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# An Epsilon Bargaining Game-Theoretic Formulation Between Carrier and Container Terminal Operators for Servicing Vessels During Unloading Operations

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*Due to globalization trends and the increasing competition between ports, the maritime policy for container shipments has witnessed a change in operations that resulted in less reliance on direct freight flows and higher transshipment operations. Motivated to investigate a soft intelligent decision-making approach using game theory in the context of servicing vessels during unloading operations in transshipment, we propose an epsilon bargaining approach between the carrier and the container terminal operator (CTO). The objective of the game is to maximize the carrier service level while minimizing operation costs for the CTO. The players' utilities, which depend on the service level and the fees for the carrier, as well as the revenues generated and the cost incurred for the CTO, are uniquely formulated and evaluated in a bargaining scenario using an ordinal ranking approach. The negotiation process is further improved between the two players based on our proposed Epsilon Bargaining Equilibrium, which to the best of our knowledge has not been used in maritime transportation problems. Results from a risk aversion case illustrate the value of the soft computing mathematical model that we formulated and motivate follow-up research.*

**Keywords** Bargaining Problem; Container Terminal; Epsilon Equilibrium; Game Theory; Unloading Operations in Transshipment; Vessel Service

## INTRODUCTION

The concept of globalization has shifted a large fraction of maritime shipments of containers from direct freight flows into transshipment flows, and has increased the investments of shipping companies in container terminals in order to sustain their current markets shares and to take over new markets (Nehme & Awad, 2010; Van de Voorde, 2009). A typical container terminal operates as illustrated in Figure 1, which was proposed by Ng and Mak (2005).

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Color versions of one or more of the figures in the article can be found online at [www.tandfonline.com/gits](http://www.tandfonline.com/gits).

A container terminal is normally divided into two main areas: the quay side and the yard side. Containers at a terminal are divided into three categories: imported, exported, and transshipped. The loading and unloading of containers to and from vessels are processed in the quay side. Once the vessel arrives in the quay side, the specified quay cranes (QC) unload both the transshipped and the imported containers into internal trucks to be forwarded to the yard. The specified yard cranes (YC) store the transshipped containers in the storage area and diffuse the imported containers into the customer-specific trucks to be logged out from the container terminal. The exported and the transshipped containers are discharged from the yard side via internal trucks to be unloaded by the quay cranes into the vessel (Choo, 2006).

To be able to attract more container vessels, container terminal operators (CTOs) around the world usually try to

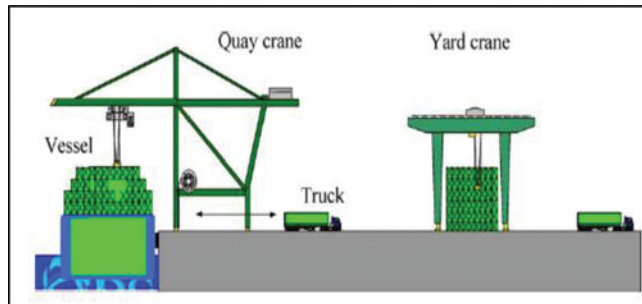


Figure 1 Typical container terminal operation.

optimize resources while offering improved customer service for the carriers. The main objective of the carrier is to optimize vessel usage by minimizing the berth time of the vessel in the container terminal. Problems arise when the carrier requests from the CTO an increase in allocated resources and a higher service level while the CTO is utilizing these resources to serve other vessels. In practice, an agreement on a yearly service rate shall be reached between the carrier and the CTO before signing a contract. The agreed-upon yearly service rate embeds many direct and indirect factors. In our study, the main factors in such an agreement are quay cranes assigned, fees charged, service efficiency, and value of time.

In an effort to rationalize decision making in servicing unloading vessels during transshipment, we investigate in this article a game-theoretic approach for the interaction between the carrier (Player 1) and the CTO (Player 2). The service level for the Carrier is considered to be as inversely proportional to the time spent by the vessel inside the container terminal including: waiting, unloading and loading times by the CTO. The lower the time spent by the vessel inside the container terminal, the higher the service level (Nehme & Awad, 2010). This article, extends the previous work of (Nehme & Awad, 2010) and contributes to the literature with an enhanced formulation of the players' utility functions, the concept of soft computing and more specifically the Epsilon Bargaining in the negotiation process between the Carrier and the CTO for unloading vessels during transshipment.

The remainder of the article is structured as follows. In the second section, the literature related to vessel servicing by container terminal operators, game theory for modeling competition among ports, and epsilon bargaining are discussed. The proposed mathematical model and the utility functions are presented in the third section. In the fourth section, the transshipment operation as a soft pseudo-bargaining problem with ordinal ranking preferences is defined and illustrated by a numerical example. Finally, the fifth section details the epsilon bargaining equilibrium, and the sixth section concludes the article with potential follow-up research.

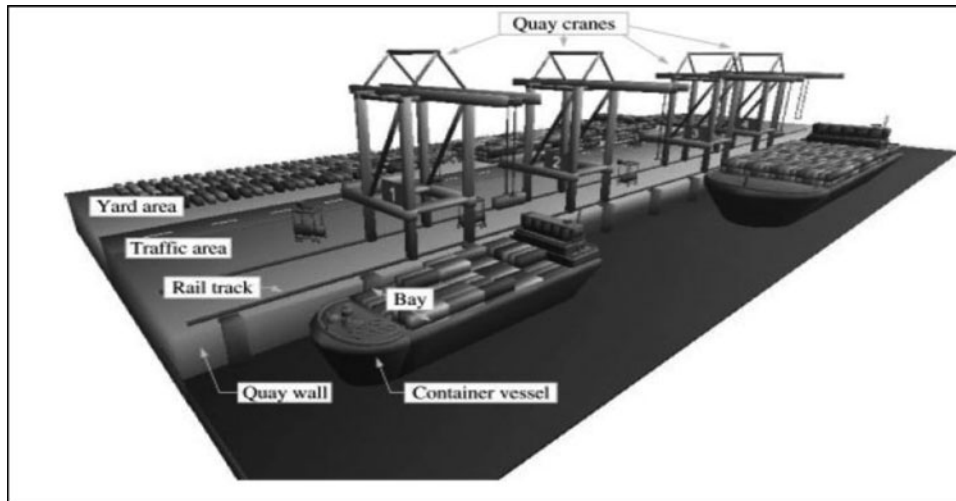
## LITERATURE REVIEW

This article combines three different areas of research: (i) vessel servicing by container terminal operators, (ii) game theory for modeling competition among ports, and (iii) epsilon bargaining. The literature review here summarizes related work in each of these three areas.

