Research Article

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Integrable generators of Lie algebras of vector fields on $\mathbb{C}^n$

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Abstract: There exist three vector fields with complete polynomial flows on $\mathbb{C}^n$, $n \geq 2$, which generate the Lie algebra generated by all algebraic vector fields on $\mathbb{C}^n$ with complete polynomial flows. In particular, the flows of these vector fields generate a group that acts infinitely transitively. The analogous result holds in the holomorphic setting.

Keywords: Density property, completely integrable vector fields, Andersen–Lempert theory

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1 Introduction

We will need the following two notions of flexibility and infinite transitivity introduced by Arzhantsev et al. [5], and the so-called density property introduced by Varolin [13, 14]. These notions describe in a precise way that the group of automorphisms $\mathrm{Aut}(X)$ of a complex variety X is "large". The subgroup $\mathrm{SAut}(X)$ generated by unipotent one-parameter subgroups, i.e., complete polynomial flows of polynomial vector fields, is called the *special automorphism group* of X. The Lie algebra of all holomorphic vector fields on X will be denoted by $\mathfrak{X}(X)$ and the Lie algebra of all holomorphic vector fields on X preserving a closed form ω will be denoted by $\mathfrak{X}_\omega(X)$. The group of ω-preserving holomorphic automorphisms is denoted by $\mathrm{Aut}_\omega(X)$.

Definition 1. (1) Let X be a complex algebraic variety. A point $x \in X_{\mathrm{reg}}$ is called *flexible* if the tangent space $T_x X$ is spanned by the orbits of unipotent one-parameter subgroups of $\mathrm{SAut}(X)$. The variety X is called *flexible* if every point $x \in X_{\mathrm{reg}}$ is flexible.

(2) Let X be a reduced Stein space. A point $x \in X_{\mathrm{reg}}$ is called *holomorphically flexible* if the completely integrable holomorphic vector fields on X span the tangent space $T_x X$. The space X is called *holomorphically flexible* if every point $x \in X_{\mathrm{reg}}$ is flexible.

Definition 2. Let X be a complex manifold and let G be a group. The action of G on X is said to be *infinitely transitive* if it acts m-transitively on X for any $m \in \mathbb{N}$.

A vector field is called *complete* or *completely integrable* if its flow map exists for all complex times. Note that the flow of a complete algebraic vector field is not necessarily algebraic, e.g., the flow of $zw^k \frac{\partial}{\partial z}$ for $(z, w) \in \mathbb{C}^2$ is complete and given by $\varphi_t(z, w) = (\exp(tw^k) \cdot z, w)$, which is not algebraic.

Definition 3. (1) Let X be a complex algebraic manifold. If the Lie algebra generated by the complete algebraic vector fields on X coincides with the Lie algebra of all algebraic vector fields on X, we say that X has the *algebraic density property*.

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(2) Let X be a complex manifold. If the Lie algebra generated by the complete holomorphic vector fields on X is dense (with respect to local uniform convergence) in the Lie algebra of all holomorphic vector fields on X, we say that X has the *density property*.

Definition 4. (1) Let X be a complex algebraic manifold with an algebraic *volume form* ω, i.e., a nowhere vanishing section of the canonical bundle. If the Lie algebra generated by the complete ω-preserving algebraic vector fields on X coincides with the Lie algebra of all ω-preserving algebraic vector fields on X, we say that (X, ω) has the *algebraic volume density property*.
(2) Let X be a complex manifold with a holomorphic *volume form* ω, i.e., a nowhere vanishing section of the canonical bundle. If the Lie algebra generated by the complete ω-preserving holomorphic vector fields on X is dense (with respect to local uniform convergence) in the Lie algebra of all ω-preserving holomorphic vector fields on X, we say that (X, ω) has the *volume density property*.

Remark 5. Note that Lie combinations of complete vector fields are in general not complete. However, we have the following approximation result [14, Proposition 2.4]: Let V, W be complete vector fields with flows φ_t, ψ_t, respectively. Then for $t > 0$, the following hold:
(1) An algorithm[1] for $V + W$ is given by $\psi_t \circ \varphi_t$.
(2) An algorithm for $[V, W]$ is given by $\psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}$.
This can be generalized [14] to show that any flow of a finite Lie combination of complete vector fields can be approximated uniformly on compacts of its maximal domain by compositions of these complete vector fields.

