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On the Teichmüller–Nitsche Problem of T. Iwaniec, L.V. Kovalev, and J. Onninen

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Abstract
In 2011, Iwaniec et al. [8] posed the following problem: For which values s, t, $1 < s, t < \infty$, does there exist a harmonic homeomorphism $f : \mathcal{T}(s) \to \mathcal{T}(t)$, where $\mathcal{T}(\cdot)$ is a Teichmüller domain? In 2022, Bshouty et al. [3] solved this problem only for harmonic homeomorphisms f satisfying the symmetric property $f(\overline{z}) = \overline{f(z)}$. The purpose of this paper is two-fold: (a) to establish for a given $s > 1$ a necessary condition for the existence of harmonic homeomorphisms $f : \mathcal{T}(s) \to \mathcal{T}(t)$, and (b) to test in view of our results Conjecture 1.4 *loc. cit.* regarding the moduli of doubly-connected domains related by harmonic homeomorphisms.

Keywords Univalent harmonic mappings · Analytic dilatation · Modulus of doubly connected domains and affine capacity · Schwarz–Christoffel transformations

Mathematics Subject Classification Primary 31A05; Secondary 53C43 · 58E20 · 30C45

In Memory of Peter Duren

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1 Introduction

Unlike with conformally equivalent multiply connected domains, two doubly connected domains Ω and Ω^* need not have the same moduli if they are harmonically equivalent [3]. In 1962, Nitsche [14] conjectured that a harmonic homeomorphism from the annulus $1 < |z| < R$ to the annulus $1 < |z| < r$ exists if, and only if,

$$R \leq r + \sqrt{r^2 - 1} \quad \text{or} \quad r \geq \frac{1}{2}\left(R + \frac{1}{R}\right).$$

This long-standing conjecture was settled positively in a pioneering work by Iwaniec, Kovalev, and Onninen [7] following many attempts by several authors [9, 10, 13, 15, 17].This work led to interesting problems related to the following question raised in [8].

Question 1 *Characterize pairs of doubly connected domains $(\Omega,\ \Omega^*)$ that admit a univalent harmonic mapping from Ω onto Ω^*.*

Evidently, Ω may be replaced by any domain conformally equivalent to Ω.

Recently, Question 1 was tackled by D. Bshouty, A. Lyzzaik, A. Rasila, and A. Vasudevarao [3], but only for the special class of harmonic homeomorphisms f between Teichmüller domains Ω and Ω^* satisfying the symmetric property $f(\overline{z}) = \overline{f(z)}$. In this paper, the question will be addressed in full generality, again between Teichmüller domains.

Like with *loc. cit.*, the main result of the paper (Theorem 1) is based on the method of extremal length whose ingredients are given henceforth and may also be found along with some requisite related concepts in [2, pp. 220–223] and [11].

For a domain $\Omega \subset \mathbb{C}$ and a non-empty collection Γ of "curves" γ, where each γ consists of countably many arcs in Ω, let ρ be a non-negative lower semicontinuous function of Ω. The ρ-length of γ is defined by

$$L(\gamma, \rho) = \int_{\gamma} \rho\, |dz|,$$

where $L(\gamma, \rho)$ is possibly ∞.The ρ-area of Ω is defined by

$$A(\Omega, \rho) = \iint_{\Omega} \rho^2\, dx\, dy.$$

Because ρ is lower semicontinuous, $A(\Omega, \rho) > 0$ unless ρ is identically 0 in Ω. Given Γ, we let

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\Gamma, \rho).$$

Definition 1 The *extremal length* of Γ in Ω is defined by

$$\lambda_\Omega(\Gamma) = \sup_\rho \frac{L(\Gamma, \rho)^2}{A(\Omega, \rho)}$$

as ρ ranges over all lower semicontinuous functions which are not identically 0.

Since this notion depends only on Ω because the curves of Γ lie in Ω, we shall be using $\lambda(\Gamma)$ instead of $\lambda_\Omega(\Gamma)$.

