J. Math. Anal. Appl. 519 (2023) 126822

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

On harmonic homeomorphisms from the punctured unit disc onto its exterior

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A R T I C L E I N F O

Article history:
Received 19 August 2022
Available online 28 October 2022
Submitted by S.G. Krantz

Keywords:
Planar harmonic mapping
Convex function in the direction of the real axis
Multivalent function

A B S T R A C T

The purpose of this paper is to study the class of harmonic sense-preserving homeomorphisms from the punctured unit disc onto its planar exterior which extend homeomorphically between the unit circle and itself and admit a simple pole at the origin.

1. Introduction

A *planar harmonic mapping* of a complex domain Ω is a complex-valued function of the form

$$f(z) = u(z) + iv(z),$$

where u and v are real harmonic functions. It is immediate that f admits the *canonical representation*

$$f = h + \overline{g},$$

where h and g are analytic functions that are single-valued if Ω is simply connected and possibly multi-valued otherwise. The mapping f is sense-preserving in Ω if, and only if, its *second complex dilatation* $\omega = g'/h'$ is analytic and satisfies $|\omega(z)| < 1$ in Ω. Furthermore, f is sense-preserving and locally univalent in Ω if, and only if, its Jacobian $|h'|^2 - |g'|^2$ is positive there. Throughout this paper, we deal only with sense-preserving harmonic mappings and assume that an analytic function is single-valued unless otherwise is stated.

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https://doi.org/10.1016/j.jmaa.2022.126822
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Let $\mathbb{C}$, $\widehat{\mathbb{C}}$, $\mathbb{D}$, and $\mathbb{T}$ denote the complex plane, extended complex plane, open unit disc $\{z : |z| < 1\}$, and the unit circle $\{z : |z| = 1\}$, respectively.

Harmonic mappings of the punctured unit disc were characterized by D. Bshouty and W. Hengartner [3] as follows.

Proposition 1. *Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ be a harmonic mapping. Then*

$$f(z) = h(z) + \overline{g(z)} + \alpha \ln |z|^2 \qquad (z \in \mathbb{D} \setminus \{0\}), \tag{1}$$

where h and g are analytic functions of $\mathbb{D} \setminus \{0\}$ and $\alpha \in \mathbb{C}$ is fixed.

Evidently, a rotation of f guarantees that α in (1) is real.

In this paper, we deal primarily with the following class of harmonic sense-preserving homeomorphisms $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ satisfying (1) and where α is real.

Definition 1. Let $\mathcal{H}_u$ denote the class of all sense-preserving homeomorphisms $f : \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ such that $f(0) = \infty$, $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is a harmonic mapping satisfying (1) with $\alpha \in \mathbb{R}$.

Clunie and Sheil-Small [4] introduced the "*shear construction*" method for constructing univalent harmonic mappings of the unit disc with prescribed properties. Here, we extend this method for constructing univalent harmonic mappings of doubly-connected domains.

Definition 2. Let f be a harmonic mapping of some complex domain D. Suppose that $f = h + \overline{g}$, where h and g are analytic functions, possibly multi-valued, of D. Then for a fixed real ϕ we can write

$$f(z) = h(z) - e^{2i\phi} g(z) + 2e^{i\phi} \Re(e^{i\phi} g(z)) = F(z) + 2e^{i\phi} \Re(e^{i\phi} g(z)) \qquad (z \in D),$$

where $F = h - e^{2i\phi} g$ is an analytic function, possibly multivalued, termed as the analytic shear in the ϕ-direction of f. In case $\phi = 0$, then $F = h - g$ is called the analytic horizontal shear of f.

It is immediate that F is the analytic horizontal shear of f in a domain D if, and only if, $\Im F(z) = \Im f(z)$ in D.

Henceforth, an analytic shear is assumed to be horizontal unless otherwise is stated.

Suppose now that $f \in \mathcal{H}_u$ be written as in (1). By invoking Definition 2, we write

$$f(z) = F(z) + 2\,\Re\,\{g(z) + \alpha \ln |z|\} \qquad (z \in \mathbb{D} \setminus \{0\}), \tag{2}$$

or equivalently,

$$F(z) = f(z) - 2\Re\,\{g(z) + \alpha \ln |z|\} \qquad (z \in \mathbb{D} \setminus \{0\}), \tag{3}$$

where $F = h - g$ is the analytic shear of f. Note that every value $F(z)$ can be obtained from $f(z)$ via a horizontal shift.

A simply connected domain $\Omega \subset \mathbb{C}$ is called *convex in the horizontal direction* (CHD) if its intersection with every horizontal line is either a line segment or empty. A univalent analytic function of a simply connected domain D is called *convex in the horizontal direction* if its image set $f(D)$ is CHD. The region Ω is called *convex in the positive direction* (CPD) if its intersection with every horizontal line is either empty, a semi-line $\{w + t : t \geq 0\}$ for some point w, or the horizontal line itself; domains *convex in the negative direction* (CND) are defined analogously.

Clunie and Sheil-Small [4, p. 37] established the following basic result.

Proposition 2. *Let $f = h + \overline{g}$ be a sense-preserving, harmonic, locally univalent mapping of $\mathbb{D}$. Then f is univalent and its range is CHD if, and only if, $h - g$ has the same properties.*

Note that $\mathbb{D}$ may be replaced by any simply connected domain.

Let $f \in \mathcal{H}_u$ be written as in (2). The first result of the paper, Theorem 1, provides a complete description of the geometry of the range $F(\mathbb{D})$ of F, from which we infer that F have angular limits everywhere in $\mathbb{T}$. Theorem 2 gives basic analytic properties of h and g: h has a simple pole at the origin and g may either have a simple pole or a removable singularity at the origin. In Theorem 3, we give an interesting analytic representation of the shear F of f. In Theorem 4, we establish a strong relationship between the class $\mathcal{H}_u$ and its counterpart the class of harmonic homeomorphisms from $\mathbb{D}$ onto itself that extend to sense-preserving homeomorphisms from $\mathbb{T}$ onto itself. Our last result, Theorem 5, gives necessary and sufficient conditions for a harmonic mapping $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ of form (1) to belong to $\mathcal{H}_u$. This leads to a method for constructing a harmonic homeomorphism $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$, provided that it exists, such that $\lim_{z\to 0} f(z) = \infty$ and f extends continuously to a prescribed sense-reversing homeomorphism from $\mathbb{T}$ onto itself.

2. Geometry of the horizontal shear F of $f \in \mathcal{H}_u$

In this section, we give a detailed description of the geometry of the horizontal shear F of $f \in \mathcal{H}_u$. To do so, we prove two lemmas.

