Classical and Quantum Gravity

PAPER

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To cite this article: Wafic A Sabra and Owen Vaughan 2016 *Class. Quantum Grav.* **33** 015010

IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. **33** (2016) 015010 (20pp) doi:10.1088/0264-9381/33/1/015010

10D to 4D Euclidean supergravity over a Calabi-Yau three-fold

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Received 1 July 2015, revised 26 October 2015
Accepted for publication 9 November 2015
Published 11 December 2015

Abstract
We dimensionally reduce the bosonic sector of 10D Euclidean type IIA supergravity over a Calabi-Yau three-fold. The resulting theory describes the bosonic sector of 4D, $\mathcal{N} = 2$ Euclidean supergravity coupled to vector- and hyper-multiplets. We show that the scalar target manifold of the vector-multiplets is projective special para-Kähler, and is therefore of split signature, whereas the target manifold of the hyper-multiplets is (positive-definite) quaternionic Kähler.

Keywords: string compactifications, supergravity, special geometry, euclidean spacetime signature

(Some figures may appear in colour only in the online journal)

1. Introduction

Supersymmetric Euclidean theories coupled to vector-multiplets have recently been a subject of interest [1–3]. It has been known for some time that the complex scalar fields of vector-multiplets in 4D, $\mathcal{N} = 2$ Lorentzian supersymmetric theories exhibit so-called special Kähler geometry [4]. This geometry has provided a useful tool in the understanding of field theory non perturbative structure, supergravity, string compactifications (see, e.g., [5]), as well as in the study and analysis of black-hole physics [6].

Both rigid and local Euclidean versions of special geometry were constructed and analyzed in terms of para-complex geometry in [1–3]. Roughly speaking, the Euclidean versions of special geometry can be obtained from the standard version appearing in Lorentzian

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theories by replacing i with the para-complex unit e, which satisfies the properties $e^2 = 1$ and $\bar{e} = -e$. In the supergravity literature such a replacement first appeared in the study of D-instantons in type IIB supergravity [7] [4]. Para-complex manifolds are necessarily of even dimension and split signature. For further details on para-complex geometry and Euclidean supersymmetric theories we refer the reader to [1]. It is important to emphasize that it is only the target geometry of the scalar fields that becomes para-complex in Euclidean theories. This is not true of the superalgebra representation itself or the geometry of superspace, which are both complex in the case of Euclidean spacetime signature.

Throughout this paper we will use the convention that the degree of supersymmetry $\mathcal{N}$ of a Euclidean superalgebra is matched to the number of real supercharges in the Lorentzian case. For example, 4D, $\mathcal{N} = 2$ Euclidean supersymmetry has eight real supercharges, despite the fact that the smallest supersymmetry representation in 4D Euclidean space has eight real degrees of freedom [8], and thus there is no 4D, $\mathcal{N}=1$ Euclidean theory in our conventions.

The rigid 4D, $\mathcal{N} = 2$ Euclidean vector-multiplet action was constructed in [1] by reducing 5D, $\mathcal{N} = 2$ vector-multiplets over a time-like circle. The Euclidean action and supersymmetry transformation rules were expressed in terms of para-holomorphic coordinates. As in in the local case, the bosonic sector of 4D, $\mathcal{N} = 2$ Euclidean supergravity coupled to vector-multiplets was constructed in [3] by reducing 5D, $\mathcal{N} = 2$ supergravity coupled to vector-multiplets [9] over a time-like circle, and the scalar target manifold was shown to be projective special para-Kähler [3]. The Killing spinor equations as well as the classification of supersymmetric gravitational instanton solutions of these theories were later analyzed in [10, 11].

Theories of $\mathcal{N} = 2$ hyper-multiplets are also of interest. The scalar target manifold in 3, 4, and 5 dimensions is hyper-Kähler in the rigid case [12] and quaternionic Kähler in the local case [13]. Since the hyper-multiplet target manifold is invariant under dimensional reduction, the target manifold of 4D local Euclidean hyper-multiplets is also quaternionic Kähler. On the other hand, 4D vector-multiplets (both rigid and local) can be mapped to 3D hyper-multiplets by dimensional reduction followed by Hodge dualization, which is known as the supergravity c-map. Reducing over a space-like circle results in a theory of 3D hyper-multiplets with Lorentzian spacetime signature and quaternionic Kähler target manifold [14]. However, reducing over a time-like circle results in a theory of 3D hyper-multiplets with Euclidean spacetime signature and para-quaternionic Kähler target manifold [15, 16]. This suggests that, at least in certain circumstances, the target geometry of Euclidean hyper-multiplets is not completely fixed by the signature of spacetime. Thus, we must be careful to identify the correct target geometry for the theory in question.

The goal of this paper is to establish the higher-dimensional origins of 4D, $\mathcal{N} = 2$ Euclidean supergravity, and, in particular, how the scalar target geometry emerges through the process of dimensional reduction. Our starting point is standard 11D supergravity with Lorentzian spacetime signature. We first reduce this theory over a time-like circle in order to obtain 10D Euclidean type IIA supergravity [17], and then reduce this 10D theory further over a Calabi-Yau three-fold. While the dimensional reduction of 11D supergravity over tori with one time-like circle has been considered in [18, 19], which, in our context, would correspond to taking the Calabi-Yau manifold to be T^6, the reduction over a time-like circle followed by an arbitrary Calabi-Yau three-fold is currently missing from the literature. The resulting effective theory describes the bosonic sector of 4D, $\mathcal{N} = 2$ Euclidean supergravity coupled to $h_{1,1}$ vector-multiplets and $(h_{2,1} + 1)$ hyper-multiplets. Our construction is summarized in figure 1.

[4] In this reference the para-complex unit e is referred to as the hyperbolic complex unit.

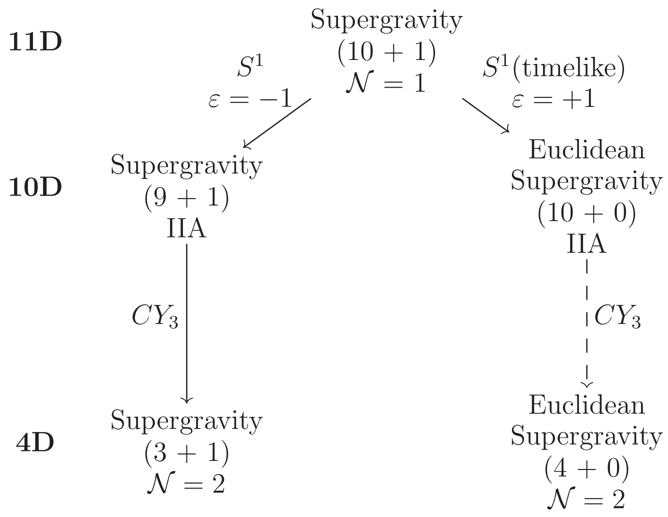

Figure 1. Dimensional reduction of supergravity from 11D to 4D over a space-like or time-like circle and Calabi-Yau three-fold. In this paper we complete the second step on the right-hand side of this diagram.

We will follow closely the original work of [20] in which the reduction of 11D supergravity over a space-like circle followed by a Calabi-Yau three-fold was first constructed. Indeed, we find that the resulting action of the 4D Euclidean theory differs from the Lorentzian case only by certain sign flips. We will keep track of these signs using the parameter ε, which is determined by the reduction of 11D supergravity over either a space-like or time-like S^1 according to the rule

$$\varepsilon = \begin{cases} -1, & S^1 \text{ spacelike} \\ +1, & S^1 \text{ timelike.} \end{cases} \tag{1}$$

Thus, after reducing further over a Calabi-Yau three-fold we end up with 4D, $\mathcal{N} = 2$ supergravity if $\varepsilon = -1$ and 4D, $\mathcal{N} = 2$ Euclidean supergravity if $\varepsilon = +1$. We will show that the target space geometry of the 4D scalar fields is given by the product $M_{\text{vector}} \times M_{\text{hyper}}$, where M_{vector} is a $2h_{1,1}$-dimensional projective special ε-Kähler manifold and M_{hyper} is a $(4h_{2,1} + 4)$-dimensional quaternionic Kähler manifold.

The (pseudo-)Riemannian structure of spacetime in our construction is given as follows. The spacetime manifolds in various dimensions are related topologically by

$$M_{11} = S^1 \times M_{10}, \qquad M_{10} = \chi \times M_4, \tag{2}$$

where χ is a Calabi-Yau three-fold. The spacetime metrics are related by

$$g_{11} = -\varepsilon e^{\frac{4}{3}\phi'}\left(d\tilde{x}^0 + V\right)^2 + e^{-\frac{1}{6}\phi'} g_{10}, \qquad g_{10} = g_\chi + \mathcal{V}^{-1} g_4, \tag{3}$$

where ϕ' and V are the 11D Kaluza-Klein scalar and vector, respectively, $\tilde{x}^0$ is the coordinate of the S^1 dimension, $\mathcal{V}$ is the volume of the Calabi-Yau three-fold, and g_χ is the Ricci-flat Calabi-Yau metric. The signatures of the various spacetime metrics are

$$\text{sig}(g_{11}) = (-\varepsilon,\, \varepsilon,\, +, \ldots,\, +), \qquad \text{sig}(g_{10}) = (\varepsilon,\, +, \ldots,\, +),$$
$$\text{sig}(g_4) = (\varepsilon,\, +, +, +).$$

The internal compact (pseudo-)Riemannian manifold $S^1 \times \chi$ has the metric $g_{S^1\times\chi} = -\varepsilon e^{\frac{4}{3}\phi'}(d\tilde{x}^0)^2 + e^{-\frac{1}{6}\phi'}g_\chi$, which depends on the base point in M_4. It has the signature $(-\varepsilon, +, \ldots, +)$.