### Vessel Servicing

Imai, Nagavia, and Tat (1997) illustrated the trade-off between the service time of the vessel and the dissatisfaction of this vessel when a berth allocation is used. They formulated a multiobjective function for the berth allocation problem (BAP) by minimizing both the time spent by the ship inside the container terminal and the dissatisfaction of the vessel in terms of berthing order compared to other ships. BAP is the allocation of berths in a port to incoming vessels to handle their cargos. Later, Imai, Nishimura, and Papadimitriou (2001) elaborated their previous static BAP model to propose a dynamic BAP model, where vessels enter the container terminal while other vessels are being served. In two more papers, Imai, Nishimura, and Papadimitriou (2003, 2008) did not take into consideration the characteristics of the vessel in term of size and capacity.

Imai, Nishimura, and Papadimitriou (2003) modified their dynamic BAP model to take into account the priority of each vessel by considering a multi-user container terminal (MUT). The MUT serves vessels based on their size, cargo volume, and handling and waiting times of the preceding and successors ships. However, their model did not consider the quay capacities of the container terminal. The authors further elaborated the model to reflect the handling capacities of the container terminal so that when the actual expected service time exceeds the time limit, the vessel is diverted into an external container terminal and the container terminal operator pays the service charge (Imai, Nishimura, & Papadimitriou, 2008). Thus, the container terminal operator had an incentive to maximize efficiency when serving the vessel so that the total service time is minimized and the service charge paid to the external container terminal is reduced. Golias, Boile, and Theofanis (2009) considered that the servicing of vessels by the container terminal operator is based on priority agreements in solving a dynamic BAP. The proposed model assumed pareto optimality with a multiobjective function that minimized the service time of each group of vessels belonging to the same owner, as well as the total service time of all vessel groups at the container terminal. Giallombardo, Moccia, Salani, and Vacca (2010) modeled the BAP by considering two decisions variables: the allocation of vessels inside the container terminal and the quay crane (QC) assignment. Their objective function aimed to maximize the resource utilization of the QC while minimizing the cost of the transshipment flows between ships.



**Figure 2** A typical container terminal during vessel servicing.

Li, Wu, Petering, Goh, and Souza (2009) argued that the container terminal efficiency is related to the performance of the yard crane (YC). The model had a multiobjective function representing a weight to minimize the retrieval earliness, storage, and retrieval delay in the yard area. Lee et al. (2009) correlated the yard truck's scheduling problem with the storage allocation problem. The proposed model minimized the makespan of discharging containers by reducing the congestion and waiting time of yard trucks in the container terminal, in other terms establishing a balance between the travel time and the waiting time. Cordeau, Gaudio, Laporte, and Moccia (2007) solved the service allocation problem inside the container terminal by minimizing the intensity of traffic service in the yard. Distances from the assignments decision were considered as optimization criteria in the objective function and a branch and bound algorithm was applied to solve real data applied to Gioia Tauro Maritime Container Terminal. Meisel (2011) presented a new approach for the quay crane scheduling problem by restricting the availability of cranes for a ship to certain time windows during loading and unloading of containers. This restriction attempts to enhance the service of high-priority ships while serving low priority ships. Figure 2 illustrates a typical container terminal during vessels servicing (Meisel, 2011).

### ***Game-Theoretic Approaches in Modeling Competition Among Ports***

Research related to application of game theory in vessel service has been limited to competition between container terminals or ports. Anderson (2008) proposed a Nash equilibrium profit of a Bertrand pricing game to analyze competition between two ports based on expansion strategy, that is, invest or not invest, in order to attract more cargo. Pricing was based on a combination of perfect competition and oligopolistic imperfect. The value-added services of being a hub and the strategic gov-

ernmental interference were not considered in this model. Saeed and Larsen (2010) presented a Bertrand game of competition between four container terminals in two ports via a two-stage model where the first stage was to decide on the level of coalition between the three container terminals in the same port and the second stage was the competition between the two ports based on the first stage results. Imai, Nishimura, Papadimitriou, and Liu (2006) analyzed the competition between container terminals using a nonzero two-players game approach. The authors considered two service scenarios: The first case, which was modeled via minimizing the travel time from origin to destination (the minimum sum location), was the container mega-ships, and the second scenario considered a multiple-port-calling network of ordinary ships and was modeled by the classic traveling salesman problem. Finally, Reyes (2005) formulated the general transshipment problem as a Shapley value in order to establish some stable conditions for a cooperative game among players. The author concluded that the verification of the apportionment of the Shapley value is essential to reach a stable solution for the cooperation among players.

The main difference between the works just reviewed and this work is the objective of the game that is formulated to maximize the Carrier service level while minimizing operations cost for the container terminal operator. The utilities defined in this work are function of the fees for the carrier, the revenues generated, and cost incurred for the container terminal operator.

### ***Bargaining Problem and Epsilon Equilibrium***

In the previous work of Nehme and Awad (2010), a mathematical model was formulated to represent pseudo bargaining for servicing vessels during transshipment operations. Three cases for the behavior of players were considered: risk averse, risk neutral, and risk taker. With the exception of Nehme and Awad (2010) and to the best of our knowledge, the bargaining

problem and the epsilon equilibrium have been researched previously in the literature but not for vessel services specifically.

The concept of a “perfect” Nash equilibrium was first introduced by Selten in 1975, where he proved that every finite extensive game with perfect recall has at least one perfect equilibrium in a one-period game (Selten, 1975). Influenced by Rubinstein (1979), Radner (1981) formulated a sequential game for the “principal–agent” relationship over a finite time period. He proved the existence of an epsilon equilibrium that offers each player an average expected utility close to his expected utility in the one-period game. Kobberling and Peters (2003) modeled the preferences of the players in the bargaining problem according to rank-dependent utility theory. Inspired by Radner (1981), Mailath, Postlewaite, and Samuelson (2005) studied perfect epsilon equilibria, where each player’s action after every round of game is assessed within an  $\epsilon$  deviation of the equilibrium of the best response.

The Mailath et al. work (2005) is different from this research in that (i) the game has a finite number of periods depending on the ordinal ranking preferences, and (ii) a new version of epsilon bargaining between the carrier and the CTO is proposed as a function of port location, time of negotiation, and customer relation. In addition, this work is unique from the vessel serving perspective: It embeds both the service efficiency and the value of time in the developed utility functions, as illustrated in the next section.