The main implication of the density property is the so-called Andersén–Lempert theorem:

Theorem 6 ([2, 9, 10, 14]). *Let X be a Stein manifold with the density property (resp. (X, ω) a Stein manifold with the volume density property). Let $\Omega \subseteq X$ be a Stein open subset (and resp. $H^{n-1}(\Omega, \mathbb{C}) = 0$) and let $\varphi\colon [0, 1] \times \Omega \to X$ be a $\mathcal{C}^1$-smooth map such that*
(1) *$\varphi_0\colon \Omega \to X$ is the natural embedding,*
(2) *$\varphi_t\colon \Omega \to X$ is holomorphic and injective (and resp. ω-preserving) for every $t \in [0, 1]$,*
(3) *$\varphi_t(\Omega)$ is a Runge subset of X for every $t \in [0, 1]$.*
Then for every $\varepsilon > 0$ and for every compact $K \subset \Omega$, there exists a continuous family $\Phi\colon [0, 1] \to \mathrm{Aut}(X)$ (resp. $\Phi\colon [0, 1] \to \mathrm{Aut}_\omega(X)$) such that $\Phi_0 = \mathrm{id}_X$ and $\|\varphi_t - \Phi_t\|_K < \varepsilon$ for every $t \in [0, 1]$.

Moreover, these automorphisms can be chosen to be compositions of flows of completely integrable generators of any dense Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}(X)$ (resp. $\mathfrak{X}_\omega(X)$).

The following Lemma goes back to Varolin. It is stated originally for jet interpolation and with approximation on a compact [13, Theorem 2 and Lemma 3.2]. For our application, a simpler version is sufficient, but we emphasize that the automorphism can be constructed using only finite compositions of flows of completely integrable generators of a dense Lie subalgebra $\mathfrak{g}$.

Lemma 7. *Let X be a Stein manifold with the density property (resp. (X, ω) a Stein manifold with the volume density property) with $\dim_{\mathbb{C}} X \geq 2$. Let $\mathfrak{g}$ be the Lie algebra generated by certain complete holomorphic vector fields on X such that $\mathfrak{g}$ is dense in $\mathfrak{X}(X)$ (resp. $\mathfrak{X}_\omega(X)$). Let $x_1, \dots, x_m, x_{m+1} \in X$ be pairwise distinct points. Then there exists a neighborhood U of x_{m+1} with the property that for all $y \in U$, there exists a holomorphic automorphism of X such that $F(x_1) = x_1, \dots, F(x_m) = x_m$ and $F(x_{m+1}) = y$. Moreover, F can be chosen to be a finite composition of flows of completely integrable generators of $\mathfrak{g}$.*

The simplified statement of this lemma allows us to give a relatively short and straightforward proof.

Proof. Around each point x_j where $j \in \{1, \dots, m + 1\}$, we choose a sufficiently small compact ball K_j inside a coordinate neighborhood such that $K_j \cap K_{j'} = \emptyset$ for $j \neq j'$ and such that the union $K = \bigcup_{j=1}^{m+1} K_j$ is $\mathcal{O}(X)$-convex. In case of the volume-density property, these coordinate neighborhoods have to be chosen so that

1 An algorithm χ_t for a vector field U satisfies weaker properties than a flow map, but $\lim_{n\to\infty} \chi_{t/n}^{\circ n}$ converges locally uniformly to the flow of U.

ω becomes the standard volume form in the respective coordinates, see, e.g., [3, Lemma 3.6]. For each point $j \in \{1, \dots, m+1\}$ and for each $\ell \in \{1, \dots, n\}$, let $V^{j,\ell}$ be the holomorphic vector field defined on K which vanishes on $K_1, \dots, K_{j-1}, K_{j+1}, \dots, K_{m+1}$ and agrees with the partial derivative $\frac{\partial}{\partial z_\ell}$ on K_j in the chosen coordinate neighborhood. The $\mathcal{O}(X)$-convex set K in the Stein manifold X admits an arbitrarily small Stein and Runge neighborhood Ω. For the volume-preserving case, note that $H^{n-1}(\Omega, \mathbb{C}) = 0$, since $n \geq 2$. For small enough time, the flow of $V^{j,\ell}$ exists on Ω and can be approximated arbitrarily well by a finite composition $F_t^{j,\ell}$ of flows of completely integrable generators of $\mathfrak{g}$ thanks to the Andersén–Lempert theorem. Consider the map $\Phi\colon (\mathbb{C}^n)^{m+1} \to X^{m+1}$ given by