2. 10D Euclidean supergravity

Our starting point is the bosonic part of the 11D supergravity action [21]

$$S^{11} = \int_{M_{11}} \left[\frac{1}{2}{*R_{11}} - \frac{1}{2}\tilde{F}_4 \wedge *\tilde{F}_4 - \frac{\sqrt{2}}{6}\tilde{F}_4 \wedge \tilde{F}_4 \wedge \tilde{A}_3 \right],$$

which has the spacetime signature $(-\varepsilon, \varepsilon, +, \ldots, +)$ in coordinates $(\tilde{x}^0, \ldots, \tilde{x}^{10})$. We will reduce this theory over the $\tilde{x}^0$ dimension, which we assume is either a space-like or time-like circle according to the rule (1). The 11D spacetime manifold and metric decompose into their 10D counterparts according to (2) and (3). The three-form is decomposed according to

$$\tilde{A}_3 = A'_3 + d\tilde{x}^0 \wedge B_2, \qquad \tilde{F}_4 = F'_4 - d\tilde{x}^0 \wedge H_3,$$

where A'_3 and B_2 are degenerate and invariant along the $\tilde{x}^0$ direction and $F'_4 = dA'_3$, $H_3 = dB_2$. This resulting 10D action is given by

$$\begin{aligned} S^{10} = 2\pi\rho \int_{M_{10}} \Bigg[&\frac{1}{2}{*R_{10}} - \frac{1}{4}d\phi' \wedge *d\phi' + \frac{1}{4}\varepsilon\, e^{\frac{3}{2}\phi'}dV \wedge *dV + \frac{1}{2}\varepsilon\, e^{-\phi'}H_3 \wedge *H_3 \\ &- \frac{1}{2}e^{\frac{1}{2}\phi'}\left(F'_4 + V \wedge H_3\right) \wedge *\left(F'_4 + V \wedge H_3\right) - \frac{1}{\sqrt{2}}\left(F'_4 \wedge F'_4 \wedge B_2\right)\Bigg], \end{aligned}$$

which has the spacetime signature $(\varepsilon, +, \ldots, +)$. Here ρ is the radius of the $\tilde{x}^0$ dimension, which we now set to $\rho = \frac{1}{2\pi}$. It is convenient to make the field redefinitions

$$A'_3 = A_3 + V \wedge B_2, \qquad F'_4 = F_4 + dV \wedge B_2 - V \wedge H_3, \qquad \phi' = \frac{3}{2}\log\phi,$$

in which case the action becomes

$$\begin{aligned} S^{10} = \int_{M_{10}} \Bigg[&\frac{1}{2}{*R_{10}} - \frac{9}{16}d\log\phi \wedge *d\log\phi + \frac{1}{4}\varepsilon\, \phi^{\frac{9}{4}}dV \wedge *dV \\ &+ \frac{1}{2}\varepsilon\, \phi^{-\frac{3}{2}}H_3 \wedge *H_3 - \frac{1}{2}\phi^{\frac{3}{4}}\left(F_4 + dV \wedge B_2\right) \wedge *\left(F_4 + dV \wedge B_2\right) \\ &- \frac{\sqrt{2}}{2}\left(F_4 + dV \wedge B_2\right) \wedge F_4 \wedge B_2 - \frac{\sqrt{2}}{6}dV \wedge B_2 \wedge dV \wedge B_2 \wedge B_2 \Bigg]. \end{aligned} \tag{4}$$

Note that the topological terms in the last line pick up a factor of ε when written in components, (see equation (32) in appendix A).

For $\varepsilon = -1$ the action (4) agrees with the bosonic sector of 10D type IIA supergravity [20]. (Note that the final term is not present in [20]. However, it is present in the earlier work [22].) For $\varepsilon = +1$ it agrees with the bosonic sector of 10D type IIA Euclidean supergravity [17]. The complete Euclidean supergravity action, including fermionic terms and supersymmetry transformation rules, can be found in [23]. We can understand the action (4) as the field theory limit of type IIA string theory with Lorentzian or Euclidean spacetime signature [17].

We would like to dimensionally reduce this theory over a compact 6D internal manifold while preserving supersymmetry. Regardless of the choice of ε, this can be achieved if and only if there exists a spinor η on the internal manifold such that the corresponding infinitesimal supersymmetry transformation of the gravitino vanishes $0 = \delta\Psi = \not{D}\eta$, i.e., the internal manifold admits a covariantly constant spinor. This motivates us to consider reduction over Calabi-Yau manifolds even in the case of Euclidean spacetime signatures.

3. Calabi-Yau reduction

In this section we present some background material, which can be found in [20, 24, 25].

We assume that the 10D spacetime manifold decomposes into $M_{10} = \chi \times M_4$, where χ is a Calabi-Yau three-fold and M_4 is a 4D (pseudo-)Riemannian manifold with the metric signature $(\varepsilon, +,+,+)$. On M_{10} we may the introduce local coordinates

$$w^{\hat{\mu}}, \qquad \hat{\mu} = 1, \ldots, 10,$$

which decompose into coordinates

$$\begin{aligned} &x^\mu, \qquad \mu = 1, \ldots, 4, \\ &y^a, \qquad a = 1, \ldots, 6, \end{aligned}$$

on M_4 and χ, respectively. It is useful to introduce complex coordinates on χ as follows:

$$\xi_1 = \frac{1}{\sqrt{2}}\left(y_1 + iy_2\right), \qquad \xi_2 = \frac{1}{\sqrt{2}}\left(y_3 + iy_4\right), \qquad \xi_3 = \frac{1}{\sqrt{2}}\left(y_5 + iy_6\right).$$

In these conventions the volume form satisfies $d^6y = id^3\xi d^3\bar{\xi} =: id^6\xi$ and the Hodge duals of (3, 0)-forms and (2, 1)-forms satisfy

$$^*\rho_{(3,0)} = -i\rho_{(3,0)}, \qquad ^*\sigma_{(2,1)} = i\sigma_{(2,1)}. \tag{5}$$

The inner product of two (p,q)-forms is defined by

$$\left(\alpha_{(p,q)}, \beta_{(p,q)}\right) = \int_\chi \alpha_{(p,q)} \wedge {}^*\beta_{(p,q)}.$$

3.1. Harmonic forms and integrals over a CY_3

On a Calabi-Yau three-fold there are nontrivial harmonic forms in the (1, 1), (2, 1), (3, 0), and (3, 3) cohomology sectors (and their Hodge duals). We use the following basis:

$$\begin{aligned} &(1,1) \qquad V^A = V^A_{i\bar{j}}\, d\xi^i \wedge d\bar{\xi}^{\bar{j}},\ A = 1, \ldots, h_{1,1} \\ &(2,1) \qquad \Phi_\alpha = \frac{1}{2}\Phi_{\alpha ij\bar{k}}\, d\xi^i \wedge d\xi^j \wedge d\bar{\xi}^{\bar{k}},\ \alpha = 1, \ldots, h_{2,1} \\ &(3,0) \qquad \Omega = \frac{1}{3!}\Omega_{ijk}\, d\xi^i \wedge d\xi^j \wedge d\xi^k, \\ &(3,3) \qquad v = \frac{1}{3!} J \wedge J \wedge J, \end{aligned}$$

where V^A are real. The Kähler form is given by $J = ig_{i\bar{j}}\, d\xi^i \wedge d\bar{\xi}^{\bar{j}} = M^A V^A$, where $M^A(x)$ are real scalar fields, and the volume by

$$\mathcal{V} = \int_\chi v = \int_\chi \sqrt{g}\ d^6y = \int_\chi i\sqrt{g}\ d^6\xi.$$

Let us first consider certain integrals relevant for the H^2 cohomology sector. Following [20] we define

$$\mathcal{K} = \int_X J \wedge J \wedge J, \qquad \mathcal{K}_{AB} = \int_X V^A \wedge V^B \wedge J,$$
$$\mathcal{K}_A = \int_X V^A \wedge J \wedge J, \qquad \mathcal{K}_{ABC} = \int_X V^A \wedge V^B \wedge V^C,$$

which satisfy $\mathcal{K} = \mathcal{K}_{ABC} M^A M^B M^C$ and $\mathcal{V} = \frac{1}{6}\mathcal{K}$. A useful formula for any real $(1, 1)$-form is given by [25]

$$*V^B = -J \wedge V^B + \frac{3}{2\mathcal{K}} J \wedge J \left(\int_X V^B \wedge J \wedge J \right),$$

from which it follows that

$$G_{AB}(M) := \frac{1}{2\mathcal{V}} \int_X V^A \wedge *V^B = -3 \left(\frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{3}{2} \frac{\mathcal{K}_A \mathcal{K}_B}{\mathcal{K}^2} \right).$$