A closely related concept to extremal length is the concept of the *modulus* of Γ defined by

$$\mathrm{Mod}(\Gamma) = \frac{2\pi}{\lambda(\Gamma)},$$

where $\mathrm{Mod}(\Gamma)$ is 0 or ∞ if $\lambda(\Gamma)$ is ∞ or 0 respectively. If Ω is a doubly connected domain and Γ is the collection of closed curves separating the boundary components of Ω, then the *modulus* of Ω is defined by $\mathrm{Mod}(\Omega) = \mathrm{Mod}(\Gamma)$. It is known that if Ω is the annulus with inner radius $r \geq 0$ and outer radius $R \leq \infty$, then $\mathrm{Mod}(\Omega) = \log(R/r)$ which is ∞ in case $r = 0$.

As the modulus of a doubly connected domain need not be preserved under a univalent affine transformation, the following definition was introduced in [8].

Definition 2 The *affine modulus* of a doubly connected domain $\Omega^* \subset \mathbb{C}$ is defined by

$$\mathrm{Mod}_@\, \Omega^* = \sup_\phi \mathrm{Mod}(\phi(\Omega^*)), \tag{1}$$

where the supremum is taken over all univalent affine transformations $\phi : \mathbb{C} \to \mathbb{C}$.

The affine modulus of an annulus equals its modulus. The proof of Nitsche's conjecture [7] inferred the truth of the following conjecture raised in [8] but only for circular annuli.

Conjecture 1 *If Ω and Ω^* are doubly connected domains with finite moduli, and if $f : \Omega \to \Omega^*$ is a harmonic homeomorphism, then*

$$\mathrm{Mod}_@\, \Omega^* \geq \log\cosh(\mathrm{Mod}\, \Omega). \tag{2}$$

For $s, t > 1$, we recall the Grötzsch and Teichmüller domains

$$\mathcal{G}(s) = \{z : |z| > 1\} \backslash [s, \infty) \text{ and } \mathcal{T}(t) = \mathbb{C} \backslash ([-1, 1] \cup [t, \infty))$$

respectively. Since the conjecture was settled for circular domains, it was proposed in *loc. cit.* to test it for other doubly connected domains whose moduli and affine moduli coincide, namely the Grötzsch and Teichmüller domains.

(a) *Grötzsch–Nitsche Problem*: For which values $s, t, 1 < s, t < \infty$, does a harmonic homeomorphism $f : \mathcal{G}(s) \to \mathcal{G}(t)$ exist?

(b) *Teichmüller–Nitsche Problem*: For which values s, t, $1 < s, t < \infty$, does a harmonic homeomorphism $f : \mathcal{T}(s) \to \mathcal{T}(t)$ exist?

Note that in (a) and (b) the domains $\mathcal{G}(s)$ and $\mathcal{T}(s)$ may be replaced by any doubly connected domain of the same modulus.

The existence question of a harmonic homeomorphism from a doubly connected domain Ω to a doubly connected domain Ω^* was raised and settled in the positive by Iwaniec, Kovalev, and Onninen [8, Thm. 1.5] provided that $\mathrm{Mod}_@\, \Omega^* > \mathrm{Mod}\, \Omega$ and $\mathbb{C}\backslash\Omega^*$ is unbounded.

Let f be a function whose domain Ω is symmetric with respect to the real axis. Throughout the paper, we say that f is *symmetric* if it satisfies the property $f(\overline{z}) = \overline{f(z)}$ for every $z \in \Omega$.

A simply connected domain $\Omega \subset \mathbb{C}$ is called *convex in the horizontal direction* (CHD) if its intersection with every horizontal line is either a line segment or empty. Also, we call Ω *convex in the positive direction of the imaginary axis* if the horizontal ray $w + it$, $t \geq 0$, lies in Ω for every point $w \in \Omega$. A univalent function f of a simply connected domain D is called *convex in the horizontal direction* if $f(D)$ is CHD. Also, we call f *convex in the positive direction of the imaginary axis* if $f(D)$ is *convex in the positive direction of the imaginary axis*.

Clunie and Sheil-Small [5, p. 37] established the following basic result.

Proposition 1 *Let $f = h + \overline{g}$ be a sense-preserving, harmonic, locally univalent mapping of $\mathbb{D}$. Then f is univalent and its range is CHD if, and only if, $h - g$ has the same properties.*

Note that $\mathbb{D}$ may be replaced by any simply connected domain.

