 0_+\}$. Then, $\mathcal{D} \in \text{SigIQC}(\Phi)$ for any memoryless operator $\Phi = \Phi^*$ defined by

$$\Phi(k) = \begin{cases} -\rho I, & k \in \mathbb{I} \\ 0, & \text{otherwise} \end{cases}$$

where $\rho > 0$. This multiplier mirrors the multiplier that characterizes the energy content of signals in the frequency-domain.[3] This time-domain multiplier can be used for systems undergoing periodic trajectories where certain disturbances are present only during specified portions of the trajectory.

5.5 | Disturbances that increase or decrease

Consider the signal set $\mathcal{D} = \{d \in \ell_2 \mid \|d(\overline{k})\|_2 \leq \|d(\underline{k})\|_2 \text{ for } \overline{k} > \underline{k} \geq 0\}$. Then, $\mathcal{D} \in \text{SigIQC}(\Phi)$, where $\Phi = \Theta^* U \Theta$, $\Theta = \left[0, I, \begin{bmatrix} I & 0 \end{bmatrix}^*, \begin{bmatrix} 0 & I \end{bmatrix}^*\right]_+$, and

$$[\![U]\!](k) = \begin{cases} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, & k = \underline{k}+1, \ldots, \overline{k} \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

If $\|d(\overline{k})\|_2 \geq \|d(\underline{k})\|_2$ in the definition of $\mathcal{D}$, the signs multiplying the identity matrices in (29) are swapped. This multiplier has a variety of potential applications. For example, when utilizing extended Kalman filters for UAS, we oftentimes observe that the error of the state estimate drastically reduces upon receiving GPS measurements and slowly increases until the next GPS measurement.

Theorem 2 provides a way for incorporating the effect of nonzero initial conditions into IQC-based robust performance analysis. Alternatively, the effect of uncertain initial conditions can be incorporated into the analysis framework using signal IQCs in one of the two ways discussed next. Consider the partitioned IC node G^{ic} defined in (4), where the uncertain initial condition $\xi^G \in \mathcal{E}(\Xi)$. For simplicity, assume that system G is LTI. This assumption can be removed by building on the work of Farhood and Dullerud.[42] Incorporating the uncertain initial condition can be performed by defining the augmented causal node

$$\tilde{G} = \left[A^G, \begin{bmatrix} B^{G1} & B^{G2} & B^{G\xi} \end{bmatrix}, \begin{bmatrix} C^{G1} \\ C^{G2} \end{bmatrix}, \begin{bmatrix} D^{G11} & D^{G12} & 0 \\ D^{G21} & D^{G22} & 0 \end{bmatrix}\right]_+$$

where $B^{G\xi} = \text{diag}(\overline{B})_{k \in 0_+}$, $\overline{B} \in \mathbb{R}^{n_G \times n_G}$, and $\overline{B}^T \overline{B} = \Xi^{-1}$. That is, there is an additional disturbance signal d_ξ that represents the uncertain initial condition. In this approach, the signal IQC defined in Section 5.4 is used to restrict the support of d_ξ and d to 1_- and 1_+, respectively. Suppose that a robust $\mathcal{D}$-to-ℓ_2 gain performance level γ is computed for $(\tilde{G}, \mathbf{\Delta})$. This implies that $\|e_{\tilde{G}}\|_{\ell_2} \leq \gamma \sqrt{\|d\|^2_{\ell_2} + \|d_\xi\|^2_{\ell_2}}$, where $e_{\tilde{G}}$ is the performance output of $(\tilde{G}, \mathbf{\Delta})$. By restricting $\|d_\xi\|_{\ell_2} \leq 1$, the previous bound can be also interpreted as a bound on the ℓ_2-norm of the performance output e of $(G^{ic}, \mathbf{\Delta})$ for all $\xi^G \in \mathcal{E}(\Xi)$, that is, $\|e\|_{\ell_2} \leq \gamma \sqrt{\|d\|^2_{\ell_2} + 1}$. The second method for incorporating the effect of uncertain initial conditions for LTI systems consists of utilizing the signal IQC multiplier to characterize (in an average sense) the output of autonomous systems.[26]