**MATHEMATICAL MODEL**

In this section, the proposed mathematical model is presented in the following three subsections. In the first subsection, parameters used and assumptions made for the proposed mathematical model are presented, while in the second subsection, players’ payoff functions are derived. In the last subsection, utility functions for all players are derived based on the previous two subsections.

**Parameters and Assumptions**

The developed mathematical model in this section is an extension of the work by Nehme and Awad (2010). The process analyzed in this article is strictly for the unloading part of the transshipment process at the time of vessel arrival at the container terminal. As mentioned earlier, we consider two players in this game: Player 1 (the carrier) and Player 2 (the CTO).

Table 1 summarizes all the mathematical notations used in this section.

Assuming  $N$  is the total number of the containers to be unloaded from the vessel, we denote by  $x_t$  the number of containers unloaded at every time period  $t$ . Player 1 is interested in maximizing service level by minimizing the time spent by the vessel inside the container terminal. Player 2 unloads the containers from Player 1’s vessel via cranes over a finite time horizon,

**Table 1** Mathematical notations used in third section.

Parameter	Description
$x_t$	Number of containers unloaded at time period $t$
$y_t$	Integer variable representing the number of cranes used at time $t$ to unload the containers
$A_t$	Number of cranes available at time period $t$ at the container terminal
$CA$	The maximum number of containers a crane can handle at any time period
$PY$	The price of the crane discounted with time
$\delta P_t$	The discount factor for the price of the crane as a function of $t$
$CY$	The operating cost of using a crane, discounted with time
$\delta C_t$	The reduction factor for the operating cost of using a crane as a function of $t$
$u_s$	Utility function for the service level of Player 1
$u_p$	Utility function for the price of the cranes used by Player 1
$u_c$	Utility function for cost of Player 2
$\pi_1$	The payoff function for Player 1
$\pi_2$	The payoff function for Player 2
SE	Service efficiency for the vessel
VOSE	Value of service efficiency
$N$	The number of containers to be unloaded
OT	The optimal time allocated to serve the vessel
RT	The real time spent by the vessel in the container terminal

which is divided into equal periods that we denote by  $t = 1, 2, \dots, T$ . Player 2 sets a price list for the cranes and charges Player 1 for cranes used during the unloading process based on the utilization time period  $t$ . The operating cost of the cranes (which includes the manpower costs, overhead costs and other incurred costs related to the unloading operation) inside the container terminal is charged to Player 2.

At every time period  $t$ , every player has a set of actions to choose from. The set of actions is the number of cranes to be used during different time periods. Let  $y_t$  be the integer variable representing the number of cranes used at time  $t$  to unload the containers. The number of cranes used at every time period  $y_t$ , cannot exceed the number of cranes available at every time period  $t$  at the container terminal,  $A_t$ , at this specific time period. A crane can handle a maximum of  $CA$  containers per unit  $y_t$ .

It is common practice that the CTO charges the carrier based on the number of container moves. The deployment of cranes by the CTO is constrained by working shifts and depends on the collective agreement between the labor union (of terminal operators or ports) and the CTO management. Considering the price of using 1 unit of crane at any time period  $t$  to be  $PY$ , the price of the crane,  $PY$ , is assumed to be relative to the

time of unloading and discounted with time by the discount factor  $\delta P_t$  as a function of  $t$ . This assumption is valid since the analysis starts at the beginning of a peak period where the need for crane deployment is the highest. Thus; the price of the crane is not constant over the required time horizon;  $\delta P_t$  is a nonincreasing function starting from 1 at time period  $t = 1$  and diminishing with increasing  $t$ . The discount price factor  $\delta P_t = [0:1]$ . Similarly, the operating cost of using 1 crane at anytime,  $CY$ , is considered to be discounted with time based on higher front loading of such costs by the CTO over a typical operating shift.  $CY$  should be always less than  $PY$  in order for the CTO to make profit.  $\delta C_t$  is defined as the reduction factor for  $CY$  as a function of  $t$ ; this reduction factor is not constant over the required time horizon but it is a nonincreasing function starting from 1 at time period  $t = 1$  and diminishing with increasing  $t$ . The discount cost factor  $\delta C_t = [0:1]$ . It is worth noting that in practice the number of cranes that the CTO deploys to a vessel is determined by the optimal crane intensity, which is the maximum number of cranes that can operate simultaneously on a vessel, without causing any crane crash.

### Players' Payoff Functions

Player 1 has two objectives. The first objective for Player 1 is to maximize the service level by minimizing time spent in the container terminal. The utility function for the service level of Player 1,  $u_s(y_t, A_t, T)$  is affected by  $y_t$ , the number of cranes used to unload containers in every time period  $t$  compared to  $A_t$  the total number of cranes available at every time period  $t$  over the time horizon  $T$ . The second objective of Player 1 is to minimize the fees for the crane usage during the unloading process. Further,  $u_p(y_t, \delta P_t, PY, T)$ , the utility function for the price of the cranes used by Player 1, depends on  $y_t$ , the number of cranes used at every time period  $t$ ,  $PY$ , and the price of each crane multiplied by the discount price factor  $\delta P_t$  over the time horizon  $T$ . Player 1's payoff function could be best defined as the difference between the profit generated from increasing the service level, such as leaving the container terminal early, and the fees accumulated from using many cranes to unload the vessel.

The proposed payoff function for Player 1 is:

$$\pi_1 = u_s(y_t, A_t, T) - u_p(y_t, \delta P_t, PY, T) \quad (1)$$

Player 2 has also two objectives: The first one is to maximize revenues by renting more cranes to Player 1, and the second one is to minimize the operating costs. The utility function for revenue of Player 2 is the same utility function as the fees of Player 1, since the revenues for Player 2 are generated from the fees charged to Player 1. The utility function for revenues of Player 2 is thus  $u_p(y_t, \delta P_t, PY, T)$ .

The operating cost of the cranes used for vessel unloading is a function of  $y_t$ , the number of cranes rented to Player 1 at every time period  $t$ ,  $CY$ , and the operating cost of cranes which is multiplied by the discount cost factor,  $\delta C_t$ , over

the time horizon  $T$ . Utility function for cost of Player 2 is  $u_c(y_t, \delta C_t, CY, T)$ .

We define Player 2's payoff function as the difference between the profit generated from charging the cranes price for Player 1 and the operating cost for the cranes used.

The payoff function for Player 2 is:

$$\pi_2 = u_p(y_t, \delta P_t, PY, T) - u_c(y_t, \delta C_t, CY, T) \quad (2)$$

Every player has some practical constraints that bound the payoff function. For both players, the number of containers unloaded from the vessel,  $x_t$ , via the cranes,  $y_t$ , could not exceed the capacity of the cranes,  $CA$ , at every time period  $t$ .