Lemma 1. *Let $f \in \mathcal{H}_u$ be written as in (1), and let F be the analytic horizontal shear of f. Then F is locally univalent in $\mathbb{D} \setminus \{0\}$.*

Proof. Fix $z_0 \in \mathbb{D} \setminus \{0\}$, and let $w_0 = f(z_0)$ and $W_0 = F(z_0)$. It is immediate that there exists a Jordan curve $\gamma \subset \mathbb{D} \setminus \{0\}$ enclosing z_0 but not the origin such that $C = f(\gamma)$ is a circle with center w_0 and not enclosing $\mathbb{T}$. Denote by D the Jordan domain bounded by γ, and by Δ the circular domain bounded by C. Then $f : D \to \Delta$ is a harmonic homeomorphism and Δ is CHD. By invoking Proposition 2, we conclude that F is a univalent function in D convex in the horizontal direction with $W_0 \in F(D)$. Therefore, F is locally univalent at z_0. $\square$

Lemma 2. *Under the assumptions of Lemma 1, F extends to a meromorphic function of $\mathbb{D}$, also denoted by F, that satisfies the following properties:*

(a) *F is either a univalent or a 2-valent function in $\mathbb{D}$.*
(b) *F has a simple pole at the origin;*

Proof. (a) Fix $W_0 \in F(\mathbb{D} \setminus \{0\})$. There exist two mutually disjoint Jordan curves γ and Γ in $\mathbb{D} \setminus \{0\}$ that satisfy the following properties:

(i) γ is negatively-directed and encloses the origin;
(ii) Γ is positively-directed and encloses γ;
(iii) $c = f(\gamma)$ and $C = f(\Gamma)$ are concentric circles centered at the origin; c positively-directed, C negatively-directed, and C encloses c;
(iv) $W_0 \notin F(\gamma \cup \Gamma)$.

It follows from (3) that $F(\gamma)$ and $F(\Gamma)$ are horizontal shears of c and C, respectively, and that, by conditions (iii) and (iv), each of $n(F(\gamma), W_0)$ and $n(F(\Gamma), W_0)$ is either $0, 1$, or -1. Hence, $n(F(\gamma \cup \Gamma), W_0)$ is either $0, 1, -1, 2, -2$. But since F is analytic in $\mathbb{D} \setminus \{0\}$ and, by (i) and (ii), $\gamma \cup \Gamma$ is homologous to zero there,

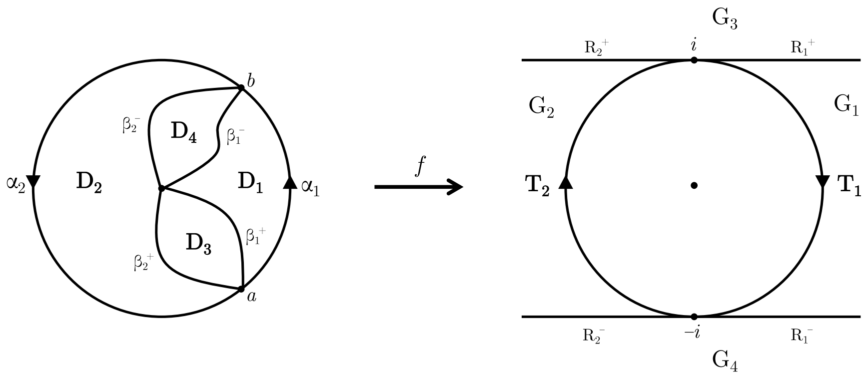

Fig. 1. The Geometry of $f \in \mathcal{H}_u$.

$n(F(\gamma \cup \Gamma), W_0)$ is either $0, 1$, or 2. Now, since γ and Γ are chosen arbitrarily with the conditions (i), ..., (iv) satisfied, the proof of (a) is complete.

(b) We conclude at once from (a) that F admits the origin as a removable singularity, a simple pole, or a double pole.

Suppose that the origin is a removable singularity of F. Then the previous argument with γ replaced by the origin implies that F is a univalent function in $\mathbb{D}$. In this case, let D be a Jordan domain containing the origin and mapping under F to a circular region G with center $F(0)$ and radius r. Then, by (2), $f(D)$ is a horizontal shear of G; that is, $f(D)$ lies in the horizontal strip

$$\{W : |\Im(W - F(0))| \leq r\} \cup \{\infty\},$$

which contradicts the fact that f has a pole at the origin. Hence, F is a meromorphic function in $\mathbb{D}$ with either a simple or double pole at the origin.

We show now that the origin is a simple pole of F. Suppose to the contrary that F has a double pole at the origin. Since f has a simple pole at the origin, there exists an open neighborhood N of the origin on which F is non-vanishing. Let γ be a Jordan curve in N such that $n(\gamma, 0) = -n(f(\gamma), 0) = 1$, where $f(\gamma)$ is a circle. Then, by using a previous argument employing (3), $n(F(\gamma), 0)$ is either $0, 1$, or -1; hence F does not have a double pole at the origin. This proves (b), and the proof is complete. $\square$

Let $a = (f^*)^{-1}(i)$ and $b = (f^*)^{-1}(-i)$. Denote by α_1 the subarc obtained by traversing $\mathbb{T}$ positively from a to b, and by α_2 the subarc obtained by traversing $\mathbb{T}$ positively from b to a. Consider the rays $R_1^{\pm} = \{x \pm i : x \geq 0\}$ and $R_2^{\pm} = \{x \pm i : x \leq 0\}$. Let $\beta_1^{\pm} = f^{-1}(R_1^{\pm})$ and $\beta_2^{\pm} = f^{-1}(R_2^{\pm})$; β_1^+ and β_2^+ are Jordan arcs joining the origin to a and lying otherwise in $\mathbb{D}$, and β_1^- and β_2^- are Jordan arcs joining the origin to b and lying otherwise in $\mathbb{D}$. These arcs subdivide $\mathbb{D}$ into four simply connected domains D_j, $j = 1, 2, 3, 4$, where $D_j, j = 1, 2$, is bounded by α_j and $\beta_j^{\pm}$, D_3 is bounded by β_1^+ and β_2^+, and D_4 is bounded by β_1^- and β_2^-. Let $\mathbb{T}_1$ be the semi-circle obtained by traversing the unit circle negatively from i to $-i$, and let $\mathbb{T}_2$ be the semi-circle obtained by traversing the unit circle negatively from $-i$ to i. Denote by G_1 the strip-like domain bounded by $R_1^{\pm} \cup \mathbb{T}_1$, and G_2 the strip-like domain bounded by $R_2^{\pm} \cup \mathbb{T}_2$, G_3 the upper half-plane bounded by $R_1^+ \cup R_2^+$, G_4 the lower half-plane bounded by $R_1^- \cup R_2^-$. It is immediate that each $f : D_j \to G_j$, $j = 1, 2, 3, 4$, is a harmonic homeomorphism. Finally, for a fixed $s \in \mathbb{R}$, define the semi-lines $\mathcal{U}_s^+ = \{x + i : x \geq s\}$, $\mathcal{U}_s^- = \{x + i : x \leq s\}$, $\mathcal{L}_s^+ = \{x - i : x \geq s\}$, and $\mathcal{L}_s^- = \{x - i : x \leq s\}$. See Fig. 1.

With this notation in mind, the geometry of F is described as follows.