In components we have the formulae

$$2\mathcal{V}\, G_{AB} = \int_X d^6 y \sqrt{g} \left[V^A_{i\bar{j}} V^{Bi\bar{j}} \right], \tag{6}$$

$$\mathcal{K}_{AB} = \int_X d^6 y \sqrt{g} \left[V^A_{i\bar{j}} V^{Bi\bar{j}} - V^A_{i\bar{j}} V^B_{k\bar{l}} g^{i\bar{j}} g^{k\bar{l}} \right]. \tag{7}$$

Let us now turn to the H^3 cohomology sector. We will follow the conventions for special Kähler geometry given in appendix A. Since the H^3 sector contains contributions from $h_{2,1}$ harmonic $(2, 1)$-forms and one harmonic $(3, 0)$-form indices run from $I, J = 0, \ldots, h_{2,1}$. Consider a real cohomology basis α_I, β^I of H^3 that satisfies

$$\int_X \alpha_I \wedge \beta^J = \delta_I^J. \tag{8}$$

In the above basis the holomorphic three-form Ω can be written as

$$\Omega = X^I \alpha_I - F_I \beta^I, \tag{9}$$

where $F_I = F_I(X)$ and X^I depends only on the 4D spacetime coordinates x^μ. Integrating gives

$$\left(\Omega, \bar{\Omega} \right) = \int_X i\Omega \wedge \bar{\Omega} = -i \left(X^I \bar{F}_I - F_I \bar{X}^I \right) = ||\Omega||^2 \mathcal{V},$$

where $||\Omega||^2 = \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$. The function $||\Omega||^2$ is in fact completely independent of the coordinates y^a and depends only on the spacetime coordinates x^μ [26, theorem 4.3.2]. The derivative $\Omega_I = \frac{\partial}{\partial X^I}\Omega$ takes the form [25]

$$\Omega_I = \Phi_I + K_I \Omega,$$

where we have introduced an additional harmonic $(2, 1)$-form Φ_0, which is a linear combination of Φ_α, defined by the above equation. This implies that

$$\int_X \Omega \wedge \Omega_I = 0,$$

and therefore $F_I = \frac{\partial}{\partial X^I} F$, where $F = F(X)$ is a homogeneous function of degree two. Using the homogeneity of F we have

$$\left(\Omega, \bar{\Omega}\right) = -N_{IJ}\bar{X}^I X^J,$$

and therefore $N_{IJ}\bar{X}^I X^J$ is strictly negative due to the positivity of the inner product $(\cdot,\cdot)$. The formula for K_I is given by

$$K_I = \frac{\left(N\bar{X}\right)_I}{\bar{X}NX} = -\frac{\partial}{\partial X^I}K,$$

where $K := -\log(-\bar{X}NX) = -\log(\Omega, \bar{\Omega})$, and we are using the notation $\bar{X}NX = N_{IJ}\bar{X}^I X^J$ and $(NX)_I = N_{IJ}X^J$. From the fact that $\Omega_I = \alpha_I - F_{IJ}\beta^J$ we obtain the following expressions for α_I, β^I and Φ_I:

$$\alpha_I = \Omega_I + iF_{IJ}N^{JK}\left(\Omega_K - \bar{\Omega}_K\right) \tag{10}$$

$$\beta^I = iN^{IJ}\left(\Omega_J - \bar{\Omega}_J\right) \tag{11}$$

$$\Phi_I = \left(\delta_I^J - K_I X^J\right)\left(\alpha_J - F_{JK}\beta^K\right), \tag{12}$$

which allows us to calculate

$$\begin{aligned}\left(\Phi_I, \bar{\Phi}_{\bar{J}}\right) &= \int_\chi -i\Phi_I \wedge \bar{\Phi}_{\bar{J}} = \int_\chi d^6y\sqrt{g}\left[\frac{1}{2}\Phi_{Iij\bar{k}}\bar{\Phi}_{\bar{J}}{}^{ij\bar{k}}\right]\\ &= \left(N_{IJ} - \frac{\left(N\bar{X}\right)_I (NX)_J}{\bar{X}NX}\right) = -\mathcal{M}_{I\bar{J}}.\end{aligned}$$

All other integrals vanish $(\Phi_I, \Omega) = (\Phi_I, \bar{\Omega}) = (\Phi_I, \Phi_J) = (\Omega, \Omega) = 0$.

We will also consider the $(0, 2)$-forms b_α defined by

$$b_\alpha = \frac{1}{2}b_{\alpha\bar{i}\bar{j}}d\bar{\xi}^i \wedge d\bar{\xi}^j := -\frac{i}{2}\frac{1}{||\Omega||^2}\bar{\Omega}_{\bar{i}}{}^{kl}\Phi_{\alpha kl\bar{j}}d\bar{\xi}^i \wedge d\bar{\xi}^j.$$

Integrating gives

$$\left(b_\alpha, \bar{b}_{\bar{\beta}}\right) = \int_\chi d^6y\sqrt{g}\left[\frac{1}{2}b_{\alpha\bar{i}\bar{j}}\bar{b}_{\bar{\beta}}{}^{\bar{i}\bar{j}}\right] = \frac{1}{\left|\left|\Omega^2\right|\right|}\left(\Phi_\alpha, \bar{\Phi}_{\bar{\beta}}\right) =: \mathcal{V}\tilde{G}_{\alpha\bar{\beta}}, \tag{13}$$

where we have used $\Omega_{ijm}\bar{\Omega}^{klm} = \epsilon_{ijm}\epsilon^{klm}\,||\Omega||^2 = 2\delta^{[k}_{[i}\delta^{l]}_{j]}\,||\Omega||^2$ and the fact that $||\Omega^2||$ is independent of the coordinates y^a. The matrix $\tilde{G}_{\alpha\bar{\beta}}$ defines a hermitian metric for the $h_{2,1}$ complex variables $z^\alpha = X^\alpha/X^0$. Due to homogeneity we have $\tilde{G}_{\alpha\bar{\beta}}(X^0, X^1, \ldots, X^{h_{2,1}}) = \tilde{G}_{\alpha\bar{\beta}}(1, z^\alpha, \ldots, z^{h_{2,1}})$. This metric is projective special Kähler with Kähler potential K defined previously. The holomorphic prepotential on the corresponding conic affine special Kähler manifold is given by $F(X)$. For further details on special Kähler geometry we refer the reader to [3].

3.2. Zero modes

We begin by considering the zero modes of the 10D metric, which we decompose according to

$$(g_{10})_{\mu\nu}(w) = \left(g_4'\right)_{\mu\nu}(x), \qquad (g_{10})_{\mu a}(w) = 0, \qquad (g_{10})_{ab}(w) = (g_\chi)_{ab}(x, y).$$

Note that the components $(g_{10})_{\mu a}$ must vanish since they correspond to a Killing vector on the Calabi-Yau three-fold, and such continuous isometries are incompatible with $SU(3)$ holonomy. See, for example, [27]. We denote the 4D metric with a prime in anticipation of a Weyl transformation that will be made later in (20). Zero modes of the wave operator correspond to deformations of the Ricci-flat Calabi-Yau metric that preserve the $SU(3)$ structure. These are given by

$$i\delta g_{i\bar{j}} = \sum_{A}^{h_{1,1}} \delta M^A V^A_{i\bar{j}}, \qquad \delta g_{ij} = \sum_{\alpha}^{h_{2,1}} \delta \bar{z}^{\bar{\alpha}} \bar{b}_{\bar{\alpha}ij},$$

with M^A and z^α defined in the previous section. Since the Calabi-Yau metric is Ricci flat we have

$$R_{ij} = R_{i\bar{j}} = R_{\bar{i}\bar{j}} = 0,$$
$$R_{10}(w) = R_4'(x) + g_4'^{\mu\nu}\left(R^{i}{}_{\mu i\nu}(x, y) + R^{\bar{i}}{}_{\mu\bar{i}\nu}(x, y)\right).$$

The Ricci scalar is explicitly calculated to be

$$\frac{1}{2}R_{10} = \frac{1}{2}R_4' - \frac{1}{2}\partial_\mu z^\alpha \partial^\mu \bar{z}^\beta b_{\alpha\bar{i}\bar{j}} \bar{b}_{\beta}^{\,\bar{i}\bar{j}} + \partial_\mu M^A \partial^\mu M^B \left(\frac{3}{2} V^A_{i\bar{j}} V^{Bi\bar{j}} - V^A_{i\bar{j}} V^B_{l\bar{k}} g^{i\bar{j}} g^{l\bar{k}}\right). \tag{14}$$

We refer the reader to [20] for details concerning this calculation.