Let f be a function from $\mathbb{D}$ to $\widehat{\mathbb{C}}$. We say that f has the *angular limit* $a \in \widehat{\mathbb{C}}$ at $\zeta \in \mathbb{T}$ if

$$f(z) \to a \quad \text{as} \quad z \in A, \quad z \to \zeta$$

for every Stolz angle A at ζ; $A = \{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < \pi/2 - \delta\}$ where $0 < \delta < \pi/2$. The *cluster set* of f at ζ is defined as the set of all $\omega \in \widehat{\mathbb{C}}$ for which there exist sequences (z_n) such that

$$z_n \in \mathbb{D}, \qquad z_n \to \zeta \qquad f(z_n) \to \omega \qquad \text{as} \qquad n \to \infty.$$

For these notions see Ch. Pommerenke [16]. These definitions extend analogously if $\mathbb{D}$ is replaced by a region Ω with an open analytic boundary arc containing ζ.

In this paper, it is assumed without prior notice that all harmonic mappings used are sense-preserving.

Let $t > 1$. Following [3] we consider the quadrilateral domain

$$\Pi_t = \{w : \mathrm{Re}\ w > -1\}\backslash\{w : 1 \leq \mathrm{Re}\ w \leq t, \mathrm{Im}\ w \leq 0\}.$$

There exist a unique Schwarz–Christoffel transformation

$$S : \overline{\mathbb{H}^+} \to \overline{\Pi_t}$$

and a unique value $\nu(t) > 1$, dependent on t, such that $S(-\infty) = -1+\infty\, i$, $S(-1) = -1-\infty\, i$, $S(1) = 1$, and $S(\nu(t)) = t$, respectively. Evidently, $\nu : (1, \infty) \to (1, \infty)$ is a strictly increasing homeomorphism.

The main result of this paper is the following.

Theorem 1 *Let $f : \mathcal{T}(s) \to \mathcal{T}(t)$ be a harmonic homeomorphism. Then*

$$t \geq \frac{\nu^{-1}(s)+1}{2} = \frac{\sqrt{2}}{4}\,\frac{s+3}{\sqrt{s+1}}. \tag{3}$$

Throughout the paper, we denote by $\mathbb{C}$, $\widehat{\mathbb{C}}$, $\widehat{\mathbb{R}}$, $\mathbb{D}$, and $\mathbb{T}$ the complex plane, the extended complex plane, the extended real line, the open unit disc, and the unit circle respectively. For $R > 1$, we denote by $\mathbb{T}_R$ and $\mathbb{A}_R$ the circle $\{z : |z| = R\}$ and the annulus $\{z : 1 < |z| < R\}$ respectively. Also, we denote by ∂S the boundary of a set S, and denote by γ° the open arc which is the interior of γ. If γ_1 and γ_2 are arcs such that the terminal point of γ_1 is the starting point of γ_2, then the product $\gamma_1\gamma_2$ is the arc obtained by traversing γ_1 followed by traversing γ_2.

The purpose of this paper is two-fold: first, to address the Teichmüller–Nitsche problem by establishing for a given $s > 1$ a necessary condition for the existence of a harmonic homeomorphism $f : \mathcal{T}(s) \to \mathcal{T}(t)$ (Theorem 1), and second, to test Conjecture 1 in light of our results.

The rest of the paper is organized as follows. In §2, we present analytic and geometric properties involving harmonic homeomorphisms $f : \mathcal{T}(s) \to \mathcal{T}(t)$. In §3, we prove Theorem 1 which solves the Teichmüller–Nitsche problem for harmonic homeomorphisms between $\mathcal{T}(s)$ and $\mathcal{T}(t)$. In §4, we test the lower bounds of t as proposed in Conjecture 1 and concluded in Theorem 1.

2 Analytic and geometric considerations involving harmonic homeomorphisms $f : \mathcal{T}(s) \to \mathcal{T}(t)$

In this section, we give requisite preliminary results and investigate some basic geometric properties involving harmonic homeomorphisms $f : \mathcal{T}(s) \to \mathcal{T}(t)$.

The following proposition is due to D. Bshouty and W. Hengartner [4].

Proposition 2 *Let f be a harmonic mapping of the annulus $\mathbb{A}_R$. Then f can be written as*

$$f(z) = h(z) + \overline{g(z)} + 2A\,\ln|z|, \tag{4}$$

where h and g are single-valued analytic functions on $\mathbb{A}_R$ and $A \in \mathbb{C}$.