Theorem 1. *Let $f \in \mathcal{H}_u$ be written as in (1), and let F be the analytic horizontal shear of f. Then*

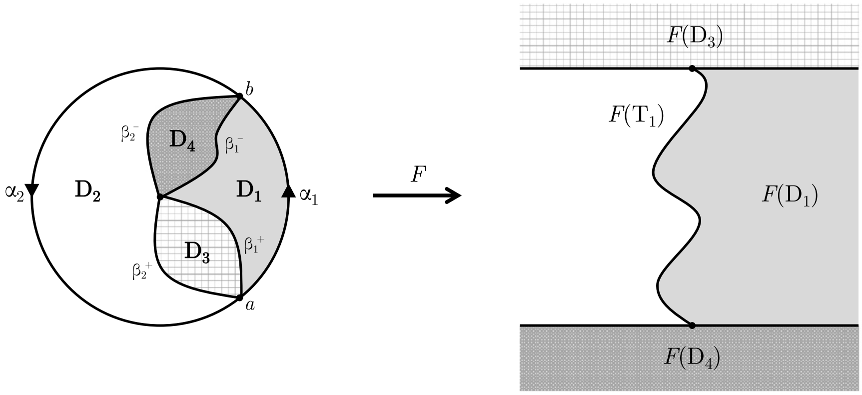

Fig. 2. The Geometry of the Shear F of $f \in \mathcal{H}_u$.

(a) *F is a conformal map from D_3 onto the upper half-plane $\Im W > 1$ and from D_4 onto the lower half-plane $\Im W < -1$.*
(b) *$F(D_1)$ is a CPD domain and $F(D_2)$ is a CND domain.*
(c) *There exist reals s, t such that $\partial F(D_1) = \mathcal{U}_s^+ \cup \mathcal{L}_t^+ \cup \Gamma_1$ and $\partial F(D_2) = \mathcal{U}_s^- \cup \mathcal{L}_t^- \cup \Gamma_2$, where Γ_1 and Γ_2 are the horizontal shears of $\mathbb{T}_1$ and $\mathbb{T}_2$ under F.*
(d) *There exists a positive real δ such that $\Re w \geq -\delta$, $w \in \Gamma_1$, and $\Re w \leq \delta$, $w \in \Gamma_2$.*
(e) *Γ_1 and Γ_2 are continua in $\widehat{\mathbb{C}}$ connecting the horizontal lines $|\Im w| = 1$.*

Proof. In the course of the proof, refer to Fig. 2.

Parts (a) and (b) follow at once from Proposition 2, the mapping properties of the boundaries of $D_j, j = 1, 2, 3, 4$, under f, and the fact that F has a simple pole at the origin.

(c) By Proposition 2 again, there exist $s_1 \in [-\infty, \infty)$ and $s_2 \in (-\infty, \infty]$ such that $\mathcal{U}_{s_1}^+ \subset \partial F(D_1)$ and $\mathcal{U}_{s_2}^- \subset \partial F(D_2)$ respectively. If $s_1 \neq s_2$, then a boundary argument implies that F is not a conformal map of D_1 onto $\Im W > 1$; hence, $s_1 = s_2 \in \mathbb{R}$. By a similar argument, we conclude that there exists a real t such that $\mathcal{L}_t^+ \subset \partial F(D_1)$ and $\mathcal{L}_t^- \subset \partial F(D_2)$. This proves (c).

(d) Since F has a simple pole at ∞, there exist δ_1 and δ_2 such that $\delta_1 > 0 > \delta_2$, $\Re w < \delta_1$ for all $w \in \Gamma_1$ and $\Re w > \delta_2$ for all $w \in \Gamma_2$. Letting $\delta = \min\{\delta_1, |\delta_2|\}$, then $\Re w < \delta$ for all $w \in \Gamma_1$ and $\Re w > -\delta$ for all $w \in \Gamma_2$. This proves (d).

(e) This follows at once since Γ_1 and Γ_2 are the cluster sets of F on $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. This completes the proof of the theorem. $\square$

Let f be a function from $\mathbb{D}$ to $\widehat{\mathbb{C}}$. We say that f has the *angular limit* $a \in \widehat{\mathbb{C}}$ at $\zeta \in \mathbb{T}$ if

$$f(z) \to a \quad \text{as} \quad z \in A, \quad z \to \zeta$$

for every Stolz angle A at ζ; $A = \{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < \pi/2 - \delta\}$ where $0 < \delta < \pi/2$. We shall use the symbol $f(\zeta)$ to denote the angular limit of f at ζ whenever it exists. For this notion see Ch. Pommerenke [7, p. 267].

As a consequence of Theorem 1, we conclude:

Corollary 1. *Let $f \in \mathcal{H}_u$ be written as in (1), and let F be the analytic horizontal shear of f. Then the angular limit $F(\zeta)$ exists for every $\zeta \in \mathbb{T}$.*

Proof. Let $\eta = f^*(\zeta) \in \mathbb{T}$. There exists a Jordan arc $\gamma \subset \mathbb{D} \setminus \{0\}$ terminating in ζ such that it maps under f homeomorphically to a straight line segment in $\mathbb{C} \setminus \overline{\mathbb{D}}$ terminating in η.

By adhering to (3), we conclude that F maps γ into the horizontal line $w = t + i\Im\eta$, $t \in \mathbb{R}$. In fact, by Lemma 1, since F is locally univalent, F maps γ homeomorphically onto a horizontal line segment terminating at an ideal boundary point of $F(\mathbb{D} \setminus \{0\})$. If $\eta \neq \pm i$, then it follows from Theorem 1 that F is univalent in a local neighborhood of ζ, and the angular limit $F(\zeta)$ exists by [7, Corollary 2.17(i)]. If $\eta = i$, then Theorem 1 implies that F is either locally univalent or 2-valent in any local neighborhood of ζ; in particular, there exists a single-valued branch of $\sqrt{F - F(i)}$ which is locally univalent at ζ. Hence, again by Corollary 2.17(i), the angular limit of $\sqrt{F - F(i)}$ at ζ exists [7], and consequently $F(\zeta)$ exists. The latter argument also applies in case $\eta = -i$. This ends the proof. $\square$

3. The analytic and co-analytic parts of $f \in \mathcal{H}_u$

The purpose of this section is to show that the analytic part of $f \in \mathcal{H}_u$ has a simple pole at the origin whereas its co-analytic part may have either a simple pole or a removable singularity at the origin.

Theorem 2. *Let $f \in \mathcal{H}_u$ be written as in (1). Then*

(a) *h has a simple pole at the origin;*
(b) *g is either analytic in $\mathbb{D}$ or has a simple pole at the origin.*

Proof. Let F be the horizontal shear of f. By Lemma 2, we can write

$$F(z) = \frac{A}{z} + k_1(z) \qquad (z \in \mathbb{D} \setminus \{0\})$$

for some complex constant $A \neq 0$ and an analytic function k_1 in $\mathbb{D}$. Then

$$F'(z) = \frac{-A}{z^2} + k_1'(z) \qquad (z \in \mathbb{D} \setminus \{0\}),$$

and

$$(zh'(z) + \alpha) - (zg'(z) + \alpha) = zh'(z) - zg'(z) = \frac{-A}{z} + zk_1'(z) \qquad (z \in \mathbb{D} \setminus \{0\}).$$

Since f is sense-preserving, $|\omega| < 1$ in $\mathbb{D}\setminus\{0\}$, and ω has a removable singularity at the origin with $|\omega(0)| < 1$. We can write

$$(zh'(z) + \alpha)(1 - \omega(z)) = \frac{-A}{z} + zk_1'(z) \quad (z \in \mathbb{D} \setminus \{0\}),$$

and

$$zh'(z) = -\alpha + \frac{1}{1 - \omega(z)} \left(\frac{-A}{z^2} + k_1'(z) \right).$$

By replacing $1/(1 - \omega)$ by its Maclaurin's series, we obtain

$$\begin{aligned} h'(z) &= \frac{-\alpha}{z} + \left(\frac{1}{1 - \omega(0)} + \frac{\omega'(0)}{(1 - \omega(0))^2} z + k_2(z) \right) \left(-\frac{A}{z^2} + k_1'(z) \right) \\ &= \left(\frac{-A}{1 - \omega(0)} \right) \frac{1}{z^2} - \left(\alpha + \frac{A\omega'(0)}{(1 - \omega(0))^2} \right) \frac{1}{z} + k_3(z), \end{aligned} \tag{4}$$

where k_2 and k_3 are analytic functions in $\mathbb{D}$. But h is an analytic function of $\mathbb{D} \setminus \{0\}$; hence

$$\alpha + \frac{A\omega'(0)}{(1-\omega(0))^2} = 0. \tag{5}$$

Here we consider two cases.