$$\begin{pmatrix} (t_{1,1}, \dots, t_{1,n}) \\ \vdots \\ (t_{m+1,1}, \dots, t_{m+1,n}) \end{pmatrix} \mapsto \begin{pmatrix} F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}}(x_1) \\ \vdots \\ F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}}(x_{m+1}) \end{pmatrix}.$$

For a sufficiently close approximation, this map is submersive in $0 \in (\mathbb{C}^n)^{m+1}$. Now by the implicit function theorem, there exists a neighborhood $U_1 \times \dots \times U_{m+1}$ of $(x_1, \dots, x_{m+1}) \in X^{m+1}$ and a neighborhood $V_1 \times \dots \times V_{m+1}$ of $(0, \dots, 0) \in (\mathbb{C}^n)^{m+1}$ such that $\Phi\colon V_1 \times \dots \times V_{m+1} \to U_1 \times \dots \times U_{m+1}$ is a surjective (in fact, bijective) holomorphic map. In particular, for each $y \in U_{m+1} =: U$, we find $(t_{1,1}, \dots, t_{1,n}) \in V_1, \dots, (t_{m+1,1}, \dots, t_{m+1,n}) \in V_{m+1}$ such that

$$\begin{pmatrix} F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}}(x_1) \\ \vdots \\ F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}}(x_m) \\ F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}}(x_{m+1}) \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y \end{pmatrix}.$$

Note that the choice of the times depends holomorphically on y without any further control, but the map $F := F^{m+1,n}_{t_{m+1,n}} \circ \dots \circ F^{m+1,1}_{t_{m+1,1}} \circ \dots \circ F^{1,n}_{t_{1,n}} \circ \dots \circ F^{1,1}_{t_{1,1}} \colon X \to X$ is a finite composition of flows of completely integrable generators of $\mathfrak{g}$. □

Corollary 8. *Let X be a Stein manifold with the density property* (*resp.* (X, ω) *a Stein manifold with the volume density property*) *with* $\dim_{\mathbb{C}} X \geq 2$. *Let $\mathfrak{g}$ be as in the preceding lemma. Then the group of holomorphic automorphisms generated by the flows of completely integrable generators of $\mathfrak{g}$ acts infinitely transitively on X.*

Proof. It is sufficient to prove that we can construct a holomorphic automorphism F of X such that for any pairwise distinct points $x_1, \dots, x_m, p, q \in X$, we find an automorphism which fixes $x_1, \dots, x_m$ pointwise and maps p to q. To this extent, we choose a path γ in X with startpoint p and endpoint q which avoids the points $x_1, \dots, x_m$. Now we apply the preceding lemma repeatedly and move p to q along γ. Since the trace of the path γ is compact, this can be achieved in finitely many steps. □

Remark 9. The following implications are well known (see [11]):

$$\begin{aligned} \text{algebraic (volume) density property} &\implies \text{(volume) density property} \\ &\implies \text{holomorphic flexibility} \wedge \text{holomorphic infinite transitivity.} \end{aligned}$$

However, the algebraic density property may not necessarily imply (algebraic) flexibility or (algebraic) infinite transitivity. The results of [5] for irreducible algebraic varieties show that

$$\text{flexible} \iff \text{SAut infinitely transitive.}$$

In [7, Theorem 2.1], Arzhantsev, Kuyumzhiyan and Zaidenberg have shown that the smooth part of any nondegenerate complex-affine toric variety of dimension at least 2 is a flexible manifold. More recently, they showed [6] that finitely many unipotent subgroups are sufficient in order to generate a subgroup of SAut which acts m-transitively for any $m \in \mathbb{N}$, provided the toric variety in question is smooth in codimension 2.

In particular, for $X = \mathbb{C}^n$, they showed in the first available preprint of their paper that 4 unipotent subgroups are sufficient. In case of $n = 2$, even 3 unipotent subgroups are sufficient, see [6, Theorem 5.17].