Therefore,

$$x_t \leq y_t \cdot CA, \quad \forall t \quad (3)$$

In addition, all the containers  $N$  should be unloaded and handled during the time horizon  $T$ , that is,

$$\sum_{t=1}^T x_t = N \quad (4)$$

### Utility Function Definition

The utility function for service of Player 1 reflects the importance of unloading containers at an earlier stage. The earlier containers are unloaded, the higher the service level is, and vice versa. Therefore, we formulate the utility function as:

$$u_s(y_t, A_t, T) = SE * VOSE \quad (5)$$

where  $SE$  is the service efficiency for the vessel and  $VOSE$  is the value of service efficiency for the carrier.

The  $SE$  reflects the time spent by the vessel inside the container terminal including loading, unloading, and waiting time. The service efficiency  $SE$  is measured by the optimal time allocated to serve the vessel ( $OT$ ), based on the number of containers to be loaded/unloaded,  $N$ , and the number of cranes available at each time period  $A_t$ , over the real time spent in the container terminal ( $RT$ ), which is affected by the number of cranes used at each time period  $y_t$ . Thus,  $SE = \frac{OT(N, A_t)}{RT(y_t)}$ .

The optimal time allocated to serve the vessel ( $OT$ ) can be deduced as follows. At time period  $t = 0$ , the number of remaining containers to be unloaded is set to be equal to  $N$ , which is the total number of containers to be unloaded from the vessel. Also,  $OT$  is set to be equal to zero. The time period is incremented by 1 unit, and during this increment the remaining number of containers to unload at each time period,  $M_t$ , is calculated via a loop, by decreasing the previous number of containers to unload by the maximum capacity of cranes to handle containers at that specific time period,  $CA * A_t$ . The loop stops once there are no more containers to unload. The purpose of defining  $M_t$  is to calculate the optimal time required to unload all containers ( $OT$ ).

The real time spent by the vessel in the container terminal ( $RT$ ) is deduced as follows.

At time period  $t = 0$ ,  $RT$  is set to be equal to 0. Then, the time is incremented by 1 unit, and during this increment a test is conducted to check whether any of the cranes is used to unload containers (by checking that the number of cranes used at time  $t$ ,  $y_t$ , is different from 0). Only if a crane is used, then  $RT$  is set to be equal to  $t$ . This process continues till  $t$  reaches  $T$ , which is the time horizon allowed to unload the containers. Thus,  $RT$  is determined.

$VOSE$  is related to the vessel size and the number of containers available on this vessel. The  $VOSE$  is estimated by the carrier.

We define the utility function for the price as being:

$$u_p(y_t, \delta P_t, PY, T) = \sum_{t=1}^T \delta P_t * PY * y_t \quad (6)$$

and the utility function for the incurred cost as:

$$u_c(y_t, \delta C_t, CY, T) = \sum_{t=1}^T \delta C_t * CY * y_t \quad (7)$$

The payoff function for Player 1 is given by:

$$\pi_1 = SE * VOSE - \sum_{t=1}^T \delta P_t * PY * y_t \quad (8)$$

and the payoff function for Player 2 by:

$$\pi_2 = \sum_{t=1}^T [(\delta P_t * PY - \delta C_t * CY) * y_t] \quad (9)$$

Both payoff functions are subject to constraints (Eqs. 3 and 4).

**ORDINAL RANKING PREFERENCES**

In this section, the transshipment operation is defined as a soft pseudo-bargaining problem with ordinal ranking preferences. This section too is divided into three subsections. In the first subsection, the mathematical model formulated in the third section is defined as a bargaining problem. Then in the second subsection, the existence of a unique Nash bargaining solution for the formulated mathematical model is proved. In the last subsection, a numerical case is presented to illustrate the formulated model by considering players with risk aversion preferences.

**Problem Definition**

With  $Y$  being the set of possible agreements players may reach, and  $\mathbf{y}$  the set of feasible solutions for  $(y_{t=1} \ y_{t=2} \ \dots \ y_{t=T})$ , the feasible solution for our work refers to the numbers of cranes that are mandatory to use in order to unload a vessel. Let  $U$  be the set of possible pairs of payoffs generated from set  $Y$ :

$$U = \{(v_1, v_2) : \pi_1(\mathbf{y}) = v_1, \pi_2(\mathbf{y}) = v_2\} \ \mathbf{y} \in Y \quad (10)$$

We denote by  $D$  the disagreement outcome and by  $d$  the pair of payoffs such that:

$$d = (\pi_1(D), \pi_2(D)) \quad (11)$$

The pair  $(U, d)$  satisfies the following four conditions:

Condition 1:  $d$  is a member of  $U$ . Both players may agree to disagree. Disagreement is a possible outcome of bargaining; in this case Player 1 diverts to another container terminal to unload the vessel, and Player 2 serves other than Player 1.

Condition 2: Some form of agreement is better than disagreement for both players; that is, unloading all containers at time  $t = 1$  is better than  $d = (0, 0)$ .

Condition 3:  $U$  is convex. Based on Lemma 1,  $\pi_1$  is a convex function in  $y$ , and Lemma 2,  $\pi_2$  is a convex function in  $y$ . Proofs for both lemmas are presented in the Appendix.

Condition 4: Since  $y_t$  varies from 0 to  $A_t$  at every time period  $t$ ,  $U$  is bounded. From Eqs. 3 and 4, there exists a lower bound by which:

$$\sum_{t=1}^T y_t \geq \left\lceil \frac{N}{CA} \right\rceil \quad (12)$$

where  $\lceil X \rceil$  is the smallest integer number larger or equal to  $X$ .

Because all the preceding four conditions are satisfied, we can define the model as a bargaining problem of pair  $(U, d)$ . Readers interested in a detailed analysis of the preceding four conditions for bargaining can refer to Osborne (2009).

**Nash Bargaining Solution**

The four axioms of Pareto efficiency, symmetry, invariance to equivalent payoff representation, and independence of irrelevant alternative are presented (Osborne, 2009) in the Appendix. Note that there exists a unique Nash bargaining solution (NBS)  $(v_1, v_2)$  such that:

$$\max_{(v_1, v_2)} (v_1 - d_1)(v_2 - d_2) \quad (13)$$

subject to,

$$(v_1, v_2) \in U \quad (14)$$

$$(v_1, v_2) \geq (d_1, d_2) \quad (15)$$

Since  $y_t \leq A_t$ , the possible values for any  $y_t = \{0, 1, \dots, A_t\}$ , and the number of possible agreements is:  $Y = \prod_{t=1}^T (A_t + 1)$ .