Case 1. $\alpha = 0$. Then, by (5) and (4), we obtain $\omega'(0) = 0$ and

$$h'(z) = \left(\frac{-A}{1-\omega(0)}\right)\frac{1}{z^2} + k_3(z).$$

It follows that

$$\begin{aligned} g'(z) &= h'(z)\omega(z) \\ &= \left[\left(\frac{-A}{1-\omega(0)}\right)\frac{1}{z^2} + k_3(z)\right]\omega(z) \\ &= \frac{-A\omega(0)}{z^2} + k_4(z), \end{aligned}$$

where k_4 is an analytic function in $\mathbb{D}$. Since $A \neq 0$, we conclude that h has a simple pole at the origin, and that g is likewise unless $\omega(0) = 0$ in which case g is analytic in $\mathbb{D}$.

Case 2. $\alpha \neq 0$. Then, by (5), $\omega'(0) \neq 0$ and

$$A = \frac{-\alpha(1-\omega(0))^2}{\omega'(0)}.$$

By (4), we obtain

$$h'(z) = \frac{\alpha(1-\omega(0))}{\omega'(0)}\frac{1}{z^2} + k_3(z).$$

By using the Maclaurin's series of $\omega(z)$ and that $zg'(z) + \alpha = (zh'(z) + \alpha)\omega(z)$, it follows that

$$\begin{aligned} g'(z) &= \left(h'(z) + \frac{\alpha}{z}\right)\omega(z) - \frac{\alpha}{z} \\ &= \frac{\alpha\omega(0)(1-\omega(0))}{\omega'(0)}\frac{1}{z^2} + k_4(z), \end{aligned}$$

where k_4 is an analytic function of $\mathbb{D}$. Again, as in Case 1, we conclude that h has a simple pole at the origin, and that g is likewise unless $\omega(0) = 0$ in which case g is analytic in $\mathbb{D}$. This ends the proof. $\square$

4. Analytic characterization of the shear of $f \in \mathcal{H}_u$

Let $f \in \mathcal{H}_u$ be as given in (2). In this section, we establish an interesting analytic characterization of the shear F of f.

For $z_0 \neq 0$, we introduce the function

$$\Psi(z; z_0) = \frac{(z - z_0)(1 - \overline{z_0}z)}{z}$$

which is analytic in $\mathbb{C}$ except for a simple pole at the origin with residue $-z_0$, and maps $\mathbb{T}$ onto the interval $[(1 - |z_0|)^2, (1 + |z_0|)^2]$.

Theorem 3. *Let $f \in \mathcal{H}_u$, F the horizontal shear of f, and $A = \operatorname{Res}(F; 0)$. Then*

$$F(z) = \Phi(z) - \Psi(z; A) \qquad (z \in \mathbb{D}),$$

where Φ is a univalent function convex in the horizontal direction. Furthermore,

$$\Phi(z) = -\frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \Im f^*(e^{it})\, dt + c \qquad (z \in \mathbb{D}) \tag{6}$$

for some real constant c.

Proof. As in the proof of Lemma 2, let $a = (f^*)^{-1}(i)$ and $b = (f^*)^{-1}(-i)$, α_1 be the subarc obtained by traversing the unit circle positively from a to b, and α_2 be the subarc obtained by traversing the unit circle positively from b to a. Since $f^* : \mathbb{T} \to \mathbb{T}$ is a sense-reversing homeomorphism, $\Im f^* : \alpha_1 \to [-1, 1]$ is a strictly decreasing homeomorphism and $\Im f^* : \alpha_2 \to [-1, 1]$ is a strictly increasing homeomorphism. In particular, $\Im f^*$ takes on every value of the open interval $(-1, 1)$ exactly twice in $\mathbb{T} \setminus \{a, b\}$.

It follows from (2) that $\Im f(z) = \Im F(z)$ for every $z \in \mathbb{D} \setminus \{0\}$. Since $\Im f(z)$ is continuous on $\overline{\mathbb{D}} \setminus \{0\}$, $\Im F(z)$ extends continuously to $\overline{\mathbb{D}} \setminus \{0\}$ such that $\Im f^* = \Im F^*$; in particular, $\Im F^* : \alpha_1 \to [-1, 1]$ is a strictly decreasing homeomorphism and $\Im F^* : \alpha_2 \to [-1, 1]$ is a strictly increasing homeomorphism.

Let $\Phi(z) = F(z) + \Psi(z; A)$, $z \in \mathbb{D}$. It is immediate that Φ is an analytic function of $\mathbb{D}$ as F and Ψ have simple poles at the origin with opposite residues there. Since $\Im \Psi$ is identically zero on $\mathbb{T}$, we infer that $\Im \Phi$, like $\Im F$, extends continuously to $\mathbb{T}$ such that $\Im \Phi^* = \Im F^*$ and $\Im \Phi^*$ takes on every value of $(-1, 1)$ exactly twice.

We contend that $\Phi(\mathbb{D})$ lies in the horizontal strip $S = \{w : |\Im w| < 1\}$. Suppose otherwise, then there exists $z_0 \in \mathbb{D}$ such that $\Phi(z_0) \in \mathbb{C} \setminus \overline{S}$. Consider the semi-line $L : \Phi(z_0) + t$, $t \geq 0$, and observe that $L \subset \mathbb{C} \setminus \overline{S}$ and $\Phi^*(\mathbb{T}) \subset S$. There exists a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D} \setminus \{0\}$ and a point $\zeta \in \mathbb{T}$ such that $\Phi(z_n) \in L$, $z_n \to \zeta$, and $\Im \Phi(z_0) = \lim_{n \to \infty} \Im \Phi(z_n) = \Im \Phi^*(\zeta)$. But $|\Im \Phi^*(\zeta)| \leq 1$ since $\Phi^*(\mathbb{T}) \subset S$; hence $|\Im \Phi(z_0)| \leq 1$, and we have a contradiction which proves that our contention is true.