6 | UAS ILLUSTRATIVE EXAMPLE

In this section, we apply Corollary 1 for the analysis of a UAS flight controller. Our previous work[21] considers a controlled small fixed-wing UAS tasked to perform an aggressive Split-S maneuver and applies IQC-based robustness analysis

to find a ball in which the UAS is guaranteed to reside at the end of the maneuver. Here, we leverage the results of this article to reduce the size of the aforementioned ball. First, as explained in Section 5, we use time-varying counterparts of the provided[21] IQC multipliers/factorizations. The purpose is to increase the number of decision variables in the SDP to be solved and reduce conservatism. Second, we use signal IQCs to incorporate the available information about the wind disturbances that affect the closed-loop system. These disturbances are considered in our related work to be general ℓ_2 signals.[21] Here, we employ appropriate signal IQCs to characterize the disturbance sets. Finally, the example in our previous work assumes zero position error for the UAS at the beginning of the maneuver. In this work, we use Corollary 1 to additionally investigate the effect of uncertain initial conditions on the UAS final position error.

As will be shown in this example, the workflow for conducting IQC analysis bears a number of steps. First, an uncertain system is modeled by injecting the nominal model with uncertainties (either to express possible ranges of uncertain parameters or to simplify the nominal model's overly complex characteristics with an uncertainty). Second, characteristics of the exogenous disturbances (such as cutoff frequencies or admissible time intervals) are determined. In both of these points, it is important that the uncertainties and disturbance sets can be characterized with IQCs. Third, the nominal model, the matrices defining the set of uncertain initial conditions, and the IQC multipliers characterizing the disturbances and uncertainties—with their parameterizations, decision variables, and constraints—are used to formulate the conditions given in Corollary 1, wherein (27) and (28) are solved for P, S, U, λ, and γ in order to obtain the robust $(\mathcal{E}(\Xi), \mathcal{D})$-to-$\ell_2$ gain performance level of (λ, γ).

6.1 | The UAS, controller synthesis, and uncertainties

We briefly describe the system dynamics, controller structure, and uncertainties for the motivating example. A more detailed explication of the scenario can be found in our previous work.[21] The system dynamics consist of the interconnection of a plant and a controller. The plant dynamics reflect the typical 12 UAS states derived from the rigid-body 6-DOF equations of motion linearized about a Split-S trajectory and discretized to the sampling time $\tau = 0.05$ second. Six additional states due to the UAS actuators are also incorporated in the plant dynamics. The LTV controller maps sensed outputs to commanded actuation, is synthesized to minimize the ℓ_2-induced norm of the closed-loop system,[30] and has 18 states. Hence, the nominal closed-loop system has 36 states. The system disturbances are represented by a 13-channel signal, wherein the first 3 channels represent wind disturbances ($[w_u, w_v, w_w]^T$), and the last 10 channels represent sensor noise. The performance output e is defined to be identically zero at each time instant except at the end of the Split-S maneuver—when time instant k equals 119—in which e is defined as the position error of the UAS. Such a construction ensures that $\|e\|_{\ell_2} = \|e(119)\|_2$, enabling the robust performance level to provide a bound on the UAS position at a specific time instant.

Fourteen perturbations are incorporated in the nominal closed-loop model to address uncertainties in the actuators and the aerodynamic model. Perturbations in the actuators express model uncertainties (Δ_E, Δ_A, Δ_R, and Δ_T) and saturation (Δ_{σ_E}, Δ_{σ_A}, Δ_{σ_R}, and Δ_{σ_T}) as shown in Figure 4. Each of the aerodynamic uncertainties (Δ_x, Δ_y, Δ_z, Δ_l, Δ_m, and Δ_n) is added to the aerodynamic coefficient $C_{(\cdot)}$ that appears in the pertinent aerodynamic force or moment. In this work, the fourteen UAS uncertainties are either rate-bounded, static, linear time-varying (RB-SLTV) uncertainties or dynamic, linear, time-invariant (DLTI) uncertainties. The uncertainty type and bounds for the UAS are given in Table 1. The bounds in Table 1 differ from those given in the most recent work,[32] due to the fact that the UAS in this article is a Junior Telemaster, rather than a Senior Telemaster Plus. As the purpose of this example is to illustrate and compare theoretical results, we use the same uncertain model[21] upon which this work builds. We further note that, due to an error in the aforementioned publication, its expressed bound on Δ_T should be corrected to be 0.15, not 0.2.