Because of this interesting representation, and if $\mathcal{T}(s)$ and $\mathbb{A}_R$ are conformally equivalent, it is convenient to address the problem in question by considering throughout harmonic homeomorphisms from $\mathbb{A}_R$ onto $\mathcal{T}(t)$ instead of harmonic homeomorphisms from $\mathcal{T}(s)$ onto $\mathcal{T}(t)$.

Our first observation is the following.

Lemma 1 *Let $f : \mathbb{A}_R \to \mathcal{T}(t)$ be a harmonic homeomorphism with boundary correspondence that associates $\mathbb{T}$ and $\mathbb{T}_R$ with the intervals $[-1, 1]$ and $[t, \infty)$ respectively. Then there exists a Jordan arc α starting from a point $q_\alpha \in \mathbb{T}_R$ and terminating at a point $p_\alpha \in \mathbb{T}$, and a Jordan arc β starting from a point $p_\beta \in \mathbb{T}$ and terminating at a point $q_\beta \in \mathbb{T}_R$, such that f^{-1} restricted to the intervals $(-\infty, -1)$ and $(1, t)$ are the open arcs α° and β° respectively.*

Proof First, we show that α° terminates at a point $p_\alpha \in \mathbb{T}$. Suppose otherwise, and that α° accumulates along a non-degenerate subarc, δ, of $\mathbb{T}$.

We contend that δ is a proper subset of $\mathbb{T}$. Suppose to the contrary that $\delta = \mathbb{T}$. By invoking [1, Thm. 2] and [6, pp. 15–17], we conclude that f admits angular limits a.e. in $\mathbb{T}$. Then $\delta = \mathbb{T}$ implies that the angular limit of f is -1 a.e. in $\mathbb{T}$. But f is a harmonic homeomorphism on $\mathbb{A}_R$, which implies that f admits the limit -1 everywhere in $\mathbb{T}$. Thus we have a contradiction since the cluster set of f on $\mathbb{T}$ is the interval $[-1, 1]$, and δ is a proper subset of $\mathbb{T}$.

Since δ is a continuum in $\mathbb{T}$, we infer that $\mathbb{T}\backslash\delta$ is an open subarc of $\mathbb{T}$. Let C be a Jordan arc whose endpoints belong to $\mathbb{T}\backslash\delta$, lies otherwise in $\mathbb{A}_R$, and separates δ from $\mathbb{T}_R$. Denote by $D \subset \mathbb{A}_R$ the Jordan domain whose boundary contains C and δ, and let $\phi : \overline{\mathbb{D}} \to \overline{D}$ be a homeomorphism such that $\phi : \mathbb{D} \to D$ is a conformal map. Then $f \circ \phi$ is a bounded harmonic homeomorphism of $\mathbb{D}$, and $\phi^{-1}(\alpha^\circ)$ accumulates along the non-degenerate subarc $\phi^{-1}(\delta)$ of $\mathbb{T}$. This contradicts [1, Cor. 2], and the desired contradiction follows.

The same proof applies to show that β° starts and terminates at points $p_\beta \in \mathbb{T}$ and $q_\beta \in \mathbb{T}_R$ respectively.

We show now that α° starts at a point $q_\alpha \in \mathbb{T}_R$. Suppose to the contrary; then α° accumulates in $\mathbb{T}_R$ along a non-degenerate closed subarc of $\mathbb{T}_R$. Let ψ be a conformal map from $\mathbb{D}$ onto $\mathbb{A}_R \setminus \beta$. It is immediate that there exists an open subarc C' of $\mathbb{T}$ such that ψ extends to a conformal map from $\mathbb{D} \cup C'$ onto $(\mathbb{A}_R\backslash\beta) \bigcup (\mathbb{T}_R\backslash q_\beta)$ and that $\psi(C') = \mathbb{T}_R\backslash q_\beta$. We conclude that $f \circ \psi$ is a harmonic homeomorphism from $\mathbb{D}$ onto $\mathcal{T}(t) \cup (1, t)$ such that $\psi^{-1}(\alpha)$ accumulates along a non-degenerate subarc $\gamma \subset C'$. Again, by applying [1, Thm. 2] and [6, pp. 15–17], we infer that $f \circ \psi$ admits finite angular limits *a.e.* in $\mathbb{T}$, in particular in γ. But $f \circ \psi$ admits infinite angular limits *a.e.* in γ since $U = (\mathrm{Re}\, f) \circ \psi$ admits angular limit a.e. $-\infty$ in γ. Thus we have a contradiction, and the proof of the lemma is complete. □

Unless otherwise is specified, we assume throughout the following:

(a) $f : \mathbb{A}_R \to \mathcal{T}(t)$ is a sense-preserving harmonic homeomorphism;
(b) The Jordan arcs α and β are as obtained in Lemma 1.
(c) α and β are endowed with the directions inherited via f^{-1} from the positive directions of $(-\infty, -1)$ and $(1, t)$ respectively.