Let us now consider the zero modes of the other bosonic fields. Recall that there are no harmonic one-forms on a Calabi-Yau manifold. The dilaton and Kaluza-Klein vector zero modes are given simply by

$$\phi(w) = \phi(x), \qquad V = V_\mu(x) dx^\mu.$$

The zero modes of the two-form B_2 and three-form A_3 are given by

$$B_2(w) = \mathcal{B}_2(x) + a^A(x) V^A(y),\ a^A \in \mathcal{C}^\infty(M_4), \qquad \mathcal{B}_2 \in \Omega^2(M_4),$$
$$A_3(w) = \mathcal{A}_3(x) + \mathcal{A}^A(x) \wedge V^A(y) + \check{A}(x, y),\ \mathcal{A}^A \in \Omega^1(M_4), \qquad \mathcal{A}_3 \in \Omega^3(M_4), \tag{15}$$

where

$$\check{A}(x, y) = 2^{\frac{1}{4}} \zeta^I(x) \alpha_I(x, y) + 2^{\frac{1}{4}} \tilde{\zeta}_I(x) \beta^I(x, y), \qquad \zeta^I,\ \tilde{\zeta}_I \in \mathcal{C}^\infty(M_4). \tag{16}$$

Recall that harmonic forms on manifold with positive definite metric are always closed, and, hence, exterior derivatives are given by $dB_2 = d\mathcal{B}_2 + da^A \wedge V^A$, etc.

It will be useful later to write the exterior derivative of $\check{A}$ in the basis Φ_I, $\bar{\Phi}_I$, Ω, $\bar{\Omega}$. This can be achieved by first taking the exterior derivative

$$d\check{A} = 2^{\frac{1}{4}} d\zeta^I \wedge \alpha_I + 2^{\frac{1}{4}} d\tilde{\zeta}_I \wedge \beta^I,$$

and then expanding α_I, β^I in terms of Φ_I, Ω and their complex conjugates[5]. Using expressions (10), (11) and the fact that $N_{IJ} = i(\bar{F}_{IJ} - F_{IJ})$ we find, after some simplifications, that

$$d\check{A} = i2^{\frac{1}{4}}\left(d\tilde{\zeta}_J + \bar{F}_{JK} d\zeta^K\right) N^{JI} \wedge \Phi_I - i2^{\frac{1}{4}} \frac{1}{\left(\bar{X}NX\right)} X^I \left(d\tilde{\zeta}_I + F_{IJ} d\zeta^J\right) \wedge \bar{\Omega} + \text{h.c.}$$

Next, observe that

$$\bar{F}_{JK} N^{JI} \Phi_I = \mathcal{N}_{JK} N^{JI} \Phi_I, \qquad X^I F_{IJ} = X^I \mathcal{N}_{IJ},$$

where in the first equation we used $X^I \Phi_I = 0$, which can easily be seen from (12). We may now write the derivative as

[5] In a previous version of this paper an alternative calculation of $d\check{A}$ was presented, which is included in appendix B. We thank one of our referees for suggesting the more concise calculation presented here.

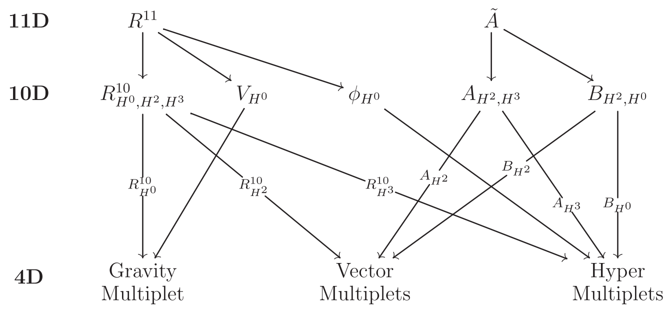

Figure 2. Composition of gravity-, vector, and hyper-multiplets.

$$d\check{A} = P^I \wedge \Phi_I + \bar{Q} \wedge \bar{\Omega} + \text{h.c.}, \tag{17}$$

where P^I, $\bar{Q} \in \Omega^1(M_4) \otimes \mathbb{C}$ are given by

$$P^I = i2^{\frac{1}{4}}\left(d\tilde{\zeta}_J + \mathcal{N}_{JK}d\zeta^K\right)N^{JI}, \qquad \bar{Q} = -i2^{\frac{1}{4}}\frac{1}{\left(\bar{X}NX\right)}X^I\left(d\tilde{\zeta}_I + \mathcal{N}_{IJ}d\zeta^J\right).$$

Having written down all zero modes our task is to construct the corresponding 4D effective action. For reduction over a Calabi-Yau three-fold it is known that this can be obtained by substituting the above expressions into the 10D action and integrating over the Calabi-Yau three-fold. We will show that, as one would expect, we obtain a theory of $\mathcal{N} = 2$ Euclidean supergravity coupled to $h_{1,1}$ vector-multiplets and $(h_{2,1} + 1)$ hyper-multiplets.

From the 4D perspective we expect to see the following field content:

$$H^0\text{-sector:} \quad \phi,\, V,\, \mathcal{B}_2 \qquad H^2\text{-sector:} \quad a^A,\, \mathcal{A}^A \qquad H^3\text{-sector:} \quad \zeta^I,\, \tilde{\zeta}_I.$$

The one-forms will only appear in the action through their field strengths $\mathcal{F}^0 = dV$ and $\mathcal{F}^A = d\mathcal{A}^A$, which form the gauge-fields of the gravity-multiplet and vector-multiplets, respectively. The two-form also only appears through the field strength $\mathcal{H}_3 = d\mathcal{B}_2$, which is then dualized to a scalar field $\tilde{\phi}$. This contributes to the hyper-multiplet sector along with ϕ and ζ^I, $\tilde{\zeta}_I$. The composition of gravity-, vector-, and hyper-multiplets is displayed schematically in figure 2.

We end this section by considering the contribution of the H^0 sector of A_3 to the 4D action. After performing the Weyl rescaling (20) this term is given by

$$S^4_{H^0(A_3)} = \int_{M_4}\left[-\frac{1}{2}\phi^{\frac{3}{4}}\mathcal{V}^3\left(\mathcal{F}_4 + dV \wedge \mathcal{B}_2\right) \wedge *\left(\mathcal{F}_4 + dV \wedge \mathcal{B}_2\right)\right]. \tag{18}$$

By adding an appropriate Lagrange multiplier the four-form $\mathcal{F}_4$ may be dualized to a constant e_0, which appears as a prefactor in front of a scalar potential of the form $(\phi^{\frac{3}{4}}\mathcal{V}^3)^{-1}$ in the resulting action. In order to avoid such a potential we will set $e_0 = 0$, after which (18) vanishes completely and plays no further role in the discussion. We remark that in the case of Lorentz spacetime signature this term corresponds to an RR-flux, and induces a gauging of the axionic scalar field $\tilde{\phi}$ dual to $\mathcal{B}_2$ (which will be introduced later in section 5.2) with charge e_0 and gauge field V [28] [6].

[6] We would like to thank one of our referees for pointing this out.

4. 4D gravity- and vector-multiplets

In this section we will consider the contributions from the gravity and H^2 cohomology sector to the reduction of the 10D action (4) over a Calabi-Yau three-fold. The resulting 4D effective action with spacetime signature $(\varepsilon, +,+,+)$ is given by

$$\begin{aligned} S^4_{grav+vector} = \int_{M_4} \Bigg[& \frac{1}{2} {*R_4} - g_{AB}(y)\big(dx^A \wedge {*dx^B} - \varepsilon dy^A \wedge {*dy^B}\big) \\ & + \varepsilon \Bigg(\frac{1}{2}(cyyy)\left(\frac{1}{6} + \frac{2}{3}(gxx)\right)\mathcal{F}^0 \wedge {*\mathcal{F}^0} - \frac{2}{3}(cyyy)(gx)_A \mathcal{F}^A \wedge {*\mathcal{F}^0} \\ & + \frac{1}{3}(cyyy) g_{AB} \mathcal{F}^A \wedge {*\mathcal{F}^B} \Bigg) \\ & + \frac{1}{6}\big(3(cx)_{AB}\mathcal{F}^A \wedge F^B - 3(cxx)_A \mathcal{F}^A \wedge \mathcal{F}^0 + (cxxx)F^0 \wedge \mathcal{F}^0\big) \Bigg], \end{aligned} \tag{19}$$

where we are using the shorthand notation $(cyyy) = c_{ABC}y^A y^B y^C$, $(cyy)_A = c_{ABC}y^B y^C$, etc. When written in components the terms in the last line pick up an overall factor of ε (see appendix A).

The above action describes a theory of 4D, $\mathcal{N} = 2$ supergravity coupled to vector-multiplets with Lorentzian spacetime signature if $\varepsilon = -1$ and Euclidean spacetime signature if $\varepsilon = +1$ [3]. The scalar fields form a nonlinear sigma model into a $2h_{1,1}$-dimensional projective special ε-Kähler manifold M_{vector}. The 4D spacetime metric is related to the ten- and even-dimensional spacetime metrics according to (3).

In the rest of this section we explain how (19) is constructed term-by-term.