Until the presented theoretical results, IQC analysis based on dissipativity arguments could not incorporate signal IQCs. Now, with Theorem 2 and Corollary 1, signal IQCs can be incorporated, and they are used to characterize the wind disturbance $[u_w, v_w, w_w]^T$. In this work, we characterize u_w, v_w, and w_w to reside in the set of band-limited signals, defined as

$$\mathcal{D} = \{d_w \in \ell_2 \mid \operatorname{supp}(\hat{d}_w(e^{j\omega})) \subseteq [-\omega_b, \omega_b]\} \tag{30}$$

where $\hat{d}_w$ represents the Fourier transform of d_w, and $\omega_b \in (0, \pi]$ defines the frequency band for d_w. This set can be closely approximated by the "band-limiting" signal IQCs.[32,36] Such a characterization of the disturbance signals u_w, v_w, and w_w

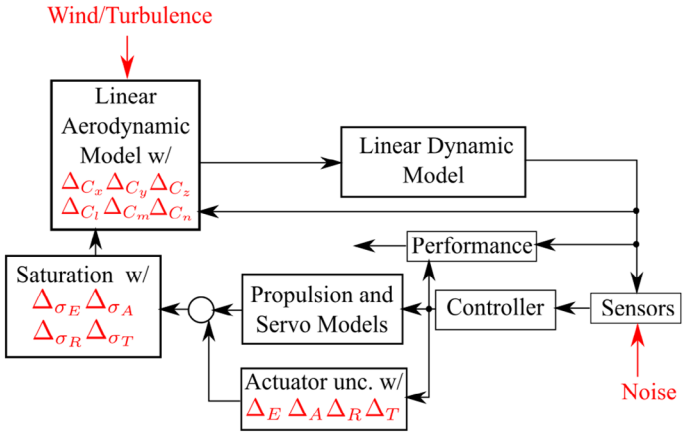


FIGURE 4 Uncertain UAS [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 UAS uncertainty types and bounds

Name	Type	Bounds
$\Delta_E, \Delta_A, \Delta_R$	DLTI	$\|\Delta_{(\cdot)}\|_\infty \le 0.05$
Δ_T	DLTI	$\|\Delta_T\|_\infty \le 0.15$
$\Delta_{\sigma_E}, \Delta_{\sigma_A}\ \Delta_{\sigma_R}, \Delta_{\sigma_T}$	RB-SLTV	$0 \le \Delta_{\sigma_{(\cdot)}}(k) \le 0.1, \quad -0.1 \le \Delta_{\sigma_{(\cdot)}}(k+1) - \Delta_{\sigma_{(\cdot)}}(k) \le 0.1$
Δ_{C_x}	RB-SLTV	$-0.0088 \le \Delta_{C_x}(k) \le 0.0072, \quad -0.01 \le \Delta_{C_x}(k+1) - \Delta_{C_x}(k) \le 0.01$
Δ_{C_y}	RB-SLTV	$-0.011 \le \Delta_{C_y}(k) \le 0.011, \quad -0.015 \le \Delta_{C_y}(k+1) - \Delta_{C_y}(k) \le 0.013$
Δ_{C_z}	RB-SLTV	$-0.038 \le \Delta_{C_z}(k) \le 0.038, \quad -0.050 \le \Delta_{C_z}(k+1) - \Delta_{C_z}(k) \le 0.058$
Δ_{C_l}	RB-SLTV	$-0.005 \le \Delta_{C_l}(k) \le 0.0054, \quad -0.005 \le \Delta_{C_l}(k+1) - \Delta_{C_l}(k) \le 0.006$
Δ_{C_m}	RB-SLTV	$-0.016 \le \Delta_{C_m}(k) \le 0.018, \quad -0.025 \le \Delta_{C_m}(k+1) - \Delta_{C_m}(k) \le 0.022$
Δ_{C_n}	RB-SLTV	$-0.0022 \le \Delta_{C_n}(k) \le 0.002, \quad -0.003 \le \Delta_{C_n}(k+1) - \Delta_{C_n}(k) \le 0.002$

Abbreviations: DLTI, dynamic, linear, time-invariant; RB-SLTV, rate-bounded, static, linear time-varying.

is congruous with the assumption that atmospheric disturbances consist of constant wind and Dryden turbulence; the bound ω_b on each signal's frequency band is determined by the cutoff frequency from the Dryden turbulence model.