Meanwhile, they have independently improved the result and show that 3 unipotent subgroups are sufficient for $\mathbb{C}^n$, $n \geq 2$.

In this short article we sharpen this result for $\mathbb{C}^n$, $n \geq 2$, and generalize it further to the holomorphic situation. We show that 3 explicitly given unipotent subgroups are always sufficient, and moreover generate the whole Lie algebra of volume-preserving algebraic vector fields. The result also holds in an algebro-holomorphic situation: 3 complete algebraic vector fields, one of them with necessarily non-algebraic flow, can be chosen so that they generate the Lie algebra of all polynomial vector fields on $\mathbb{C}^n$, $n \geq 2$.

2 Three generators

The following lemma is well known.

Lemma 10. *A triangular derivation*

$$D = \sum_{k=1}^{n} p_k(z_1, \dots, z_{k-1}) \frac{\partial}{\partial z_k} = p_1 \frac{\partial}{\partial z_1} + p_2(z_1) \frac{\partial}{\partial z_2} + \cdots + p_n(z_1, \dots, z_{n-1}) \frac{\partial}{\partial z_n}$$

of the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ *is locally nilpotent.*

Proof. We prove the claim by induction in $n \in \mathbb{N}$. The case $n = 1$ with $D = p_1 \frac{\partial}{\partial z_1}$, $p_1 \in \mathbb{C}$, is obvious. Since each action of D on $f \in \mathbb{C}[z_1, \dots, z_{n-1}, z_n]$ decreases the degree in z_n, eventually in $D^k(f) \in \mathbb{C}[z_1, \dots, z_{n-1}]$ for k large enough. The restriction of D to $\mathbb{C}[z_1, \dots, z_{n-1}]$ is locally nilpotent by the induction hypothesis. □

Theorem 11. *The Lie algebra of polynomial vector fields on* $\mathbb{C}^n$, $n \geq 2$, *is generated by the following three complete polynomial vector fields:*

$$\begin{aligned} U &= \frac{\partial}{\partial z_n}, \\ V &= \frac{\partial}{\partial z_n} + z_n^7 \frac{\partial}{\partial z_{n-1}} + z_n^3 z_{n-1}^7 \frac{\partial}{\partial z_{n-2}} + \cdots + z_n^3 z_{n-1}^3 \cdots z_3^3 z_2^7 \frac{\partial}{\partial z_1}, \\ W &= z_1^3 \cdots z_{n-1}^3 \cdot z_n \frac{\partial}{\partial z_n}. \end{aligned}$$

Proof. By Lemma 10, the derivations U and V are locally nilpotent and hence complete vector fields. The flow of W is given by $\varphi_t(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, \exp(t \cdot z_1^3 \cdots z_{n-1}^3) z_n)$ and complete as well.

The polynomial vector fields will be constructed inductively in several steps. It is sufficient to construct all monomial vector fields for each coordinate direction. We first need to take care of low degrees.

(1) By acting 4, 5, 6 and 7 times, respectively, with $[U, \cdot\,]$ on V, we obtain

$$z_n^p \frac{\partial}{\partial z_{n-1}} \quad \text{for } p = 0, 1, 2, 3.$$

(2) We now continue by induction in $k = n-1, \dots, 2$ by acting 4 times with $[\frac{\partial}{\partial z_{k-1}}, \cdot\,]$ on V and obtain

$$z_n^3 z_{n-1}^3 \cdots z_k^3 \frac{\partial}{\partial z_{k-1}},$$

and finally

$$z_n^3 \frac{\partial}{\partial z_{k-1}}, \qquad \frac{\partial}{\partial z_{k-1}},$$

by acting on the previously obtained field with $[\frac{\partial}{\partial z_\ell}, \cdot\,]$ 3 times for each $\ell = k, \dots, n-1$ (resp. n).

(3) Note that we can now get all lower degrees of already obtained monomials by forming a Lie bracket with partial derivatives. The left-hand side contains only terms for which we have established that they can be generated.