The lower bound for  $\sum_{t=1}^T y_t$  is  $\lceil \frac{N}{CA} \rceil$ ; therefore, the feasible set of agreements between both players  $\mathbf{y}$  is any combination where  $\mathbf{y} \in Y$  and the lower bound condition is satisfied.

In our bargaining procedure, a player is not allowed to repeat the offer once proposed. In other terms, a player is not allowed to be persistent on a certain preference. In addition, a tie in Player 1's payoff function, when it happens, is resolved by choosing the preference that has the maximum number of cranes. This is justified by the fact that when more resources are used, it is

more beneficial for Player 1 since it enhances the service level. The opposite applies for Player 2.

**Risk Aversion Case**

In this section, a numerical example that shows the importance of the discount factor for the cranes' price in the existence of the NBS is presented. The case of risk aversion for the behavior of the discount price factor  $\delta P_t$  is considered. Other cases such as risk-neutral and risk-taker cases were studied in Nehme and Awad (2010), using different utility functions. Compared to the previous work, the formulated utility function for service level from a carrier's perspective is more realistic in this article.

For illustration purposes, the scenario when  $N = 40$ ,  $T = 3$ ,  $CA = 10$ ,  $A_1 = A_2 = A_3 = 2$ ,  $PY = 3$ ,  $CY = 1$ , and  $VOSE$  for Player 1 is equal to 40 is selected. For simplicity, 1 unit for each container and  $\delta P_t = \delta C_t$  are assumed.

In this case, the possible values for  $y_1$ ,  $y_2$ , and  $y_3$  are  $\{0,1,2\}$ . The number of possible agreements such that  $y = (y_1 y_2 y_3)$ ,  $y_i \in \{0,1,2\}$  is  $\prod_{i=1}^T (A_i + 1) = 27$ .

The lower bound for  $\sum_{t=1}^T y_t$  is  $\lceil \frac{N}{CA} \rceil = 4$ . Therefore, the feasible set of agreements between both players is restricted to  $\{(2 2 0), (2 0 2), (0 2 2), (2 1 1), (1 2 1), (1 1 2), (2 2 1), (2 1 2), (1 2 2), (2 2 2)\}$ .

Since the function of  $\delta P_t$  in this case should have a risk aversion behavior; therefore the proposed function is considered to be  $\delta P_t = \exp^{1-\sqrt{t}}$ , which will ensure that at time  $t = 1$  the value of  $\delta P_t$  equals 1, and the function is decreasing in  $t$  and converging to 0 at  $T = \infty$ . Table 2 presents the payoff values of the risk aversion discount price factor for every player with respect to every feasible agreement sorted by preferences, from the most to the least desirable preference. The first column presents the preferences of every player sorted from the most desirable preference to the least preference for every player. The second and the third column present the feasible agreement of Player 1 with the payoff value associated to this agreement sorted from the highest payoff to the lowest payoff. The same applies for the fourth and the fifth column for Player 2. From

Table 2, the first choice for Player 1 is the lower bound case (2 2 0), where Player 2 allocates all the available resources for servicing Player 1. In this case, Player 1 accomplished the service in two time units, which is the minimum time required to unload the transshipped containers. For Player 2, the feasible agreement (2 2 2) is the most preferable agreement since it will generate the highest payoff.

Every player bargains according to the most desirable preference that maximizes his or her payoff. The bargaining rounds start with the first preference of every player. Player 1 proposes the feasible agreement that generates the highest payoff value. If the proposed agreement has the same preference for Player 2, an agreement is reached and the bargaining process stops. Otherwise, Player 2 rejects the offer and suggests a new proposal based on his or her next preference. This is repeated until an agreement is reached. Figure 3 illustrates the bargaining process.

In the first round, Player 1 proposes an offer based on maximizing the payoff function, in this case (2 2 0). Player 2 rejects it since it is not his or her first preference and proposes his or her first preference (2 2 2) at the first round. In the second round, Player 1 rejects the offer proposed by Player 2 in round 1 and offers his or her second best preference, which is (0 2 2). Player 2 rejects the proposed offer in round 2 and proposes his or her second preference (2 2 1) at the second round. In the third round, Player 1 rejects the offer proposed by Player 2 in round 2 and offers his or her third best preference (1 1 2). Player 2 rejects the proposed offer in round 3 and proposes his or her third preference (2 1 2) at the third round. In the fourth round, Player 1 rejects the offer proposed by Player 2 in round 3 and offers his or her fourth best preference (1 2 1). Player 2 rejects the proposed offer in round 4 and proposes his or her fourth preference (2 2 0) at the fourth round. In this case Player 1 accepts the offer of Player 2, (2 2 0), since it matches with his or her first preference and offer in round 1. In this case, the NBS  $y = (2 2 0)$  is unique and appears at the fourth round of bargaining, which is the first best preference for Player 1 and fourth best preference for Player 2.

**Table 2** Payoffs value for each player: risk aversion case.

Preference	Player 1		Player 2	
	feasible agreement	$\pi_1$	feasible agreement	$\pi_2$
1	(2 2 0)	30.03	(2 2 2)	8.57
2	(0 2 2)	19.82	(2 2 1)	7.61
3	(1 1 2)	18.80	(2 1 2)	7.25
4	(1 2 1)	18.26	(2 2 0)	6.64
5	(2 0 2)	17.78	(1 2 2)	6.57
6	(2 1 1)	17.24	(2 1 1)	6.28
7	(1 2 2)	16.82	(2 0 2)	5.92
8	(2 1 2)	15.79	(1 2 1)	5.61
9	(2 2 1)	15.26	(1 1 2)	5.25
10	(2 2 2)	13.82	(0 2 2)	4.57

**EPSILON BARGAINING EQUILIBRIUM**

This section details the epsilon bargaining equilibrium. The section is divided into three subsections. In the first subsection, the bargaining equilibrium for the formulated model in earlier (in the third section) is discussed. Then in the second subsection, an epsilon equilibrium formulation is defined for the proposed mathematical model. A theorem for the epsilon bargaining solution is derived and proved. In the third subsection, a numerical example is presented to illustrate the application of the derived theorem and to highlight the contrast between the ordinal ranking preferences and the epsilon bargaining equilibrium.