Along with model uncertainties, we consider uncertainties in the initial position of the UAS via the ellipsoid $\mathcal{E}(\Xi)$, where $\Xi = (1/100)I_3$. Such an ellipsoid describes a sphere with a 10 m radius. Uncertainties in the entire UAS state may also be considered, but we use position errors to easily visualize the results. For all time instances after $k = 119$, we set the system matrices to zero, thereby providing a $(120, 1)$-eventually periodic IC node G^{ic}, whereby the uncertain system $(G^{ic}, \mathbf{\Delta})$ is amenable to IQC analysis via Corollary 1.

6.2 | Results

The following results are generated using 64-bit MATLAB R2015a, YALMIP,[43] and MOSEK[44] on a computer with 128 GB of RAM and 2 Intel Xeon E5-2683v4 2.1 GHz CPUs (16 cores per CPU), with solution times ranging from 8 minutes to 2.5 hours. When considering uncertain initial conditions, IQC analysis involves solving for the vector (λ, γ) instead of an optimal scalar γ. Hence, Figure 5 displays the Pareto optimal curve of the robust performance level for the uncertain UAS having an uncertain initial condition. We found that setting the scalarizing constant $\kappa = 160$ in the SDP objective $\gamma^2 + \kappa\lambda^2$ provides a balanced (λ, γ)-value of $(3.13, 0.694)$ in the knee of the curve. While the Pareto-optimal curve in Figure 5 is generated by redefining the SDP objective with different values for κ, the results in Tables 2 and 3 are derived by using the same scalarizing constant $\kappa = 160$.

A comparison of results obtained by applying Corollary 1 with those obtained in our previous work[21] is given in Table 2. For context, only the uppermost row of Table 2 could be obtained previously.[21] All remaining analyses required application of the theoretical results derived herein. It is immediately clear by Table 2 that the incorporation of signal IQCs has a significant effect in reducing the robust performance level. For example, the application of signal IQCs reduces the λ-values by 10% to 12% and the γ-values by 29% to 36%, as compared with IQC analysis results without signal IQCs. We also observe that utilizing time-varying multipliers instead of time-invariant multipliers further reduces conservatism,

FIGURE 5 Pareto-optimal curve for the uncertain UAS robust performance level [Colour figure can be viewed at wileyonlinelibrary.com]

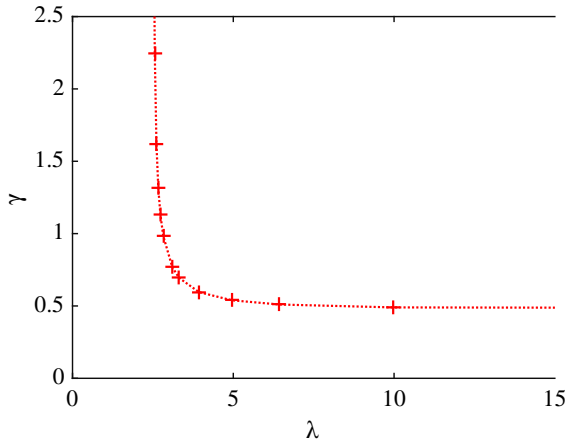


TABLE 2 Robust $(\mathcal{E}(\Xi), D)$-to-ℓ_2 performance levels for UAS example when IQC analysis is conducted with LTI IQC multipliers (Π is LTI), or with LTV IQC multipliers (Π is LTV), or with Signal IQCs and LTV IQC multipliers (Signal IQCs, Π is LTV)

		No uncertainties	All uncertainties
$x^G(0) = 0$	Π is LTI	$(-, 0.689)$	$(-, 0.753)$
	Signal IQCs, Π is LTI	$(-, 0.44)$	$(-, 0.488)$
	Signal IQCs, Π is LTV	$(-, 0.44)$	$(-, 0.479)$
$x^G(0) \in \mathcal{E}(\Xi)$	Π is LTI	$(2.91, 0.876)$	$(4.06, 1.0)$
	Signal IQCs, Π is LTI	$(2.55, 0.611)$	$(3.67, 0.715)$
	Signal IQCs, Π is LTV	$(2.55, 0.611)$	$(3.13, 0.694)$

Abbreviation: IQC, integral quadratic constraint.