Next, for each $k = 1, \dots, n-1$, we form

$$\Big[\frac{\partial}{\partial z_n}, \Big[W, z_n \frac{\partial}{\partial z_k}\Big]\Big] + 2\Big[\frac{\partial}{\partial z_k}, W\Big] = z_1^3 \cdots z_{n-1}^3 \frac{\partial}{\partial z_k},$$
$$\Big[z_n^2 \frac{\partial}{\partial z_k}, z_1 \cdots z_{k-1} \cdot z_k^2 \cdot z_{k+1} \cdots z_{n-1} \frac{\partial}{\partial z_k}\Big] = 2 z_1 \cdots z_{n-1} \cdot z_n^2 \frac{\partial}{\partial z_k}.$$

One can get further $z_k^p \frac{\partial}{\partial z_k}$ for any $p \in \{0, 1, 2, 3\}$ and $k \in \{1, \dots, n-1\}$. Using W, one can also get $z_k^p \frac{\partial}{\partial z_n}$ for any $p \in \{0, 1, 2, 3\}$ and $k \in \{1, \dots, n-1\}$.

Moreover, we also want to obtain $z_n^2 \frac{\partial}{\partial z_n}$ and $z_n^3 \frac{\partial}{\partial z_n}$:

$$\Big[z_n \frac{\partial}{\partial z_k}, z_k z_n \frac{\partial}{\partial z_n}\Big] + z_k z_n \frac{\partial}{\partial z_k} = z_n^2 \frac{\partial}{\partial z_n},$$
$$\Big[z_n^2 \frac{\partial}{\partial z_k}, z_k z_n \frac{\partial}{\partial z_n}\Big] + 2 z_k z_n^2 \frac{\partial}{\partial z_k} = z_n^3 \frac{\partial}{\partial z_n}.$$

(4) We are now able to obtain all monomials by a two-step inductive process. First we get $z_k^p \frac{\partial}{\partial z_k}$ for any $p \geq 3$ and $k = 1, \dots, n$ by induction in $p \in \mathbb{N}$:

$$\Big[z_k^2 \frac{\partial}{\partial z_k}, z_k^p \frac{\partial}{\partial z_k}\Big] = (2 - p_k) z_k^{p+1} \frac{\partial}{\partial z_k}.$$

Let $p_1, \dots, p_n \in \mathbb{N}_0$, and let now $k, \ell \in \{1, \dots, n\}$ with $k \neq \ell$. Let f be a monomial in all other variables but z_ℓ, with power p_k in z_k. Proceeding by induction in ℓ, one gets

$$\Big[z_\ell^{p_\ell} \frac{\partial}{\partial z_\ell}, z_\ell \frac{\partial}{\partial z_k}\Big] = z_\ell^{p_\ell} \frac{\partial}{\partial z_k},$$
$$\Big[z_\ell^{p_\ell} \frac{\partial}{\partial z_k}, f(z) \cdot z_k \frac{\partial}{\partial z_k}\Big] = (p_k + 1) \cdot z_\ell^{p_\ell} f(z) \frac{\partial}{\partial z_k}. \qquad \square$$

Corollary 12. *The group of the holomorphic automorphisms generated by the flows of U, V and W acts infinitely transitively on $\mathbb{C}^n$.*

Proof. This is a consequence of the preceding theorem and Corollary 8. □

Remark 13. One should compare this theorem and its corollary also to the result by Wold and the author [4] that already 2 holomorphic automorphisms are sufficient to generate a dense subgroup of the holomorphic automorphism group of $\mathbb{C}^n$. However, one of these automorphisms was not obtained as a flow of a vector field. The method of proof is not related and cannot be used to further reduce the number of complete vector fields needed for generating the Lie algebra.

Theorem 14. *The Lie algebra of volume-preserving polynomial vector fields on $\mathbb{C}^n$, $n \geq 2$, is generated by the following three complete vector fields:*

$$U = \frac{\partial}{\partial z_n},$$
$$V' = \frac{\partial}{\partial z_n} + z_n^5 \frac{\partial}{\partial z_{n-1}} + z_n^2 z_{n-1}^5 \frac{\partial}{\partial z_{n-2}} + \cdots + z_n^2 z_{n-1}^2 \cdots z_3^2 z_2^5 \frac{\partial}{\partial z_1},$$
$$V'' = \frac{\partial}{\partial z_1} + z_1^5 \frac{\partial}{\partial z_2} + z_1^2 z_2^5 \frac{\partial}{\partial z_3} + \cdots + z_1^2 z_2^2 \cdots z_{n-2}^2 z_{n-1}^5 \frac{\partial}{\partial z_n}.$$

Proof. By Lemma 10 each of the derivations V' and V'' is locally nilpotent and induces an algebraic $\mathbb{C}_+$-action.