We show now that Φ is locally univalent in $\mathbb{D}$. Suppose not, then there exists $z_0 \in \mathbb{D}$ such that $\Phi'(z_0) = 0$ and $|\Im \Phi(z_0)| < 1$. Let ℓ be the horizontal line passing through $\Phi(z_0)$. Since $\Phi'(z_0) = 0$, there exist two crosscuts γ_1 and γ_2 of $\mathbb{D}$ crossing at z_0 and are otherwise disjoint. For if one of the curves γ_1 or γ_2 intersects itself, or if γ_1 and γ_2 share a common point in $\overline{D}$ other than z_0, then we would obtain a Jordan subdomain, G, of $\mathbb{D}$ whose boundary maps under Φ into ℓ. This implies that $\Phi(G)$ contains $\mathbb{C} \setminus \overline{S}$, and we have a contradiction. We conclude that γ_1 and γ_2 are crosscuts of $\mathbb{D}$ that meet only at z_0 in $\overline{\mathbb{D}}$. Denote by ζ an endpoint of either γ_1 or γ_2. It follows at once that $\Im \Phi^*(\zeta) = \Im \Phi(z_0)$. But since the endpoints of γ_1 and γ_2 are distinct, we conclude that $\Im \Phi^*$ takes on the value $\Im \Phi(z_0)$ exactly four times in $\mathbb{T} \setminus \{a, b\}$, which yields a contradiction. Therefore, Φ is locally univalent in $\mathbb{D}$.

Next, we show that Φ is univalent in $\mathbb{D}$. Suppose to the contrary that there exist distinct values z_1 and z_2 such that $\Phi(z_1) = \Phi(z_2)$. Let ℓ be the horizontal line passing through $\Phi(z_1)$. Since Φ is locally univalent in $\mathbb{D}$, one of the following two cases arise.

Case 1. $\Phi^{-1}(\ell)$ contains a crosscut γ passing through z_1 and z_2. Since $\Phi(\gamma) \subset \ell$, there exists a point $z_3 \in \gamma$ lying in the subarc of γ terminating at z_1 and z_2 at which $\Re \Phi$ admits an extreme value; consequently Φ admits a critical point in $\mathbb{D}$ thus contradicting the fact that Φ is locally univalent.

Case 2. $\Phi^{-1}(\ell)$ contains two crosscuts γ_1 and γ_2 passing through z_1 and z_2. By invoking the fact that $\Im \Phi^*$ takes on every value in $[-1, 1]$ at most twice, we argue as above to conclude that γ_1 and γ_2 are crosscuts of $\mathbb{D}$ that can neither meet in $\mathbb{D}$ nor have different terminal points in $\mathbb{T}$. This once again would yield a Jordan subdomain, G, of $\mathbb{D}$ whose boundary lies in $\gamma_1 \cup \gamma_2$ and maps under Φ into ℓ; hence $\Phi(G)$ contains $\mathbb{C} \setminus \overline{S}$, and we have a contradiction. Thus Φ is a univalent function of $\mathbb{D}$.

In view of the argument in Case 2, we also conclude that for any horizontal line ℓ in the strip S, $\Phi^{-1}(\ell)$ is a crosscut of $\mathbb{D}$. Hence, Φ is a univalent function of $\mathbb{D}$ convex in the horizontal direction.

It remains to establish (6). Since $\Im\Phi^* = \Im f^*$ and that each of $\Im\Phi^* : \alpha_j \to [-1, 1], j = 1, 2$, is a homeomorphism, by Poisson's theorem [1, pp. 166-168], we can write:

$$\begin{aligned}\Im\Phi(z) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\right\}\Im\Phi^*(e^{it})\,dt\\ &= \Re\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\Im f^*(e^{it})\,dt\right)\\ &= \Im\left(-\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\Im f^*(e^{it})\,dt\right) \qquad (z\in\mathbb{D}).\end{aligned}$$

Therefore, (6) holds and the proof is complete. □

5. The class $\mathcal{H}_u$ and the Radó-Kneser-Choquet Theorem

In this section, we establish a strong relationship between the class $\mathcal{H}_u$ and the class of harmonic homeomorphisms from $\mathbb{D}$ onto itself that extend to sense-preserving homeomorphisms from $\mathbb{T}$ onto itself.

For convenience, we begin with the following definition.

Definition 3. Let $f^* : \mathbb{T} \to \mathbb{T}$ be a sense-preserving homeomorphism. By the Radó-Kneser-Choquet Theorem [5, pp. 29-34], f^* extends continuously to the function

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}f^*(e^{it})\,dt \qquad (z\in\mathbb{D})$$

which is a harmonic homeomorphism of $\mathbb{D}$ onto itself. We call f the *harmonic extension of* f^* *to* $\mathbb{D}$.

Let $f \in \mathcal{H}_u$ be written as in (2), and let f^* be its boundary function. Then $1/f^* : \mathbb{T} \to \mathbb{T}$ is a sense-preserving homeomorphism, and the function

$$\chi(z) = H + \overline{G} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\,\frac{1}{f^*(e^{it})}\,dt, \tag{7}$$

where H and G are analytic functions of $\mathbb{D}$, is the harmonic extension of $1/f^*$ to $\mathbb{D}$. The following theorem establishes an interesting relationship between f and χ.

Theorem 4. *Let $f \in \mathcal{H}_u$, F the horizontal shear of f, χ the harmonic extension of $1/f^*$ given in (7), and $A = \operatorname{Res}(F; 0) = a + b$, where $a = \operatorname{Res}(h; 0) \neq 0$, $b = \operatorname{Res}(g; 0)$. Then the following hold.*

(a) *There exists a real ρ such that*

$$F(z) + \Psi(z; A) = G(z) - H(z) + \rho \qquad (z\in\mathbb{D}).$$

(b) *There exists a constant ϱ, with $\Re\,\varrho = (\rho - (1+|A|^2))/2$, such that*

$$h(z) = \frac{a}{z} + \overline{b}z + G(z) + \varrho \text{ and } g(z) = -\frac{b}{z} - \overline{a}z + H(z) - \overline{\varrho} \qquad (z \in \mathbb{D}).$$

Proof. (a) By (7), we can write:

$$\begin{aligned}
\chi(z) &= H + \overline{G} \\
&= \frac{1}{4\pi}\int_0^{2\pi} \left\{ \frac{e^{it}+z}{e^{it}-z} + \overline{\left(\frac{e^{it}+z}{e^{it}-z}\right)} \right\} \frac{1}{f^*(e^{it})}\,dt \\
&= \frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{1}{f^*(e^{it})}\,dt + \overline{\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{1}{\overline{f^*(e^{it})}}\,dt}.
\end{aligned}$$

It follows that there exists a complex constant d such that

$$H = \frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{1}{f^*(e^{it})}\,dt + d \qquad (z \in \mathbb{D})$$

and

$$G = \frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{1}{\overline{f^*(e^{it})}}\,dt - \overline{d} \qquad (z \in \mathbb{D}).$$

Therefore, by (6),

$$\begin{aligned}
G(z) - H(z) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\Im\left(\frac{1}{f^*(e^{it})}\right)dt - 2\Re\,d \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\Im f^*(e^{it})\,dt - 2\Re\,d \\
&= \Phi(z) - c - 2\Re\,d \\
&= F(z) + \Psi(z;A) - \rho,
\end{aligned}$$

where $\rho = c + 2\Re\,d$. This proves (a).