4.1. Einstein-Hilbert term

Consider the Einstein-Hilbert term and the kinetic term for dilaton in the 10D action (4):

$$S^{10}_{EH+\phi} = \int_{M_{10}} \left[\frac{1}{2} {*R_{10}} - \frac{9}{16} d \log\phi \wedge {*d} \log\phi \right].$$

Substituting the expression for the 10D Ricci scalar (14) into this action we find

$$\begin{aligned} S^{10}_{EH+\phi} = \int_{M_{10}} \Bigg[& \frac{1}{2} {*R'_4} - \frac{9}{16} d\log\phi \wedge {*d}\log\phi - dz^\alpha \wedge b_\alpha \wedge {*\big(d\bar{z}^\beta \wedge \bar{b}_\beta\big)} \\ & + \frac{1}{2} dM^A \wedge V^A \wedge \left({*\big(dM^B \wedge V^B\big)} + \frac{1}{2} dM^B \wedge V^B \wedge J \right) \Bigg], \end{aligned}$$

where we have made use of the component expressions (6), (7), and (13). We now integrate over the Calabi-Yau manifold to obtain the 4D action

$$\begin{aligned} S^4_{EH+\phi} = \int_{M_4} \mathcal{V} \Bigg[& \frac{1}{2} {*R'_4} - \frac{9}{16} d\log\phi \wedge {*d}\log\phi - \tilde{G}_{\alpha\bar{\beta}} dz^\alpha \wedge {*d\bar{z}^{\bar{\beta}}} \\ & + \frac{1}{2}\left(G_{AB} + \frac{\mathcal{K}_{AB}}{\mathcal{V}} \right) dM^A \wedge {*dM^B} \Bigg]. \end{aligned}$$

In order to write the action in the Einstein frame we perform the Weyl rescaling

$$\left(g'_4\right)_{\mu\nu} = \mathcal{V}^{-1}(g_4)_{\mu\nu}. \tag{20}$$

Notice that in four dimensions

$$\sqrt{g_4'} = \mathcal{V}^{-2}\sqrt{g_4}, \qquad \sqrt{g_4'}\,g_4'^{\mu\nu} = \mathcal{V}^{-1}\sqrt{g_4}\,g_4^{\mu\nu}, \qquad \sqrt{g_4'}\,g_4'^{\mu\nu}g_4'^{\rho\sigma} = \sqrt{g_4}\,g_4^{\mu\nu}g_4^{\rho\sigma}.$$

After this transformation the 10D and 4D metrics are related via (3). The action is now written in the Einstein frame

$$S^4_{EH+\phi} = \int_{M_4}\left[\frac{1}{2}{*R_4} - \frac{3}{4}d\log\mathcal{V}\wedge *d\log\mathcal{V} - \frac{9}{16}d\log\phi\wedge *d\log\phi \right.$$
$$\left. - \tilde{G}_{\alpha\bar{\beta}}dz^\alpha\wedge *d\bar{z}^{\bar{\beta}} + \frac{1}{2}\left(G_{AB} + \frac{\mathcal{K}_{AB}}{V}\right)dM^A\wedge *dM^B\right].$$

Using the fact that $\mathcal{V} = \frac{1}{3!}\mathcal{K}$ and $d\mathcal{V} = \frac{1}{2}\mathcal{K}_A dM^A$ this can be written as

$$S^4_{EH+\phi} = \int_{M_4}\left[\frac{1}{2}{*R_4} - \frac{9}{16}d\log\phi\wedge *d\log\phi - \tilde{G}_{\alpha\bar{\beta}}dz^\alpha\wedge *d\bar{z}^{\bar{\beta}}\right.$$
$$\left. - \frac{1}{2}\left(G_{AB} + \frac{9}{4}\frac{\mathcal{K}_{AB}}{\mathcal{K}^2}\right)dM^A\wedge *dM^B\right].$$

Let us now make the field redefinition

$$M^A = \sqrt{2}\,\phi^{-3/4}v^A. \tag{21}$$

Since $\mathcal{K}$ is homogeneous of degree three in M^A we have

$$\mathcal{K}(M) = 2\sqrt{2}\,\mathcal{K}(v)\phi^{-9/4}, \qquad \mathcal{K}_A(M) = 2\mathcal{K}_A(v)\phi^{-3/2}, \qquad \text{etc.}$$

The action then takes the form

$$S^4_{EH+\phi} = \int_{M_4}\left[\frac{1}{2}{*R_4} - \frac{1}{2}G_{AB}(v)dv^A\wedge *dv^B - \frac{1}{4}d\varphi\wedge *d\varphi - \tilde{G}_{\alpha\bar{\beta}}(z,\bar{z})dz^\alpha\wedge *d\bar{z}^{\bar{\beta}}\right], \tag{22}$$

where we have defined

$$\varphi = \log\left(2\mathcal{V}(v)\phi^{-3}\right). \tag{23}$$

Since there are no factors of ε in the action (22) it is the same in both Lorentzian and Euclidean spacetime signatures.

The first term in (22) is simply the 4D Einstein-Hilbert term that appears in (19). The second term contributes to the scalar sigma model appearing in (19), which we will discuss next. The last two terms contribute to the action of the hyper-multiplets and will be dealt with in section 5.

4.2. Sigma model

We now consider the contribution from the H^2 cohomology sector of the B_2 field in the 10D action (4)

$$S^{10}_{H^2(B_2)} = \int\left[\frac{1}{2}\varepsilon\,\phi^{-\frac{3}{2}}H_3\wedge *H_3\big|_{H^2}\right].$$

We anticipate that the overall factor of ε will remain in place after dimensional reduction over the Calabi-Yau manifold.

Substituting $H_3|_{H^2} = da^A\wedge V^A$ into the above action and integrating over the Calabi-Yau three-fold results in the 4D effective action

$$S^4_{H^2(B_2)} = \int_{M_4} \left[\varepsilon\ \phi^{-\frac{3}{2}} \mathcal{V} G_{AB}(M) da^A \wedge *da^B \right].$$

Performing the Weyl rescaling (20) and making the field redefinition (21) we obtain

$$S^{10}_{H^2(B_2)} = \int_{M_4} \left[\varepsilon\ \frac{1}{2} G_{AB}(v) da^A \wedge *da^B \right]. \tag{24}$$

As expected, the overall factor of ε has survived the 4D action.

We may combine (24) with the H^2 contribution from (22) (which is given by the second term) to obtain the enlarged sigma model

$$S^4_{H^2(EH+\phi)} + S^4_{H^2(B_2)} = \int_{M_4} \left[-\frac{1}{2} G_{AB}(v) \left(dv^A \wedge *dv^B - \varepsilon\ da^A \wedge *da^B \right) \right].$$

In order to compare this expression with the existing literature it is convenient to make the field redefinition

$$v^A = \frac{1}{2^{\frac{1}{6}}} y^A, \qquad a^A = -\frac{1}{2^{\frac{1}{6}}} x^A, \qquad \mathcal{K}_{ABC} = c_{ABC}. \tag{25}$$

Due to the homogeneity properties of G_{AB} the factors of $2^{\frac{1}{6}}$ are irrelevant to the above action, but they will be useful later when considering terms involving the gauge fields. After the field redefinitions (25) the action is given by

$$S^4_{H^2(EH+\phi)} + S^4_{H^2(B_2)} = \int_{M_4} \left[-g_{AB}(y) \left(dx^A \wedge *dx^B - \varepsilon\ dy^A \wedge *dy^B \right) \right], \tag{26}$$

where we have defined

$$g_{AB}(y) \coloneqq -\varepsilon\ \frac{1}{2} G_{AB}(y) = \varepsilon \frac{3}{2} \left(\frac{(cy)_{AB}}{(cyyy)} - \frac{3}{2} \frac{(cy)_A (cy)_B}{(cyyy)^2} \right). \tag{27}$$

This agrees with the expression for the vector-multiplet sigma model for Lorentzian or Euclidean spacetime signatures given in [3], which corresponds to the second term in the action (19).

We remark that the target manifold M_{vector} described by the sigma model (26) is a $2h_{1,1}$-dimensional projective special ε-Kähler manifold. In order to expose this property we may define the ε-complex coordinates[7] $w^A = x^A + i_\varepsilon y^A$, where i_ε is the ε-complex unit. In these coordinates the ε-Kähler potential is given by

$$K = -\log \mathcal{V}(y), \qquad \mathcal{V}(y) = \frac{1}{3!} c_{ABC} y^A y^B y^C,$$

where it is understood that $y^A = \mathrm{Im}(w^A)$. The ε-holomorphic prepotential on the corresponding conic-affine special ε-Kä hler manifold is given by $F = -\frac{1}{6} c_{ABC} Z^A Z^B Z^C / Z^0$, where $(Z^0, \ldots, Z^{h_{1,1}})$ are homogeneous special ε-holomorphic coordinates satisfying $w^A = Z^A / Z^0$. Since the coefficients c_{ABC} are real there is a 1–1 correspondence between holomorphic prepotentials $F_{\varepsilon=-1}$ and para-holomorphic prepotentials $F_{\varepsilon=+1}$, at least in the the context of dimensional reduction over a Calabi-Yau three-fold considered in this paper. This is not true for holomorphic and para-holomorphic functions in general.