TABLE 3 The number of constraints, scalar decision variables, matrix decision variables, and the solution time pertaining to analyses conducted in Table 2, as reported by MOSEK

		No uncertainties	All uncertainties
$x^G(0) = 0$	Π is LTI	80587 \ 0 \ 121 \ 0.21 hour	251743 \ 10 \ 135 \ 1.36 hour
	Signal IQCs, Π is LTI	142303 \ 9 \ 121 \ 0.43 hour	354115 \ 19 \ 135 \ 2.24 hour
	Signal IQCs, Π is LTV	142303 \ 9 \ 121 \ 0.43 hour	360115 \ 1219 \ 1335 \ 2.17 hour
$x^G(0) \in \mathcal{E}(\Xi)$	Π is LTI	80588 \ 0 \ 122 \ 0.14 hour	241744 \ 10 \ 136 \ 1.36 hour
	Signal IQCs, Π is LTI	142304 \ 9 \ 122 \ 0.28 hour	354116 \ 19 \ 136 \ 2.25 hour
	Signal IQCs, Π is LTV	142304 \ 9 \ 122 \ 0.28 hour	360116 \ 1219 \ 1336 \ 1.90 hour

Abbreviation: IQC, integral quadratic constraint.

though not as dramatically as signal IQCs. Although this decrease is small (a modest 14% reduction for λ-values and a mere 2% to 3% reduction for γ-values), it demonstrates the advantage of using time-varying multipliers for uncertainties that may still be characterized with time-invariant multipliers. The relevance of the reductions gained by both signal IQCs and time-varying multipliers is best seen by comparing the γ-value from the lower-rightmost cell (pertaining to the uncertain UAS exposed to constrained disturbances and beginning in an uncertain initial condition) to the γ-value from the upper-leftmost cell (pertaining to the nominal UAS exposed to any disturbance in ℓ_2 while beginning with zero initial conditions), which is only greater by about 1%.

In Table 3, we study the computational effort required to generate the analysis results presented in Table 2. Table 3 shows that the reduction in conservatism afforded by signal IQCs comes with a computational price-tag; comparing the values of Table 3 under "Signal IQCs, Π is LTI" with those under "Π is LTI," we see that the number of constraints and the solution time increased by 40% to 76% and 64% to 105%, respectively. Curiously, we do not see an increase in the solution time when using time-varying multipliers. Though there is a slight increase in equality constraints (2%) and a significant increase in scalar and matrix variables (about 630% and 890%, respectively), the solution times actually *decreased*. Similarly, we see a decrease in solution times when incorporating uncertain initial conditions for IQC analysis with time-invariant multipliers. While we only report these observations here, it is clearly important to look closely into the

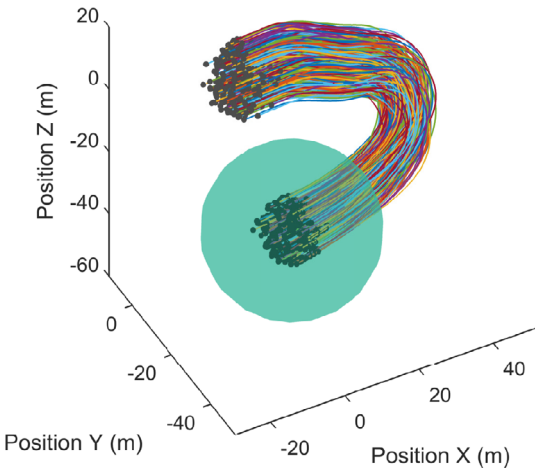


FIGURE 6 Comparison of simulations with worst-case bound obtained from Corollary 1 [Colour figure can be viewed at wileyonlinelibrary.com]

underlying reasons, which we hope to accomplish in future work. For now, this example demonstrates that, if the additional decision variables do not cause memory problems, the incorporation of time-varying multipliers may not increase solution times, and in some cases it may even reduce solution times.