(b) By using (a), we can write

$$\left(h(z) - \frac{a}{z} - \overline{b}z\right) - \left(g(z) + \frac{b}{z} + \overline{a}z\right) = G(z) - H(z) + c \qquad (z \in \mathbb{D}), \tag{8}$$

where $c = \rho - (1+|A|^2)$. Given that $a = \mathrm{Res}(h;0) \neq 0$ and $b = \mathrm{Res}(g;0)$, we may assume that there exist analytic functions μ and ν of $\mathbb{D}$ such that

$$h(z) = \frac{a}{z} + \overline{b}z + G(z) + \mu(z) \text{ and } g(z) = -\frac{b}{z} - \overline{a}z + H(z) + \nu(z) \qquad (z \in \mathbb{D}). \tag{9}$$

Then, by (8), we obtain $\mu - \nu = c$ on $\mathbb{D}$. On the other hand, $h + \overline{g}$ and $G + \overline{H}$ are harmonic maps of $\mathbb{D} \setminus \{0\}$ and $\mathbb{D}$, respectively, that extend continuously to $\mathbb{T}$ such that

$$(h + \overline{g})(\zeta) = (G + \overline{H})(\zeta) \qquad (\zeta \in \mathbb{T}).$$

In view of (9), this implies that the function $\mu + \overline{\nu}$ is a harmonic map of $\mathbb{D}$ that extends continuously to $\mathbb{T}$ such that $(\mu + \overline{\nu})(\zeta) = 0$ for every $\zeta \in \mathbb{T}$. But $(\mu - \nu)(\zeta) = \rho - (1 + |A|^2)$ for every $\zeta \in \mathbb{T}$. Since $\mu - \nu$ is real and both functions $\mu + \overline{\nu}$ and $\mu - \nu$ are continuous on $\overline{\mathbb{D}}$, we conclude that

$$\Re\, \mu = \frac{(\overline{\mu} - \overline{\nu}) + (\mu + \overline{\nu})}{2}$$

is a harmonic function of $\mathbb{D}$ that extends continuously to $\mathbb{T}$ such that $\Re\, \mu(\zeta) = (\rho - (1 + |A|^2))/2$ on $\mathbb{T}$. Hence $\Re\, \mu = (\rho - (1 + |A|^2))/2$ on $\overline{\mathbb{D}}$ by the maximum principle, and μ is a constant function, say $\mu = \varrho$ on $\overline{\mathbb{D}}$. This implies that ν is also a constant function on $\overline{\mathbb{D}}$, in particular $\nu = -\overline{\varrho}$ since $\mu + \overline{\nu} = 0$. This completes the proof. □

Corollary 2. *Let $f^* : \mathbb{T} \to \mathbb{T}$ be a sense-reversing homeomorphism. Suppose that f^* extends to a homeomorphism $f : \overline{\mathbb{D}} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}$ such that $\lim_{z\to 0} f(z) = \infty$ and $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is a harmonic mapping. Also, suppose that $\chi = H + \overline{G}$ is the harmonic extension of $1/f^*$ to $\mathbb{D}$ as given in (7). Then there exist complex numbers $a \neq 0, b, \alpha$ such that*

$$f(z) = \frac{a}{z} + \overline{b}z + G(z) + \overline{-\frac{b}{z} - \overline{a}z + H(z)} + \alpha \ln |z|^2 \qquad (z \in \mathbb{D} \setminus \{0\}). \tag{10}$$

Proof. There exists a real ϕ such that $e^{i\phi} f \in \mathcal{H}_u$. Then, by Theorem 4, there exist real values $a_1 \neq 0, b_1, \rho$ and real α_1 such that

$$e^{i\phi} f(z) = h_1(z) + \overline{g_1(z)} + \alpha_1 \ln |z|^2, \tag{11}$$

and

$$h_1(z) = \frac{a_1}{z} + b_1 z + G_1(z) + \varrho \text{ and } g_(z) = -\frac{b_1}{z} - a_1 z + H_1(z) - \overline{\varrho} \qquad (z \in \mathbb{D}),$$

where $\chi_1 = H_1 + \overline{G_1}$ is the harmonic extension of $1/(e^{i\phi} f^*(\zeta))$ to $\mathbb{D}$.

On other hand, let $\chi = H + \overline{G}$ be the homeomorphism from $\overline{\mathbb{D}}$ which is harmonic in $\mathbb{D}$ and satisfies $\chi(\zeta) = 1/f^*(\zeta)$ for every $\zeta \in \mathbb{T}$. Then, by (7), we have:

$$\begin{aligned} H_1(z) + \overline{G_1(z)} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \frac{1}{e^{i\phi} f^*(e^{it})}\, dt \\ &= e^{-i\phi} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \frac{1}{f^*(e^{it})}\, dt \\ &= e^{-i\phi} (H(z) + \overline{G(z)}) \qquad (z \in \mathbb{D}), \end{aligned}$$

and $H = e^{i\phi} H_1$ and $G = e^{-i\phi} G_1$.

By virtue of Proposition 1, we assume that f is written as in (1). But (11) yields

$$f(z) = e^{-i\phi} h_1(z) + \overline{e^{i\phi} g_1(z)} + e^{-i\phi} \alpha_1 \ln |z|^2.$$

It follows that there exists a real c such that $h = e^{-i\phi}h_1 + c$, $g = e^{i\phi}g_1 - c$, and $\alpha = e^{-i\phi}\alpha_1$, and consequently f can be written as desired in (10). This completes the proof. $\square$

Before giving examples on Theorem 4, we give the following useful result.

Proposition 3. *Suppose that $f^* : \mathbb{T} \to \mathbb{T}$ is a sense-reversing homeomorphism such that $f^*(\overline{\zeta}) = \overline{f^*(\zeta)}$, and that there exists a harmonic homeomorphism $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ of form (1) such that $\lim_{z\to 0} f(z) = \infty$ and that f extends continuously to $\mathbb{T}$ where it coincides with f^*. Then $f \in \mathcal{H}_u$, $h(z) = \overline{h(\overline{z})}$ and $g(z) = \overline{g(\overline{z})}$ for all $z \in \mathbb{D} \setminus \{0\}$. Furthermore, $a = \operatorname{Res}(h; 0)$ and $b = \operatorname{Res}(g; 0)$ are real.*

Proof. According to (1),

$$f(z) = h(z) + \overline{g(z)} + \alpha \ln |z|^2 \qquad (z \in \mathbb{D} \setminus \{0\}),$$

for some $\alpha \in \mathbb{C}$. Note that

$$\overline{f(\overline{z})} = \overline{h(\overline{z})} + g(\overline{z}) + \overline{\alpha} \ln |z|^2 \qquad (z \in \mathbb{D} \setminus \{0\})$$

is a sense-preserving harmonic homeomorphism of $\mathbb{D} \setminus \{0\}$ with boundary function that of f. Hence, $f(z) - \overline{f(\overline{z})}$ is a harmonic mapping of $\mathbb{D} \setminus \{0\}$ that extends continuously to $\overline{\mathbb{D}} \setminus \{0\}$ such that it is identically zero on $\mathbb{T}$. It can be shown that there exists $\beta \in \mathbb{C}$ such that $f(z) - \overline{f(\overline{z})} = \beta \ln |z|^2$. Consequently,

$$h(z) - \overline{h(\overline{z})} + (2i\Im\alpha - \beta) \ln z = \overline{\overline{g(\overline{z})} - g(z) + (\overline{\beta} + 2i\Im\alpha) \ln z} \qquad (z \in \mathbb{D} \setminus \{0\}).$$