[7] In [10] different conventions are used for the ε-complex coordinates. They can be matched with the conventions used here by setting $y^A \to -y^A$, $c_{ABC} \to -c_{ABC}$, and $\epsilon_{\mu\nu\rho\sigma} \to -\epsilon_{\mu\nu\rho\sigma}$.

4.3. Gauge fields

We now turn our attention to the terms involving gauge fields in (19). The starting point is the contribution from the H^2 cohomology sector of the A_3 fields to the nontopological part of the 10D action:

$$S^{10}_{H^2(A_3)} = \int_{M_{10}} \left[-\frac{1}{2}\phi^{\frac{3}{4}}\left(F_4 + dV \wedge B_2\right) \wedge *\left(F_4 + dV \wedge B_2\right)|_{H^2} \right].$$

The individual terms decompose according to

$$F_4\big|_{H^2} = \mathcal{F}^A \wedge V^A, \qquad B_2\big|_{H^2} = a^A V^A, \tag{28}$$

where we have defined $\mathcal{F}^A = d\mathcal{A}^A$. Plugging this into the action we get

$$S^{10}_{H^2(A_3)} = \int_{M_4} \left[-\frac{1}{2}\phi^{\frac{3}{4}}\left(\mathcal{F}^A + a^A dV\right) \wedge *\left[\left(F^B + a^B dV\right)\right] \right] \int_{\chi} V^A \wedge *V^B.$$

Integrating over the Calabi-Yau three-fold and making the field redefinition (21) gives us

$$S^4_{H^2(A_3)} = \int_{M_4} \left[-\frac{\sqrt{2}}{3!}\mathcal{K}(v) G_{AB}(v)\left(\mathcal{F}^A + a^A dV\right) \wedge *\left(\mathcal{F}^B + a^B dV\right) \right]. \tag{29}$$

Notice that this action is invariant under Weyl transformations.

Consider now the contribution from the 11D Kaluza-Klein vector kinetic term

$$S^{10}_V = \int_{M_{10}} \left[\frac{1}{4}\varepsilon\, \phi^{\frac{9}{4}} dV \wedge *dV \right],$$

which upon integrating over the Calabi-Yau manifold and making the field redefinition (21) gives

$$S^4_V = \int_{M_4} \left[\varepsilon\, \frac{\sqrt{2}}{2}\frac{1}{3!}\mathcal{K}(v)\mathcal{F}^0 \wedge *\mathcal{F}^0 \right],$$

where we have defined $\mathcal{F}^0 = dV$. Combining this with (29) we get

$$\begin{aligned} S^4_{H^2(A_3)} + S^4_V = \int_{M_4} \Bigg[& \sqrt{2}\left(\varepsilon\, \frac{1}{12}(\mathcal{K}vvv) - \frac{1}{6}(\mathcal{K}vvv) G_{AB} a^A a^B \right) \mathcal{F}^0 \wedge *\mathcal{F}^0 \\ & - \frac{\sqrt{2}}{3}(\mathcal{K}vvv) G_{AB} a^B \mathcal{F}^A \wedge *\mathcal{F}^0 - \frac{\sqrt{2}}{6}(\mathcal{K}vvv) G_{AB}\mathcal{F}^A \wedge *\mathcal{F}^B \Bigg], \end{aligned}$$

where $(\mathcal{K}vvv) \coloneqq \mathcal{K}_{ABC} v^A v^B v^C = \mathcal{K}(v)$. We now make the field redefinitions (25) and (27) along with

$$\mathcal{F}^A = \frac{1}{2^{\frac{1}{6}}}\mathcal{F}'^A, \tag{30}$$

to get (dropping the primes)

$$\begin{aligned} & S^4_{H^2(A_3)} + S^4_V \\ & = \int_{M_4} \varepsilon \Bigg[\frac{1}{2}(cyyy)\left(\frac{1}{6} + \frac{2}{3}(gxx) \right)\mathcal{F}^0 \wedge *\mathcal{F}^0 - \frac{2}{3}(cyyy)(gx)_A \mathcal{F}^A \wedge *\mathcal{F}^0 \\ & + \frac{1}{3}(cyyy) g_{AB}\mathcal{F}^A \wedge *\mathcal{F}^B \Bigg], \end{aligned}$$

which corresponds to the second line of (19).

Finally, we consider the contributions of the H^2-sector to the topological term in the 10D action:

$$S^{10}_{H^2(top)} = \int_{M_{10}} \left[-\frac{\sqrt{2}}{2}\left(F_4 + dV \wedge B_2\right) \wedge F_4 \wedge B_2 - \frac{\sqrt{2}}{6} dV \wedge B_2 \wedge dV \wedge B_2 \wedge B_2\big|_{H^2}\right].$$

Substituting in (28) we find

$$S^{10}_{H^2(top)} = \int_{M_4} \left[-\frac{\sqrt{2}}{2}\left(\mathcal{F}^A + \mathcal{F}^0 a^A\right) \wedge \mathcal{F}^B a^C - \frac{\sqrt{2}}{6}\mathcal{F}^0 a^A \wedge \mathcal{F}^0 a^B a^C\right] \int_\chi V^A \wedge V^B \wedge V^C,$$

which upon integration gives us

$$S^{4}_{H^2(top)} = \int_{M_4} \left[-\frac{\sqrt{2}}{2}\left((\mathcal{K}a)_{AB} F^A \wedge \mathcal{F}^B + (\mathcal{K}aa)_A \mathcal{F}^A \wedge \mathcal{F}^0 + \frac{1}{3}(\mathcal{K}aaa)\mathcal{F}^0 \wedge \mathcal{F}^0\right)\right].$$

After making the field redefinitions (25) and (30) we get

$$S^{4}_{H^2(top)} = \int_{M_4} \left[\frac{1}{6}\left(3(cx)_{AB}\mathcal{F}^A \wedge \mathcal{F}^B - 3(cxx)_A \mathcal{F}^A \wedge \mathcal{F}^0 + (cxxx)\mathcal{F}^0 \wedge \mathcal{F}^0\right)\right],$$

which produces the last line of (19).

5. 4D hyper-multiplets

In this section we will consider the hyper-multiplet part of the reduction of the 10D action (4) over a Calabi-Yau three-fold. We will show that the contribution from the H^3-sector, the dilaton, and H^0-sector of the B_2 field results in the 4D effective action with spacetime signature $(\varepsilon, +,+,+)$:

$$\begin{aligned} S^4_{hyper} = \int_{M_4} \Big[& -\tilde{G}_{\alpha\bar{\beta}} dz^\alpha \wedge *d\bar{z}^{\bar{\beta}} - \frac{1}{4} d\varphi \wedge *d\varphi \\ & - e^{-2\varphi}\left(d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right) \wedge *\left(d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right) \\ & - \lambda \frac{1}{2} e^{-\varphi}\left(\mathcal{I}_{IJ} d\zeta^I \wedge *d\zeta^J + \mathcal{I}^{IJ}\left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K\right) \wedge *\left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K\right)\right)\Big], \end{aligned} \tag{31}$$

where $\lambda = -1$ and the coupling matrices $\tilde{G}_{\alpha\bar{\beta}}$, $\mathcal{I}_{IJ}$, $\mathcal{R}_{IJ}$ depend on z^α. Note that in our conventions $\mathcal{I}_{IJ}$ is negative definite.

For the case $\varepsilon = -1$ the above action, when combined with the vector-multiplet action (19), describes the bosonic part of $\mathcal{N} = 2$ hyper-multiplets coupled to supergravity with the Lorentzian spacetime signature [14]. For the case $\varepsilon = +1$ it is expected to describe the bosonic part of $\mathcal{N} = 2$ hyper-multiplets coupled to supergravity with Euclidean spacetime signature. Indeed, it is clear that this action can be obtained by the reduction of 5D, $\mathcal{N} = 2$ local hyper-multiplets over a time-like circle, since the bosonic part of the hyper-multiplet action does not change upon dimensional reduction.

The action (31) describes a non-linear sigma model into a positive-definite $(4h_{2,1} + 4)$-dimensional quaternionic Kähler target manifold M_{hyper} [14]. Notice that the parameter ε does not appear in front of any terms in the action, nor does it appear in the definitions of the scalar fields or coupling matrices. Thus, the coupling matrices, and, hence, the scalar target geometry (quaternionic Kähler) is the same regardless of whether the reduction from 11D to 10D was performed over a space-like or time-like circle.

We remark that the bosonic sector of 3D, $\mathcal{N}=2$ local Euclidean hyper-multiplets obtained by the dimensional reduction (followed by dualization) of 4D, $\mathcal{N}=2$ vector-multiplets over time takes the same form as the above action with $\lambda=+1$. In this case the target manifold has split signature and the metric is para-quaternionic Kähler [15, 16]. We also anticipate that the above action with $\lambda=+1$ can be obtained by the reduction of 10D type IIA* supergravity with Lorentzian spacetime signature, as described in [29], over a Calabi-Yau three-fold. This is because the sign flip in front of the G_4^2 term in the IIA* action, given by expression (4.8) of [29], corresponds to setting $\lambda=+1$ in (31). On the other hand, the sign flips in front of the H^2 and the topological terms of the IIA* action will be compensated by a sign flip in the Hodge dualization procedure when $\varepsilon=-1$.