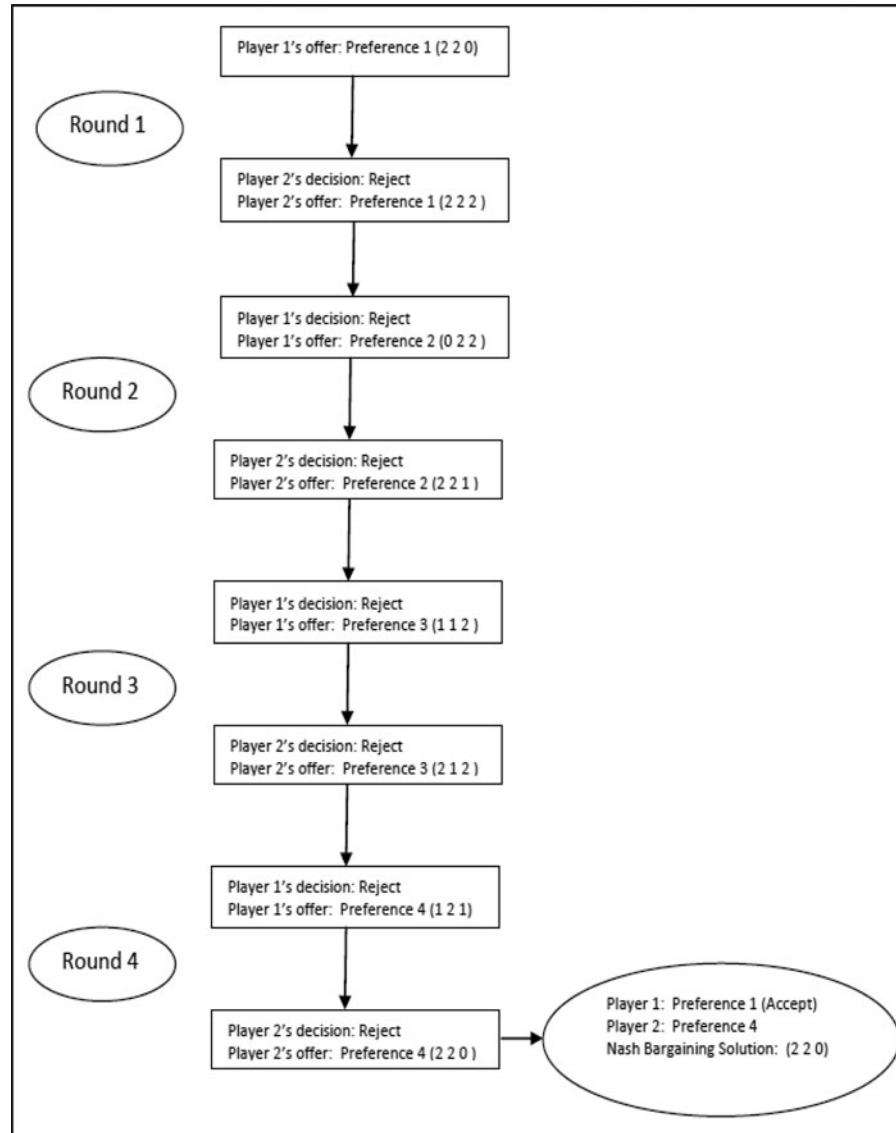


Figure 3 Bargaining process for a risk-averse case.

**Bargaining Equilibrium**

The bargaining process is typically illustrated as a zero-sum game where players compete over finite resources in a finite or infinite horizon of time or rounds until convergence to a perfect equilibrium (Rubinstein, 1982). The disagreement payoff is presented as zero value.

This is not the case in the payoff functions generated in earlier, in the third section. First the bargaining process has a finite number of rounds, which is the set of feasible agreements that both players can handle. The disagreement payoff of every player is the set of agreements excluded from the feasible agreements, including the zero payoffs. Thus, it is difficult to formulate a closed form of the disagreement payoff functions. In addition, the major different component introduced in our bargaining formulation is the time factor of the resource utiliza-

tion. Players are interested not only in the amount of resources used or acquired but also in the time usage of these resources.

One might argue that solving Eqs. 13, 14, and 15 of this bargaining problem as an optimization problem reduces directly to the Nash bargaining solution (NBS). That is usually true in the case where the disagreement functions have a closed form or have a zero value, that is, a zero-sum game of pie sharing. However, in real-life scenarios, including this case, it is not easy to determine a closed form for the disagreement, as mentioned previously. In addition, since it is not a zero-sum game, the maximization of the total player payoff functions leads to increasing the payoff for the total welfare of both players as an optimal system approach but not as an the third optimal user approach, which breaches the Pareto efficiency. The payoff functions generated in section represent a tangible value for every player as a function of the revenues and costs occurring

from every possible feasible agreement but it is not correlated with the preference of the other player:

**Lemma 3:** Any preference  $j \in y$  for player  $i$  with a negative payoff value shall be rejected by the player and shall be substituted by the disagreement value of zero. In other words, if  $\pi_i^j < 0$  then  $\pi_i^j = 0$ .

*Proof.* Let  $(v_1, v_2)$  be in  $U$  and  $(d_1, d_2)$  be in  $D$ .

If  $v_1 < d_1$ , then Player 1 will deviate to  $d_1$ . Since  $d_1 = 0$  is a disagreement payoff value, any value of  $v_1 < 0$ , will make Player 1 deviate to  $d_1 = 0$ . The same applies for Player 2.

### Epsilon Equilibrium

Finding the NBS might be time-consuming when multiple scenarios are available for every player, which would not be attractive for players constrained by time to find a solution. In the example presented in the fourth section, players reached an NBS in four rounds, even though the final solution was basically the first preference of Player 1. In most real-life scenarios, negotiators tend to recognize agreeable situations and are willing to seal an agreement even though it has a small deviation from the bargaining equilibrium. To capture this soft quasi-equilibrium, we define  $\varepsilon_k$  as the percentage of deviation from the bargaining equilibrium and that is accepted by players at negotiation round  $k$ . Here, it is assumed that in every round of negotiation each player submits his or her preference in a decreasing order, starting at round  $k = 1$ .

According to Lirn, Thanopoulou, Beynon, and Beresford (2004), the port geographical location represents a main criterion in port selection: It is about 35% of the total weight for port selection. A port with an enhanced geographical location is less likely to deviate from the bargaining equilibrium compared to another port with less advantage in its geographical location. Thus,  $\varepsilon_k$  is affected by the port geographical location. Both players have an intention to reach an agreement at the earliest time period. Also, the relation between both players affects the margin of negotiation; the port is more likely to increase its margin of negotiation to satisfy the customer, especially a customer with a good record of business transactions. Since the margin of negotiation is affected by the time of negotiation and the relation between both players, it is reasonable to include those two factors in  $\varepsilon_k$ .