The polynomial vector fields will be constructed inductively in several steps. It is sufficient to construct all monomial shear vector fields of the form $f(z) \frac{\partial}{\partial z_k}$, where f is a monomial with $\frac{\partial f}{\partial z_k} = 0$. We first need to take care of low degrees.

(1) By acting 3-times (resp. 5-times) with $[U, \cdot]$ on V', we obtain

$$z_n^2 \frac{\partial}{\partial z_{n-1}}, \qquad \frac{\partial}{\partial z_{n-1}}.$$

(2) We now continue by induction in $k = n-1, \dots, 2$ by acting 3-times with $[\frac{\partial}{\partial z_{k-1}}, \cdot]$ on V' and obtain

$$z_n^2 z_{n-1}^2 \cdots z_k^2 \frac{\partial}{\partial z_{k-1}}.$$

Note again that we can now get all lower degrees of already obtained monomials by forming a Lie bracket with a previously obtained partial derivative. We obtain, in particular, $\frac{\partial}{\partial z_{k-1}}$ in the induction step.

Finally, we obtain, in particular, all partial derivatives $\frac{\partial}{\partial z_k}$ for $k = 1, \dots, n$.

(3) By acting similarly on V'', we obtain also

$$z_1^2 z_2^2 \cdots z_{k-1}^2 \frac{\partial}{\partial z_k}$$

for $k = 2, \dots, n$. Lowering the degrees, we get $z_\ell^2 \frac{\partial}{\partial z_k}$ for any $k, \ell \in \{1, \dots, n\}$ with $\ell \neq k$.

(4) For any indices $k, \ell \in \{1, \dots, n\}$ with $k \neq \ell$, any $p \in \mathbb{N}$ and any polynomial f in all other variables except z_k and z_ℓ the following holds:

$$\left[\frac{\partial}{\partial z_\ell}, \left[z_k^2 \frac{\partial}{\partial z_\ell}\left[z_\ell^2 \frac{\partial}{\partial z_k}, z_k^p \cdot f(z) \frac{\partial}{\partial z_\ell}\right]\right]\right] = 2(p+2) z_k^{p+1} \cdot f(z) \frac{\partial}{\partial z_\ell}.$$

By taking linear combinations, this allows us to construct, by induction in p, every polynomial in the variables $z_1, \dots, z_n$ except z_k in front of $\frac{\partial}{\partial z_k}$ for each k. This is sufficient to obtain all the desired polynomial vector fields according to the result of Andersén [1, Lemma 5.7] and the subsequent remark therein. See also the textbook of Forstnerič [8, Proposition 4.9.7]. □

Remark 15. Note that this theorem does not claim that the group generated by the flows of U, V' and V'' contains the group of tame algebraic automorphisms of $\mathbb{C}^n$. However, since U, V' and V'' generate at least the corresponding Lie algebra, every tame algebraic automorphism of $\mathbb{C}^n$ can be approximated uniformly on compacts by compositions of flows of U, V' and V'', see Remark 5.

Corollary 16. *The group generated by the (algebraic) flows of U, V' and V'' acts infinitely transitively on $\mathbb{C}^n$, i.e., the group generated by the three unipotent one-parameter subgroups arising as flows of U, V' and V'' acts infinitely transitively on $\mathbb{C}^n$.*

Proof. This is a consequence of the preceding theorem and Corollary 8 in the volume-preserving case. □

Given the initially discussed result [6, Theorem 2.1] of Arzhantsev, Kuyumzhiyan and Zaidenberg for finitely generated, infinitely transitive actions on toric varieties and the positive results for the (relative) density property for certain toric varieties [12] by Kutzschebauch, Leuenberger and Liendo, the following question arises naturally:

Question 17. Can the Lie algebra of polynomial vector fields on a toric variety with the density property be generated by finitely many complete polynomial vector fields, and by how many?

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