The first line in (31) is taken from the H^3 and the dilaton terms in the gravity sector (22). In the remainder of this section we will explain the origins of the second and third lines. We will use the conventions for special Kähler geometry given in appendix A.

5.1. ζ^I, $\tilde{\zeta}_I$ terms

Let us consider the nontopological part of the H^3-sector of the 10D action

$$S^{10}_{H^3(ntop)}=\int_{M_{10}}\left[-\frac{1}{2}\phi^{\frac{3}{4}}F_4\wedge{}^*F_4\Big|_{H^3}\right].$$

Substituting in the expression (17) for $F_4|_{H^3}=d\check{A}$ into the action gives

$$S^{10}_{H^3(ntop)}=\int_{M_4}\left[-\phi^{\frac{3}{4}}P^I\wedge{}^*\bar{P}^J\right]\int_\chi\Phi_I\wedge{}^*\bar{\Phi}_J+\int_{M_4}\left[\phi^{\frac{3}{4}}\bar{Q}\wedge{}^*Q\right]\int_\chi\bar{\Omega}\wedge{}^*\Omega.$$

Integrating over the Calabi-Yau three-fold we obtain the 4D action

$$\begin{aligned}S^4_{H^3(ntop)}&=\int_{M_4}\left[\phi^{\frac{3}{4}}\left(\mathcal{M}_{I\bar{J}}P^I\wedge{}^*\bar{P}^J-\left(\bar{X}NX\right)\bar{Q}\wedge{}^*Q\right)\right]\\&=\int_{M_4}\left[\frac{\sqrt{2}}{2}\phi^{\frac{3}{4}}\mathcal{I}^{IJ}\left(d\tilde{\zeta}_I+\mathcal{N}_{IK}d\zeta^K\right)\wedge{}^*\left(d\tilde{\zeta}_J+\bar{\mathcal{N}}_{JL}d\zeta^L\right)\right],\end{aligned}$$

where in the last line we used the expression for $\mathcal{I}^{-1}$ given in (33). We now make the Weyl transformation (20) to get

$$S^4_{H^3(ntop)}=\int_{M_4}\left[\frac{1}{2}e^{-\varphi}\mathcal{I}^{IJ}\left(d\tilde{\zeta}_I+\mathcal{N}_{IK}d\zeta^K\right)\wedge{}^*\left(d\tilde{\zeta}_J+\bar{\mathcal{N}}_{JL}d\zeta^L\right)\right].$$

where φ was defined in (23). Substituting $\mathcal{N}_{IJ}=\mathcal{R}_{IJ}+i\mathcal{I}_{IJ}$ followed by a straightforward rewriting gives the third line of (31).

5.2. $\tilde{\phi}$ term

We now consider the topological part of the H^3-sector of the 10D action

$$S^{10}_{H^3(top)}=\int_{M_{10}}\left[-\frac{\sqrt{2}}{2}F_4\wedge F_4\wedge B_2\Big|_{H^3}\right].$$

Substituting in (17) we find

$$S^{10}_{H^3(top)} = \int_{M_4}\left[-\sqrt{2}\,\mathcal{B}_2 \wedge P^I \wedge \bar{P}^J\right]\int_\chi \Phi_I \wedge \bar{\Phi}_J + \int_{M_4}\left[\sqrt{2}\,\mathcal{B}_2 \wedge \bar{Q} \wedge Q\right]\int_\chi \bar{\Omega} \wedge \Omega.$$

Making use of (5) we integrate over the Calabi-Yau three-fold to obtain

$$\begin{aligned} S^4_{H^3(top)} &= \int_{M_4}\left[i\sqrt{2}\mathcal{B}_2 \wedge \left(\mathcal{M}_{I\bar{J}} P^I \wedge \bar{P}^J - \left(\bar{X}NX\right)\bar{Q} \wedge Q\right)\right] \\ &= \int_{M_4}\left[i\mathcal{B}_2 \wedge \mathcal{I}^{IJ}\left(d\tilde{\zeta}_I + \mathcal{N}_{IK} d\zeta^K\right) \wedge \left(d\tilde{\zeta}_J + \bar{\mathcal{N}}_{JL} d\zeta^L\right)\right] \\ &= \int_{M_4}\left[-2\mathcal{B}_2 \wedge d\zeta^I \wedge d\tilde{\zeta}_I\right], \end{aligned}$$

where in the last line we used $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$. Note that this term is invariant under Weyl rescalings. Integrating by parts and adding the contribution from the H^0-sector of the B_2 field gives

$$S^4_{H^3(top)} + S^4_{H^0(B_2)} = \int_{M_4}\left[2\mathcal{H}_3 \wedge \zeta^I d\tilde{\zeta}_I + \varepsilon e^{2\varphi}\mathcal{H}_3 \wedge {*\mathcal{H}_3}\right].$$

We now dualize the three-form $\mathcal{H}_3$ by adding the Lagrange multiplier

$$S^4_{Lm} = \int_{M_4}\left[2\mathcal{H}_3 \wedge d\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right)\right].$$

Solving the Euler-Lagrangian equations of $S^4_{H^3(top)} + S^4_{H^0(B_2)} + S^4_{Lm}$ for $\mathcal{H}_3$ gives

$${*\mathcal{H}_3} = -\varepsilon e^{-2\varphi}\left(d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right).$$

Substituting back into the action we get

$$\begin{aligned} &S^4_{H^3(top)} + S^4_{H^0(B_2)} + S^4_{Lm} \\ &= \int_{M_4}\left[-e^{-2\varphi}\left(d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right) \wedge *\left(d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)\right], \end{aligned}$$

where we have used the fact that $**\alpha = -\varepsilon\alpha$ for any one-form or three-form on M_4. This produces the second line of (31).

Acknowledgments

We would like to thank Thomas Mohaupt and Chris Hull for useful discussions. We would also like to thank the referees for useful comments and suggestions. The work of W S is supported in part by the National Science Foundation under grant number PHY- 1415659. The work of O V is supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 'Particles, Strings and the Early Universe.'

Appendix A. Conventions and identities

Consider an m-dimensional pseudo-Riemannian manifold with signature (k, ℓ), where k represents the number of time-like dimensions. We take the epsilon symbol and tensor, respectively, to be

$$\epsilon_{12\ldots m} = 1, \qquad \varepsilon_{12\ldots m} = \sqrt{|g|}\,\epsilon_{12\ldots m}.$$

Note that the epsilon tensor $\varepsilon_{\mu_1\ldots\mu_m}$ will always be written with indices to avoid confusion with the parameter $\varepsilon = \pm 1$ introduced in (1). We may use the metric to raise the indices of the epsilon tensor

$$\varepsilon^{\mu_1\cdots\mu_m} := g^{\mu_1\nu_1}\ldots g^{\mu_m\nu_m}\varepsilon_{\nu_1\ldots\nu_m} = (-1)^k\sqrt{|g|}^{-1}\epsilon_{\mu_1\ldots\mu_m}.$$

It follows that

$$dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_m} = (-1)^k\sqrt{|g|}\,\varepsilon^{\mu_1\cdots\mu_m}dx^1\wedge\ldots\wedge dx^m = (-1)^k\sqrt{|g|}\,\varepsilon^{\mu_1\cdots\mu_m}d^m x,$$

and

$$\varepsilon_{\mu_1\cdots\mu_p\rho_{p+1}\cdots\rho_m}\varepsilon^{\nu_1\cdots\nu_p\rho_{p+1}\cdots\rho_m} = (-1)^k p!(m-p)!\,\delta^{[\nu_1}_{[\mu_1}\ldots\delta^{\nu_p]}_{\mu_p]}.$$

Differential p-forms are expanded according to

$$\alpha_p = \frac{1}{p!}(\alpha_p)_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}.$$

The Hodge star is defined by

$$*\alpha_p = \frac{1}{p!(m-p)!}(\alpha_p)_{\mu_1\ldots\mu_p}\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\ldots\nu_m}dx^{\nu_{p+1}}\wedge\ldots\wedge dx^{\nu_m},$$

and therefore

$$\alpha_p\wedge *\beta_p = \frac{1}{p!}(\alpha_p)_{\mu_1\ldots\mu_p}\left(\beta_p\right)^{\mu_1\cdots\mu_p}\sqrt{|g|}\,d^m x.$$