Therefore,  $\varepsilon_k$  could be a function of (i) port geographical location, (ii) time of negotiation, and (iii) customer relation. The behavior of  $\varepsilon_k$  is an increasing function in term of negotiation round  $k$ .

The epsilon bargaining equilibrium (EBE) is defined as the first preference  $j$  such that:

$$\left| \frac{u_i^j - u_{i'}^j}{u_i^j} \right| \leq \varepsilon_k \tag{16}$$

where  $u_i^j$  is defined as the utility value for preference  $j \in y$  for player  $i$  such that:

$$u_i^j = \frac{r_i^j}{\max_{j \in y} r_i^j} \tag{17}$$

where  $r_i^j$  is the ranking payoff for player  $i$  determined based on the number of available feasible agreements and  $y$  is already defined as the feasible set of agreements between both players, where  $y \in Y$ .

**Theorem:** The epsilon NBS for players  $i$  and  $i'$  is the preference  $j \in y$  that minimizes  $|\frac{u_i^j - u_{i'}^j}{u_i^j}|$  given that there is no other preference  $j' \in y$  that has higher value than preference  $j \in y$  for both players  $i$  and  $i'$ .

*Proof.* The objective of player  $i$  is to maximize his or her utility value for preference  $j$  for  $j \in y$ . The same holds for the other player  $i'$ . In addition, the preference  $j \in y$  should have the highest value preference for players  $i$  and  $i'$ . Thus, the model can be formulated as follows:

$$\begin{aligned} & \max u_{i'}^j \\ & \max u_i^j \\ & \text{subject to: } u_i^j > u_{i'}^j \text{ for all } j \in y \text{ for player } i \\ & u_{i'}^j > u_i^j \text{ for all } j \in y \text{ for player } i' \end{aligned}$$

Using goal constraints, setting the objectives of both players as soft constraints, objectives can be reformulated as follows:

$$u_i^j + d_i^- - d_i^+ = 0$$

$$u_{i'}^j + d_{i'}^- - d_{i'}^+ = 0$$

where  $d_i^-$  and  $d_{i'}^-$  are the negative deviation from the target  $t_i$  and  $t_{i'}$  for both players.  $d_i^+$  and  $d_{i'}^+$  are the positive deviation from the target  $t_i$  and  $t_{i'}$  for both players.

Formulating the preceding using goal programming objective functions theory leads to:

$$\min \frac{d_i}{t_i} + \frac{d_{i'}}{t_{i'}}$$

where  $d_i$  and  $d_{i'}$  are total deviations for both players.

Considering the perspective of players' target, this will lead to minimizing  $|\frac{u_i^j - u_{i'}^j}{u_i^j}|$ .

for the constraints  $u_i^j > u_{i'}^j$  for all  $j \in y$  for player  $i$  and  $u_{i'}^j > u_i^j$  for all  $j \in y$  for player  $i'$ .

By contradiction, using the Nash equilibrium and assuming a rational behavior, a player will always deviate to the preference with the highest payoff value.

The technique of the preceding theorem is illustrated in the example of risk aversion case presented in the fourth section.

Table 3 tabulates the feasible agreement in the first column. The second and third columns present the ranking payoff for

**Table 3** Ranking payoffs.

Feasible agreement	Player 1 ranking payoff $r_1^j$	Player 2 ranking payoff $r_2^j$	$ \frac{u_1^j - u_2^j}{u_1^j} $	$ \frac{u_2^j - u_1^j}{u_2^j} $
(2 2 0)	10	7	0.30	0.43
(0 2 2)	9	1	0.89	8.00
(1 2 2)	8	2	0.75	3.00
(1 2 1)	7	3	0.57	1.33
(2 0 2)	6	4	0.33	0.50
(2 1 1)	5	5	0.00	0.00
(1 2 2)	4	6	0.50	0.33
(2 1 2)	3	8	1.67	0.63
(2 2 1)	2	9	3.50	0.78
(2 2 2)	1	10	9.00	0.90

Player 1 and Player 2 for every feasible agreement respectively while the fourth column calculate the EBE for every ranking payoff from Player 1’s perspective, The fifth column calculate the EBE for every ranking payoff from Player 2’s perspective.

From Table 3, the feasible agreement (2 1 1) has the minimum for both  $|\frac{u_1^j - u_2^j}{u_1^j}|$  and  $|\frac{u_2^j - u_1^j}{u_2^j}|$  with a value of zero. However, this preference is the sixth preference for both players and occurs at round 6 of the negotiation process, which is beyond the NBS presented in the fourth section. Figure 4 illustrates the payoff difference between the two players for each feasible agreement. In this figure, the ranking payoff is presented for both players for each feasible agreement. The difference between the payoff ranking is highlighted with an arrow. From this figure, agreement (2 1 1) has a zero difference between players.

Figure 4 illustrates the impact of  $\epsilon_k$  in reaching an agreement between players at early negotiation rounds.

As mentioned previously in this section,  $\epsilon_k$  is a function of (i) port geographical location, (ii) time of negotiation, and (iii) customer relation. This statement is fortified in the literature (Brooks, 2000; Porcari, 1999; Slack, 1985; Ugboma, Ugboma, and Ogwude, 2006). In addition; interested reader can refer to Lirn, Thanopoulou, Beynon, and Beresford (2004) where the authors presented a comprehensive technique to deduce and calculate factors affecting port selection.

To determine  $\epsilon_k$  we propose a linear model as a function of (i) port geographical location, (ii) time of negotiation, and (iii) customer relation such that:

$$\epsilon_k = \sum_j weight_j * factor_j \tag{18}$$

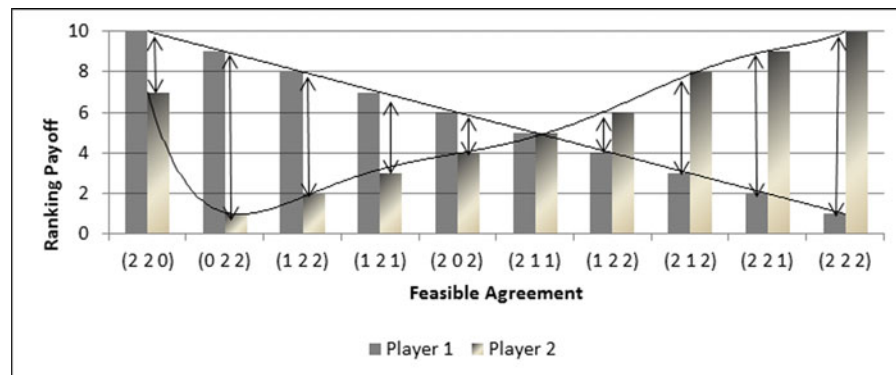
$$\sum_j weight_j = 1 \tag{19}$$

where  $weight_j$  represents the weight for each of the factors  $j$  ( $j =$  port geographical location), time of negotiation, and customer relation. Weights are determined by players themselves in a similar methodology used by Lirn, Thanopoulou, Beynon, and Beresford (2004).