To further demonstrate the utility of time-varying multipliers, we conducted IQC analysis with signal IQCs and time-varying multipliers wherein the RB-SLTV uncertainties have properties that cannot be characterized with time-invariant multipliers. In these analyses, we linearly ramp the lower and upper bounds and rate-bounds of the RB-SLTV uncertainties from 0 to the values given in Table 1. Though such a characterization of uncertainties does not have a strong physical basis, we consider this scenario to demonstrate the utility of time-varying multipliers while still studying the same Split-S example. The robust performance levels (λ, γ) for this system with zero initial conditions and uncertain initial conditions are, respectively $(-, 0.456)$ and $(2.97, 0.658)$. These values fall approximately in the middle of the ranges prescribed by analysis results pertaining to the system without uncertainties and the system with all uncertainties. For example, the γ-values from the third row of Table 2 enclose the γ-value 0.456 obtained by analyzing the system with "ramped" RB-SLTV uncertainties. The same can be observed by comparing the sixth row with the aforementioned (λ, γ)-value of $(2.97, 0.658)$.

A number of simulations are also carried out to further clarify the usefulness of Corollary 1. In these simulations, the DLTI and SLTV perturbations are pseudo-randomly generated using the MATLAB commands `drss` and `rand`, respectively, along with appropriate scalings. The truncated disturbance signal d in these simulations consists of a 2 m/s steady wind, where the wind direction varies between simulations, along with light turbulence generated from the low-altitude Dryden turbulence model, and zero-mean, unit-variance, Gaussian sensor noise across all measurement noise channels. Given the aforementioned disturbances, the average ℓ_2-norm of d is 41.3. Using the robust performance level $(6.4357, 0.510)$ from Figure 5, the lowest physical bound on the final position error is 22.02 m. This is significant since the initial position may be in error by as much as 10 m, but the final position error bound is less than the bound of 31.1 m obtained from our previous work.[21] A sample of the most deviant trajectories is displayed in Figure 6, along with the guaranteed ball where the UAS will finally reside. Though it appears that the bound from IQC analysis is conservative, it is quite possible that there exist admissible disturbances and initial conditions with a corresponding performance output at the boundary. Recent work[45] on calculating worst-case disturbances from ℓ_2 may be adaptable to incorporate signal IQCs, in which case such worst-case conditions could be found.

These case studies demonstrate the utility of Corollary 1 in (1) incorporating uncertain initial conditions, (2) reducing conservatism via signal IQCs, and (3) reducing conservatism by more tightly characterizing certain uncertainties with time-varying multipliers. Theoretically speaking, when applying Corollary 1 to analyze systems, the inclusion of signal IQCs and time-varying multipliers can only reduce the conservatism of the analysis results. However, such inclusions increase the computational complexity of the resulting SDP. As with many computational problems, the competing goals of reduced conservatism and computational tractability will need to be met in balance.

As is common with IQC analysis, the results given here rely in part on the parameterizations of the utilized IQC multipliers. While the operators S, U can be defined with a structure of constrained decision variables, in this work we set Ψ to be a known operator whose definition is subject to a variety of parameters, such as its realization and the pole locations of its basis functions. At times we have observed the robustness results to be sensitive to the pole location, though we have not observed a consistently "best" location that provides the least conservative results. Veenman et al[37] provide a

numerical discussion on the effect of pole locations on analysis results, and in practice it appears that the sensitivity and optimality of this parameter is dependent on the system to be analyzed.

7 | CONCLUSION

This article presents a dissipativity approach to IQC-based robustness analysis of uncertain, discrete-time, time-varying systems. Signal IQC multipliers are incorporated into robust performance analysis to allow for characterizing the sets of disturbances acting on the uncertain system. The analysis also accounts for the effect of uncertain initial conditions in the nominal system. Dichotomic nodes are introduced and used to define general, time-varying, static and dynamic, IQC and signal IQC multipliers. Such multipliers can be used to describe uncertainty and disturbance sets with time-domain characteristics. Finally, using an uncertain UAS in an uncertain initial state as an example case study, the utility of these results is demonstrated by computing worst-case bounds on the final position of the UAS after executing a Split-S maneuver.

ACKNOWLEDGEMENTS

This material is based upon work supported by the National Science Foundation (NSF) under Grant No. CMMI-1351640, the Office of Naval Research under Award No. N00014-18-1-2627, and the Center for Unmanned Aircraft Systems (C-UAS), an NSF Sponsored Industry/University Cooperative Research Center (I/UCRC) under NSF Grant No. CNS-1650465 along with significant contributions from C-UAS industry members. The MIT Lincoln Laboratory portion of this work is supported by the Under Secretary of Defense for Research and Engineering under Air Force Contract No. FA8702-15-D-0001. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the Under Secretary of Defense for Research and Engineering.