Notice that

$$\alpha_p\wedge\gamma_{(m-p)} = \frac{1}{p!(m-p!)}(\alpha_p)_{\mu_1\ldots\mu_p}(\gamma_{(m-p)})_{\nu_{p+1}\ldots\nu_m}\varepsilon^{\mu_1\cdots\mu_p\nu_{p+1}\cdots\nu_m}(-1)^k\sqrt{|g|}\,d^m x,$$

so, for example, the last line of (4) is written in components as

$$\int_{M_{10}} d^{10}w\sqrt{|g_{10}|}\,\varepsilon^{\hat\mu_1\cdots\hat\mu_{10}}\Bigg[-\varepsilon\,\frac{\sqrt{2}}{48^2}\Big(F_{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4} + 6V_{\hat\mu_1\hat\mu_2}B_{\hat\mu_3\hat\mu_4}\Big)F_{\hat\mu_5\hat\mu_6\hat\mu_6\hat\mu_7}B_{\hat\mu_9\hat\mu_{10}}$$
$$-\,\varepsilon\,\frac{\sqrt{2}}{192}V_{\hat\mu_1\hat\mu_2}B_{\hat\mu_3\hat\mu_4}V_{\hat\mu_5\hat\mu_6}B_{\hat\mu_7\hat\mu_8}B_{\hat\mu_9\hat\mu_{10}}\Bigg]. \tag{32}$$

We use the following conventions for special Kähler geometry:

$$
\begin{aligned}
N_{IJ} &= 2\,\mathrm{Im}\big(F_{IJ}\big) = \frac{1}{i}\big(F_{IJ} - \bar{F}_{IJ}\big)\\
\mathcal{N}_{IJ} &= \bar{F}_{IJ} + i\frac{(NX)_I (NX)_J}{XNX}\\
\mathcal{I}_{IJ} &= \mathrm{Im}\big(\mathcal{N}_{IJ}\big) = -\frac{1}{2}N_{IJ} + \frac{1}{2}\frac{(NX)_I (NX)_J}{XNX} + \frac{1}{2}\frac{\big(N\bar{X}\big)_I \big(N\bar{X}\big)_J}{\bar{X}N\bar{X}}\\
\mathcal{R}_{IJ} &= \mathrm{Re}\big(\mathcal{N}_{IJ}\big) = \frac{1}{2}\big(F_{IJ} + \bar{F}_{IJ}\big) + \frac{i}{2}\frac{(NX)_I (NX)_J}{XNX} - \frac{i}{2}\frac{(NX)_I (NX)_J}{XNX}\\
K_I &= \frac{\partial}{\partial X^I}\log\big(\bar{X}NX\big) = \frac{\big(N\bar{X}\big)_I}{\bar{X}NX}\\
\mathcal{I}^{IJ} &= -\,2N^{IJ} + 2\frac{X^I\bar{X}^J}{\bar{X}NX} + 2\frac{\bar{X}^I X^J}{\bar{X}NX} = 2N^{IK}\Big(-\delta^J_K + \bar{K}_K\bar{X}^J + K_K X^J\Big)\\
\mathcal{M}_{I\bar{J}} &= -\,N_{IJ} + \frac{\big(N\bar{X}\big)_I (NX)_J}{\bar{X}NX}\\
\mathcal{F}_{IJ} &= F_{IJ},
\end{aligned}
\tag{33}
$$

where $(NX)_I = N_{IJ}X^J$ and $\bar{X}NX = N_{IJ}\bar{X}^I X^J$, etc. The matrix N_{IJ} has complex Lorentz signature and $\mathcal{I}_{IJ}$ is negative definite. We will often omit writing indices explicitly when the meaning is clear from the order. Some useful identities are

$$
\frac{\partial\bar{K}}{\partial\bar{X}}d\bar{X} = i\big(d\bar{\mathcal{F}}\big)N^{-1}\bar{K} - \bar{K}\big(\bar{K}d\bar{X}\big)
\tag{34}
$$

$$
\begin{aligned}
d\mathcal{I}^{-1} = {}& 2iN^{-1}\big(d\bar{\mathcal{F}}\big)N^{-1}(Id - KX) + 2N^{-1}\left(\frac{\partial\bar{K}}{\partial X}dX\right)\bar{X}\\
&+ 2\big(N^{-1}\bar{K}\big)d\bar{X}\big(Id - \bar{K}\bar{X}\big) + \text{h.c.}
\end{aligned}
\tag{35}
$$

$$
\mathcal{I}^{-1}(d\mathcal{N})\mathcal{I}^{-1} = \frac{4i}{\bar{X}NX}\Big[-\big(dX\bar{X} + \bar{X}dX\big) + (KdX)\big(X\bar{X} + \bar{X}X\big)\Big] + 4N^{-1}\big(d\bar{\mathcal{F}}\big)N^{-1}.
\tag{36}
$$

Appendix B. Alternative calculation of $d\check{A}$

In a previous version of this paper a different calculation for the exterior derivation of $\check{A}$ was presented, which closely followed the original calculation of [20]. The calculation that now appears in the main text is far more concise and does not involve evaluating differentials. We include here our original calculation for the purpose of continuity with previous versions of this paper, and because it provides a complementary approach to this calculation using complex forms.

We start by writing $\check{A}$ as

$$
\check{A} = \Psi(a\alpha + b\beta) + \bar{\Psi}\big(\bar{a}\alpha + \bar{b}\beta\big) = i\,\mathrm{Im}(\Psi)2a\alpha + i\,\mathrm{Im}(\Psi)\big(b - \bar{b}\big)\beta + \mathrm{Re}\,\Psi\big(b + \bar{b}\big)\beta.
$$

Here $\Psi_I(x)$ are complex fields and $(a^{IJ}(x))$, $(b_J^I(x))$ are complex matrices, where we have chosen a to be purely imaginary. Substituting in the expression for α_I, β^I given in (10), (11) we get

$$
\check{A} = i\Psi\Big(\big(a\bar{\mathcal{F}} + b\big)N^{-1}\Omega - (a\mathcal{F} + b)N^{-1}\bar{\Omega}\Big) + \text{h.c.}
$$

We now make the ansatz

$$(a\mathcal{F} + b)N^{-1}\bar{\Phi} = 0, \qquad (a\mathcal{F} + b)N^{-1}\bar{K}\bar{\Omega} \propto \bar{K}\bar{\Omega},$$

which can satisfied by setting

$$(a\mathcal{F} + b)N^{-1} = d\bar{K}\bar{X} \qquad \Rightarrow \qquad b = d\bar{K}\bar{X}N - a\mathcal{F}.$$

Substituting this into the expression for $\breve{A}$ we get

$$\breve{A} = i\Psi\Big(\big(-ia + d\bar{K}\bar{X}\big)(\Phi + K\Omega) - d\bar{K}\bar{\Omega}\Big) + \text{h.c.}$$

It is convenient to choose $d = 2^{\frac{1}{4}}N^{-1}$, in which case

$$\breve{A} = 2^{\frac{1}{4}}i\Psi N^{-1}\big(\Phi - \bar{K}\bar{\Omega}\big) + \text{h.c.}$$

Comparing with (16) gives

$$\begin{aligned}\zeta^I &= \big(\text{Im}(\Psi)\mathcal{I}^{-1}\big)^I,\\ \tilde{\zeta}_I &= \Big(\text{Im}(\Psi)N^{-1}\Big[(Id - 2KX)\mathcal{F} + \big(Id - 2\bar{K}\bar{X}\big)\bar{\mathcal{F}}\Big] + \text{Re}(\Psi)\Big)_I.\end{aligned} \tag{37}$$

The complex fields Ψ_I are related to the real fields ζ^I, $\tilde{\zeta}_I$ by

$$\Psi_I = \tilde{\zeta}_I + \mathcal{N}_{IJ}\zeta^J. \tag{38}$$

Taking derivatives we find

$$\Big(d\Psi + \tfrac{i}{2}\big(\Psi - \bar{\Psi}\big)\mathcal{I}^{-1}d\mathcal{N}\Big)_I = d\tilde{\zeta}_I + \mathcal{N}_{IJ}d\zeta^J. \tag{39}$$

We now take derivative of $\breve{A}$ using the expressions (16) and (37). After some simplifications using identities (34) and (35) along with (9) and (12) we get

$$\begin{aligned}d\breve{A} = {}& 2^{\frac{1}{4}}i\Big(d\Psi N^{-1} - \big(\Psi - \bar{\Psi}\big)N^{-1}\Big((KdX) + i\big(d\bar{\mathcal{F}}\big)N^{-1}\Big)\Big)^I\Phi_I\\ &- 2^{\frac{1}{4}}i\left(d\Psi N^{-1}\bar{K} + \big(\Psi - \bar{\Psi}\big)N^{-1}\left(\frac{\partial\bar{K}}{\partial X}dX\right)\right)\bar{\Omega} + \text{h.c.}\end{aligned}$$

Simplifying further using (36) we can write this more concisely as

$$\begin{aligned}d\breve{A} = {}& \Big[i2^{\frac{1}{4}}N^{IJ}\Big(d\Psi + \tfrac{i}{2}\big(\Psi - \bar{\Psi}\big)\mathcal{I}^{-1}d\mathcal{N}\Big)_J\Big] \wedge \Phi_I\\ &+ \left[-i2^{\frac{1}{4}}\frac{1}{\big(\bar{X}NX\big)}X^I\Big(d\Psi + \tfrac{i}{2}\big(\Psi - \bar{\Psi}\big)\mathcal{I}^{-1}d\mathcal{N}\Big)_I\right] \wedge \bar{\Omega} + \text{h.c.}\end{aligned} \tag{40}$$

Substituting (39) we find precisely the same expressions as (17) in the main text.

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