As one side of the equation is analytic and the other is anti-analytic, there is a real constant c such that

$$h(z) - \overline{h(\overline{z})} + (2i\Im\alpha - \beta) \ln z = c = \overline{g(\overline{z})} - g(z) + (\overline{\beta} + 2i\Im\alpha) \ln z \qquad (z \in \mathbb{D} \setminus \{0\}).$$

But since $h(z) - \overline{h(\overline{z})}$ and $\overline{g(\overline{z})} - g(z)$ are both single-valued analytic,

$$2i\Im\alpha - \beta = 0 = \overline{\beta} + 2i\Im\alpha,$$

$\beta = \Im\alpha = 0$ which gives

$$h(z) - \overline{h(\overline{z})} = c = \overline{g(\overline{z})} - g(z) \qquad (z \in \mathbb{D} \setminus \{0\}).$$

Letting z be real, we conclude that $c = 0$; hence $h(z) = \overline{h(\overline{z})}$ and $g(z) = \overline{g(\overline{z})}$. As for $a = \operatorname{Res}(h; 0)$ and $b = \operatorname{Res}(g; 0)$ being real is immediate, and the proof is complete. $\square$

Example 1. Suppose that there exists a harmonic extension f to $\mathbb{D} \setminus \{0\}$ of the boundary function $f^*(\zeta) = 1/\zeta, \zeta \in \mathbb{T}$. Then, by Proposition 3, $f \in \mathcal{H}_u$. Since the identity function is the harmonic extension of $1/f^*$ to $\mathbb{D}$ with analytic and co-analytic parts H and G, we obtain

$$H(z) = z \;\text{ and }\; G(z) = 0 \qquad (z \in \mathbb{D}).$$

By invoking Theorem 4, we conclude that there exist reals $a \neq 0, b$ and α such that

$$f(z) = \frac{a}{z} + bz - \overline{\frac{b}{z} + (a-1)z} + \alpha \ln |z|^2 \qquad (z \in \mathbb{D} \setminus \{0\}). \tag{12}$$

Example 2. Fix $z_0 \in \mathbb{D}$, and assume that there exists $f \in \mathcal{H}_u$ that extends harmonically the boundary function $f^*(\zeta) = (1 - \overline{z_0}\zeta)/(\zeta - z_0), \zeta \in \mathbb{T}$. Since the Möbius transformation $(z - z_0)/(1 - \overline{z_0}z)$ is the harmonic extension of $1/f^*$ to $\mathbb{D}$ with analytic and co-analytic parts H and G, we obtain

$$H(z) = \frac{z - z_0}{1 - \overline{z_0}z} \text{ and } G(z) = 0 \qquad (z \in \mathbb{D}).$$

Then, by Theorem 4, we conclude that there exist complex numbers $a \neq 0$ and b and real α such that

$$f(z) = \frac{a}{z} + \bar{b}z + \overline{\frac{-b}{z} - \bar{a}z + \frac{z - z_0}{1 - \overline{z_0}z}} + \alpha \ln |z|^2. \tag{13}$$

Theorem 4 leads to the following necessary and sufficient conditions for a sense-reversing homeomorphism $f^* : \mathbb{T} \to \mathbb{T}$ to extend to a homeomorphism $f : \overline{\mathbb{D}} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}$ which is harmonic in $\mathbb{D} \setminus \{0\}$.

Theorem 5. *Let $f^* : \mathbb{T} \to \mathbb{T}$ be a sense-reversing homeomorphism, and let χ be the harmonic extension of $1/f^*$ in $\mathbb{D}$ as given by (7). Then $f^* : \mathbb{T} \to \mathbb{T}$ extends to a homeomorphism $f : \overline{\mathbb{D}} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}$ which is harmonic in $\mathbb{D} \setminus \{0\}$ if, and only if, f has form (10) and*

$$\left| \frac{b + \alpha z - \bar{a}z^2 + z^2 H'(z)}{-a + \alpha z + \bar{b}z^2 + z^2 G'(z)} \right| < 1 \qquad (z \in \mathbb{D} \setminus \{0\}) \tag{14}$$

for some complex numbers $a \neq 0, b$, and α.

Proof. The "only if" part follows at once from Corollary 2 and the fact that (14) stipulates that the dilatation of f in $\mathbb{D} \setminus \{0\}$ has modulus less than 1 which is true as f is a homeomorphism in $\mathbb{D} \setminus \{0\}$.

Conversely, suppose that $f^* : \mathbb{T} \to \mathbb{T}$ extends to a homeomorphism $f : \overline{\mathbb{D}} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}$ which is harmonic in $\mathbb{D} \setminus \{0\}$ and has form (10), and that (14) is satisfied. Then f is a local homeomorphism in $\mathbb{D} \setminus \{0\}$ except possibly for branch points.

By using (10), we can write

$$f(re^{i\theta}) = (ae^{-i\theta} - \bar{b}e^{i\theta}) \left(\frac{1}{r} - r \right) + 2\alpha \ln r + \overline{H(re^{i\theta}) + \overline{G(re^{i\theta})}} \qquad (0 < r < 1),$$

which implies that $\lim_{z \to 0} f(z) = \infty$.

Thus $1/f$ is an open map in $\mathbb{D} \setminus \{0\}$ that extends continuously to $\overline{\mathbb{D}}$ such that $\lim_{z \to 0} 1/f(z) = 0$ and $1/f^* : \mathbb{T} \to \mathbb{T}$ is a sense-preserving homeomorphism. It follows that $1/f$ is bounded in $\overline{\mathbb{D}}$ and $|1/f(z)| < 1$ for all $z \in \mathbb{D}$.

We conclude that the pair $(\mathbb{D} \setminus \{0\}, 1/f)$ is a complete covering of $\mathbb{D} \setminus \{0\}$ which covers every point of $\mathbb{D} \setminus \{0\}$ the same number of times, n, provided that the branch points are counted as many times as their multiplicities [2, pp. 41-43]. Since $1/f$ extends continuously to a homeomorphism from $\mathbb{T}$ onto itself, it follows that $n = 1$, and that $1/f : \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$ is a homeomorphism. Therefore, $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is a harmonic homeomorphism that extends continuously to $\overline{\mathbb{D}}$ as desired such that $\lim_{z \to 0} f(z) = \infty$ and $f : \mathbb{T} \to \mathbb{T}$ is a sense-reversing homeomorphism. This completes the proof. $\square$

Theorem 5 provides an effective way for constructing a harmonic homeomorphism f from the punctured disc $\mathbb{D} \setminus \{0\}$ onto its exterior $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that $\lim_{z \to 0} f(z) = \infty$ and f extends continuously to a prescribed sense-reversing homeomorphism from $\mathbb{T}$ onto itself.

Example 3. We construct infinitely many harmonic homeomorphisms $f \in \mathcal{H}_u$ for which $f^*(\zeta) = 1/\zeta$ for all $\zeta \in \mathbb{T}$. According to Example 1, the functions f must be of form (10). By Theorem 5 and Proposition 3, f belongs to $\mathcal{H}_u$ if there exist real values $a \neq 0, b$, and α that satisfy

$$\left|\frac{b+\alpha z-(a-1)z^2}{-a+\alpha z+bz^2}\right|<1 \qquad (z\in\mathbb{D}). \tag{15}$$

Some of the cases for which (15) can be verified to be true are the following:

(a) $a = 1$ and $b = \alpha = 0$. In this case $f(z) = 1/z$.
(b) $a = 2, b = 1$, and $\alpha = 0$. In this case $f(z) = 2/z + z - \overline{1/z + 2z}$.
(c) $a = 1, b = 1/2, \alpha = 0$. In this case $f(z) = 1/z + z/2 - \overline{1/2z}$.
(d) $a = 1, b = 0, \alpha \in [0, 1/2]$. In this case $f(z) = 1/z + \alpha \ln|z|^2$.