**Comparison Between Ordinal Ranking and EBE**

In the fourth section, for the ordinal ranking preferences, Player 1 proposes (2 2 0). Player 2 rejects it since it is not his or her first preference and proposes his or her first preference (2 2 2). In the second round, Player 1 rejects the offer proposed by Player 2 in round 1 and offers (0 2 2). Player 2 rejects the proposed offer proposes (2 2 1) at the second round. In the third round, Player 1 rejects the offer proposed by Player 2 in round 2 and offers (1 1 2). Player 2 rejects the proposed offer in round 3 and proposes (2 1 2) at the third round. In the fourth round, Player 1 rejects the offer proposed by Player 2 in round 3 and offers (1 2 1). Player 2 rejects the proposed offer in round 4 and proposes (2 2 0) at the fourth round. In this case Player 1 accepts the offer of Player 2, (2 2 0), since it matches with his or her first preference and offer in round 1. Four rounds of negotiations are needed in this case to reach the Nash bargaining solution (NBS).

Considering the EBE for the same numerical application by setting  $\epsilon_k$  to be 0.5 for the first period round of negotiation, the EBE value for (2 2 0) is  $|\frac{u_1^j - u_2^j}{u_1^j}| = 0.3$  for Player 1 and  $|\frac{u_2^j - u_1^j}{u_2^j}| = 0.43$  for Player 2. Player 1 proposes (2 2 0) at the first round of negotiation as the first preference. In this case, Player 2 accepts the offer of Player 1 at the first round because the EBE value for this offer is within the margin value of  $\epsilon_k$ .



**Figure 4** Payoff ranking difference between players.

Using the EBE, the NBS is obtained from the first round of negotiation instead of four rounds, as illustrated in the ordinal ranking preferences.

## CONCLUSION

This article formulates a soft bargaining problem model that addresses possible agreements between the carrier and the container terminal operator (CTO) during unloading operations. The problem has a unique NBS that satisfies the preferences of both players. Agreement between players is considered better than a disagreement; therefore, cooperation is needed to maximize payoffs.

The notion of an epsilon bargaining equilibrium (EBE) for negotiation during unloading of containers in a transshipment operation was introduced. The epsilon solution is a function of the port geographical location, time of negotiation, and customer relation.

Because of the promising experimental results and the complexity of the transshipment to be streamlined, the application of such models by CTO, especially in the first business contact between a potential new carrier and an emerging CTO, is recommended. An in-depth study and analysis of the EBE factors is needed in the future to assess the NBS between the Carrier and the CTO in the transshipment process. The integration of more than one vessel per player and a multi player game, such as introducing different carriers with different terminal operators, is another possible extension for this research. Also, adding binding constraints, which render some possible agreements infeasible for some players, would be interesting to analyze in order to generate a more realistic work policy between the Carrier and the CTO.

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**APPENDIX**

**Lemma 1:**  $\pi_1$  is a convex function in  $y$ .

*Proof.* Let  $f(y_t) = SE * VOT - \delta P_t * PY * y_t$ .

For every  $y_t$ ,  $f(y_t)$  is of the form  $f(x) = Ax + B$  for any  $x' \subset S$  (set of feasible solutions) and  $x'' \subset S$ , and for  $c = [0:1]$ ,

$$f(c x' + (1 - c)x'') \leq c f(x') + (1 - c) f(x'').$$

Therefore,  $f(y_t)$  is a convex function in  $y_t$ :

$$\pi_1 = SE * VOT - \sum_{t=1}^T \delta P_t * PY * y_t = \sum_{t=1}^T f(y_t)$$

The summation of convex functions is a convex function; therefore  $\pi_1$  is convex in  $y$ .

**Lemma 2:**  $\pi_2$  is a convex function in  $y$ .

*Proof.* Let  $f(y_t) = (\delta P_t * PY - \delta C_t * CY) * y_t$ .

For every  $y_t$ ,  $f(y_t)$  is of form  $f(x) = Ax$ , which is a linear function. Any linear function is a convex function:

$$\begin{aligned} \pi_2 &= \sum_{t=1}^T [(\delta P_t * PY - \delta C_t * CY) * y_t] \\ &= \sum_{t=1}^T f(y_t) \end{aligned}$$

The summation of convex functions is also a convex function. Therefore  $\pi_2$  is a convex function in  $y$ .

From Lemmas 1 and 2,  $U$  is thus convex.

Axioms:

**Axiom 1:** Pareto efficiency.

Let  $(v_1, v_2)$  and  $(v_1', v_2')$  be a feasible set of  $U$ .

If  $v_1 > v_1'$  and  $v_2 > v_2'$ , then the bargaining solution does not allocate  $(v_1', v_2')$ .

**Axiom 2:** Symmetry.

Let  $(v_1, v_2)$  is in  $U$  if and only if  $(v_2, v_1)$  is in  $U$ , and  $d_1 = d_2$ .

Then the pair  $(v_1^*, v_2^*)$  of payoffs the bargaining solution allocated to  $(U, d)$  satisfies  $v_1^* = v_2^*$ .

**Axiom 3:** Invariance to equivalent payoff representation.

Let  $(v_1, v_2)$  be a member of  $U$ ,  $U' = (\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2)$ , and  $d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$  where  $\alpha_i$  and  $\beta_i$  are numbers such that  $\alpha_i > 0$ .

If the bargaining solution allocates  $(w_1, w_2)$  to  $(U, d)$ , then it allocates  $(\alpha_1 w_1 + \beta_1, \alpha_2 w_2 + \beta_2)$  to  $(U', d')$ .

**Axiom 4:** Independence of irrelevant alternative.

Let  $(U', d')$  be a bargaining problem such that  $U'$  is a subset of  $U$  and  $d = d'$ .

If the agreement of the bargaining solution allocated to  $(U, d)$  is in  $U'$ , then the bargaining solution allocates the same agreement to  $(U', d')$ .