ORCID

Mazen Farhood https://orcid.org/0000-0001-9855-7250

REFERENCES

1. Yakubovich VA. Frequency conditions for the absolute stability of control systems with several nonlinear or linear nonstationary blocks. *Avtomatika i Telemekhanika*. 1967;6:5-30.
2. Yakubovich VA. S-procedure in nonlinear control theory. *Vestnik Leningrad Univ Math*. 1977;4:73-93.
3. Megretski A. Power distribution approach in robust control. *IFAC Proc*. 1993;26(2):27-30.
4. Megretski A, Rantzer A. System analysis via integral quadratic constraints. *IEEE Trans Autom Control*. 1997;42(6):819-830.
5. Megretski A, Rantzer A. *System Analysis Via Integral Quadratic Constraints: Part 1 TFRT-7531*. Sweden: Department of Automatic Control, Lund Institute of Technology; 1995.
6. Rantzer A, Megretski A. *System Analysis Via Integral Quadratic Constraints: Part 2 TFRT-7559*. Sweden: Department of Automatic Control, Lund Institute of Technology; 1997.
7. Kao C-Y, Megretski A, Jonsson UT. A cutting plane algorithm for robustness analysis of periodically time-varying systems. *IEEE Trans Autom Control*. 2001;46(4):579-592.
8. Jonsson UT, Chung-Yao K, Megretski A. Robustness of periodic trajectories. *IEEE Trans Autom Control*. 2002;47(11):1842-1856.
9. Jonsson UT, Kao C-Y, Megretski A. Analysis of periodically forced uncertain feedback systems. *IEEE Trans Circ Syst I Fund Theory Appl*. 2003;50(2):244-258.
10. Hagiwara T, Ohara Y. Noncausal linear periodically time-varying scaling for robust stability analysis of discrete-time systems: frequency-dependent scaling induced by static separators. *Automatica*. 2010;46(1):167-173.
11. Rantzer A. On the Kalman–Yakubovich–Popov lemma. *Syst Control Lett*. 1996;28(1):7-10.
12. Willems JC. Dissipative dynamical systems part I: general theory. *Arch Ration Mech Anal*. 1972;45(5):321-351.
13. Willems JC. Dissipative dynamical systems part II: linear systems with quadratic supply rates. *Arch Ration Mech Anal*. 1972;45(5):352-393.
14. Cantoni M, Jonsson UT, Khong SZ. Robust stability analysis for feedback interconnections of time-varying linear systems. *SIAM J Control Optim*. 2013;51(1):353-379.
15. Liu L, Lu Y. Stability analysis for time-varying systems via quadratic constraints. *Syst Control Lett*. 2011;60(10):832-839.
16. Willems JC, Takaba K. Dissipativity and stability of interconnections. *Int J Robust Nonlinear Control*. 2007;17(5-6):563-586.
17. Veenman J, Scherer CW. Stability analysis with integral quadratic constraints: a dissipativity based proof. Paper presented at: 52nd IEEE Conference on Decision and Control, Florence, Italy; 2013: 3770-3775.
18. Seiler P. Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE Trans Autom Control*. 2015;60(6):1704-1709.
19. Scherer CW, Veenman J. Stability analysis by dynamic dissipation inequalities: on merging frequency-domain techniques with time-domain conditions. *Syst Control Lett*. 2018;121:7-15.