Example 4. Following Example 2, we attempt to construct for a given $z_0 \in \mathbb{D}$ harmonic homeomorphisms $f \in \mathcal{H}_u$ of form (2) for which $f^*(\zeta) = (1-\overline{z_0}\zeta)/(\zeta - z_0)$ for every $\zeta \in \mathbb{T}$. By Theorem 5, f belongs to $\mathcal{H}_u$ if there exist complex values $a \neq 0, b$, and real α that satisfy

$$\left|\frac{-\overline{a}z^2+\alpha z+b+(1-|z_0|^2)[z/(1-\overline{z_0}z)]^2}{-\overline{b}z^2+\alpha z-a}\right|<1 \qquad (z\in\mathbb{D}).$$

We consider only the case when $z_0 = 1/R, R > 1$; in this case a, b, and α are real by Proposition 3. Then, by (10),

$$f(z)=\frac{a}{z}+bz+\overline{\frac{-b}{z}-az+\frac{1-Rz}{z-R}}+\alpha\ln|z|^2 \qquad (z\in\mathbb{D}\setminus\{0\}).$$

Assuming $b = 0$, then

$$f(z)=\frac{a}{z}+\overline{-az+\frac{1-Rz}{z-R}}+\alpha\ln|z|^2 \qquad (z\in\mathbb{D}\setminus\{0\}). \tag{16}$$

Now we consider two cases:

Case 1. $\alpha = 0$. Then

$$f(z)=\frac{a}{z}+\overline{-az+\frac{1-Rz}{z-R}} \qquad (z\in\mathbb{D}\setminus\{0\}), \tag{17}$$

and, by Theorem 5, f belongs to $\mathcal{H}_u$ if there exists a real $a \neq 0$ such that $|q(z)| < 1, z \in \mathbb{D}$, where

$$q(z)=1-\left(\frac{R^2-1}{a}\right)\left(\frac{1}{R-z}\right)^2 \qquad (z\in\mathbb{D}). \tag{18}$$

Observe that the circle $1/(R - e^{i\theta})$, $0 \le \theta \le 2\pi$, has center at $R/(R^2 - 1)$, radius of length $1/(R^2 - 1)$, and lie in the wedge $\{z : |\arg z| \le \sin^{-1}(1/R)\}$, where $\arg(.)$ is the principle argument. Then the curve $C : (R^2 - 1)/(R - e^{i\theta})^2$, $0 \le \theta \le 2\pi$, is a Jordan analytic curve that is symmetric about the real axis, lies in the closed wedge $W = \{z : |\arg z| \le 2\sin^{-1}(1/R)\}$, and meets the real axis at the points $(R-1)/(R+1)$ and $(R+1)/R-1)$. Here we consider two cases.

(i) $R > \sqrt{2}$. In this case, $W \setminus \{0\}$ falls within the open right-half plane, and for values a sufficiently larger than $(R + 1)/(R - 1)$ there exists $k, 0 < k < 1$ such that the curve $a^{-1}C$ falls inside the circle centered at 1 and has radius $1 - k$. This proves at once that $|q(z)| < 1 - k$, and consequently there exists $f \in \mathcal{H}_u$ of form (16).

(ii) $1 < R \le \sqrt{2}$. In this case, the closed right-half plane lies in W, in particular C meets the imaginary axis away from the origin. It is immediate that for every $a \neq 0$ the curve $a^{-1}C$ meets the imaginary axis away from the origin. It follows that $|q(z)| < 1$ fails to hold in $\mathbb{D}$, and that there does not exist $f \in \mathcal{H}_u$ of form (16).

Case 2. $\alpha \neq 0$. Then f has form (16), and, by Theorem 5, f belongs to $\mathcal{H}_u$ if there exist nonzero reals a, α such that $|\omega| < 1$, where ω is the dilatation of f. We may write

$$\begin{aligned}\omega(z) &= \frac{-az^2 + \alpha z + (R^2-1)(z/(z-R))^2}{-a+\alpha z} \\ &= \frac{z}{z-(a/\alpha)} - \frac{a/\alpha}{z-(a/\alpha)} z^2 q(z).\end{aligned} \tag{19}$$

Obviously, a and α must be chosen so that $|a/\alpha| > 1$ otherwise ω would not be analytic in $\mathbb{D}$. Here we only consider the case $R > \sqrt{2}$ for which we conclude as above in Case 1(i) that $|q(z)| < 1 - k$ for some $0 < k < 1$. Then, by fixing a and taking α sufficiently close to zero, we can find values $k', k < k' < 1$, such that $|\omega(z)| < k' < 1$ for all $z \in \mathbb{D}$. Therefore, whenever $R > \sqrt{2}$ there exists $f \in \mathcal{H}_u$ of form (16) with $\alpha \neq 0$.

As for the case when $1 < R \leq \sqrt{2}$, no conclusion has been reached.

In 2020, D. Kalaj [6] introduced the following conjecture.

Conjecture 1. *Let $f^* : \mathbb{T} \to \mathbb{T}$ be a sense-preserving homeomorphism. Then f^* extends continuously to a homeomorphism $f : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ which is harmonic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that $f(\zeta) = \infty$ and $f(\infty) = \eta$ for points $\zeta, \eta \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

By pre-composing the homeomorphism $f : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ by an appropriate Möbius transformation, Conjecture 1 can be reformulated as follows.

Conjecture 2. *Let $f^* : \mathbb{T} \to \mathbb{T}$ be a sense-reversing homeomorphism. Then f^* extends continuously to a homeomorphism $f : \overline{\mathbb{D}} \setminus \{0\} \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ such that f is harmonic in $\mathbb{D} \setminus \{0\}$ and $\lim_{z \to 0} f(z) = \infty$.*

Suppose now that f^* is given as in Conjecture 2, and that $\chi = H + \overline{G}$ is the harmonic extension of $1/f^*$ to $\mathbb{D}$ as given by (7). Then Corollary 2 and Theorem 5 imply that the conjecture is true if, and only if, there exist complex constants $a \neq 0, b, \alpha$ such that (14) is true.

References

[1] L.V. Ahlfors, Complex Analysis, third edition, McGraw-Hill, New York, 1979.
[2] L.V. Ahlfors, L. Sario, Riemann Surfaces, Princeton Univ. Press, Princeton, N.J., 1974.
[3] D. Bshouty, W. Hengartner, Univalent solutions of the Dirichlet problem for ring domains, Complex Var. Theory Appl. 21 (1993) 159–169, MR 95h:30018, Zbl 794.30014.
[4] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A.I. 9 (1984) 3–25.
[5] P. Duren, Harmonic Mappings in the Plane, first edition, Cambridge University Press, 2004.
[6] D. Kalaj, Quasiconformal harmonic mappings between domains containing infinity, Bull. Aust. Math. Soc. 102 (2020) 109–117.
[7] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag, Berlin, 1992.