20. Hu B, Lacerda MJ, Seiler P. Robustness analysis of uncertain discrete-time systems with dissipation inequalities and integral quadratic constraints. *Int J Robust Nonlinear Control*. 2017;27(11):1940-1962.
21. Fry JM, Farhood M, Seiler P. IQC-based robustness analysis of discrete-time linear time-varying systems. *Int J Robust Nonlinear Control*. 2017;27(16):3135-3157.
22. Pfifer H, Seiler P. Less conservative robustness analysis of linear parameter varying systems using integral quadratic constraints. *Int J Robust Nonlinear Control*. 2016;26(16):3580-3594.
23. Abou Jaoude D, Muniraj D, Farhood M. Robustness analysis of uncertain time-varying interconnected systems using integral quadratic constraints. Paper presented at: American Control Conference (ACC), Philadelphia, Pennsylvania; 2019:3087-3092.
24. Fry JM, Farhood M. Robustness analysis of eventually periodic systems using integral quadratic constraints with periodic multipliers. Paper presented at: IEEE 56th Annual Conference on Decision and Control (CDC), Melbourne, Australia; 2017:2967-2972.
25. Halanay A, Ionescu V. *Time-Varying Discrete Linear Systems*. Basel: Birkhauser; 1994.
26. Jonsson U, Megretski A. IQC characterizations of signal classes. Paper presented at: European Control Conference (ECC), Karlsruhe, Germany; 1999:1481-1486.
27. Seiler P, Moore RM, Meissen C, Arcak M, Packard A. Finite horizon robustness analysis of LTV systems using integral quadratic constraints. *Automatica*. 2019;100:135-143.
28. Yakubovich VA. Necessity in quadratic criterion for absolute stability. *Int J Robust Nonlinear Control*. 2000;10(11-12):889-907.
29. Khong SZ, Kao CY. Converse theorems for integral quadratic constraints; 2019. arXiv preprint arXiv:1905.10600.
30. Dullerud GE, Lall S. A new approach for analysis and synthesis of time-varying systems. *IEEE Trans Autom Control*. 1999;44(8):1486-1497.
31. Farhood M, Dullerud GE. LMI tools for eventually periodic systems. *Syst Control Lett*. 2002;47(5):417-432.
32. Fry JM, Farhood M. A comprehensive analytical tool for control validation of fixed-wing unmanned aircraft. *IEEE Trans Control Syst Technol*. 2020;28(5):1785-1801.
33. Palframan MC, Fry JM, Farhood M. Robustness analysis of flight controllers for fixed-wing unmanned aircraft systems using integral quadratic constraints. *IEEE Trans Control Syst Technol*. 2019;27(1):86-102.
34. Fry JM, Farhood M. IQC-based robustness analysis with time-varying multipliers. Paper presented at: 2019 American Control Conference (ACC), Philadelphia, Pennsylvania; 2019:1786-1791.
35. Abou Jaoude D, Farhood M. Guaranteed output bounds using performance integral quadratic constraints. Paper presented at: American Control Conference, Denver, Colorado; 2020:1557-1562.
36. Jonsson U. *Lecture Notes on Integral Quadratic Constraints*. Stockholm, Sweden: Royal Institute of Technology; 2001.
37. Veenman J, Scherer CW, Koroglu H. Robust stability and performance analysis based on integral quadratic constraints. *Eur J Control*. 2016;31:1-32.
38. Fetzer M, Scherer CW. Full-block multipliers for repeated, slope-restricted scalar nonlinearities. *Int J Robust Nonlinear Control*. 2017;27(17):3376-3411.
39. Hutson V, Pym JS, Cloud Michael J. *Applications of Functional Analysis and Operator Theory*. 2nd ed. Amsterdam, Netherlands: Elsevier; 2005.
40. Engwerda J. *LQ Dynamic Optimization and Differential Games*. Hoboken, NJ: John Wiley and Sons; 2005.
41. Koroglu H, Scherer CW. Robust performance analysis for structured linear time-varying perturbations with bounded rates-of-variation. *IEEE Trans Autom Control*. 2007;52(2):197-211.
42. Farhood M, Dullerud GE. Control of systems with uncertain initial conditions. *IEEE Trans Autom Control*. 2008;53(11):2646-2651.
43. Lofberg J. YALMIP: a toolbox for modeling and optimization in MATLAB. Paper presented at: IEEE International Symposium on Computer Aided Control Systems Design, New Orleans, Louisiana; 2004:284-289.
44. MOSEK ApS The MOSEK optimization toolbox for MATLAB manual. Version. 8.0; 2017. https://docs.mosek.com/8.0/intro/index.html.
45. Iannelli A, Seiler P, Marcos A. Worst-case disturbances for time-varying systems with application to flexible aircraft. *J Guid Control Dyn*. 2019;42(6):1261-1271.

How to cite this article: Fry JM, Abou Jaoude D, Farhood M. Robustness analysis of uncertain time-varying systems using integral quadratic constraints with time-varying multipliers. *Int J Robust Nonlinear Control*. 2021;31:733–758. https://doi.org/10.1002/rnc.5299