Proposition 2 leads at once to the following.

Lemma 2 *f can be written as*

$$f(z) = F(z) + 2\,\mathrm{Re}\{g(z) + \overline{A}\log_\alpha z\}, \qquad z \in \mathbb{A}_R\backslash\alpha, \tag{5}$$

where

$$F(z) = h(z) - g(z) + 2i\,(\mathrm{Im}\,A)\log_\alpha z, \tag{6}$$

h and g are single-valued analytic functions of $\mathbb{A}_R$, $\log_\alpha(\cdot)$ *is a single-valued branch analytic of the logarithm in* $\mathbb{A}_R\backslash\alpha$, *and* $A \in \mathbb{C}$.

The function F defined in (6) is called the $\alpha-$*horizontal shear* of f. Since f is a harmonic homeomorphism from $\mathbb{A}_R\backslash\alpha$ onto the CHD domain $\mathcal{T}(t) \setminus (-\infty, -1)$, we conclude by Proposition 1 that F restricted to $\mathbb{A}_R \setminus \alpha$ is a univalent function whose range $F(\mathbb{A}_R \setminus \alpha)$ is also CHD.

It is evident that $\alpha^\circ \cap \beta^\circ = \emptyset$. Being disjoint, α° and β° divide $\mathbb{A}_R$ into two mutually disjoint simply connected domains: $\mathcal{A}^+ = f^{-1}(\mathbb{H}^+)$ lying on the left-hand sides of both α and β, and $\mathcal{A}^- = f^{-1}(\mathbb{H}^-)$ lying on the right-hand sides of both α and β; note that the notation $\mathcal{A}^\pm$ is being used conveniently without involving R since R is assumed henceforth fixed. Obviously, one of the following four cases may arise: (i) $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$, (ii) $p_\alpha = p_\beta$ and $q_\alpha \neq q_\beta$, (iii) $p_\alpha \neq p_\beta$ and $q_\alpha = q_\beta$, and (iv) $p_\alpha = p_\beta$ and $q_\alpha = q_\beta$.

In the next lemma, we prove that the latter two cases fail to hold.

Lemma 3 $q_\alpha \neq q_\beta$.

Proof We establish the proof by contradiction. Suppose $q_\alpha = q_\beta$; then exactly two cases may occur. Each of these cases is investigated henceforth separately and is shown to be impossible.

1. $p_\alpha \neq p_\beta$. Let δ_1 be the circular arc obtained by traversing $\mathbb{T}$ clockwise from p_α to p_β, and let δ_2 be the circular arc obtained by traversing $\mathbb{T}$ clockwise from p_β to p_α. Also, let γ be the circular arc obtained by traversing $\mathbb{T}_R$ counterclockwise from q_α back to itself. Exactly two possibilities arise:

(a) $\mathcal{A}^+$ is the triangle whose boundary is $\alpha\delta_1\beta$, and $\mathcal{A}^-$ is the quadrilateral whose boundary is $\alpha\delta_2^{-1}\beta\gamma^{-1}$; see Fig. 1.
(b) $\mathcal{A}^+$ is the quadrilateral whose boundary is $\alpha\delta_2\beta\gamma$, and $\mathcal{A}^-$ is the triangle whose boundary is $\alpha\delta_1^{-1}\beta$; see Fig. 2.

Assume that 1(a) holds. Denote by $(\alpha^+)^\circ$ and $(\alpha^-)^\circ$ the arc α° as viewed from $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively.

It follows from (5) that $\partial F(\mathcal{A}^+)$ is a subset of $\mathbb{R}$, and consequently the conformality of F on $\mathcal{A}^+$ implies that there exist reals a^+ and b such that $a^+ < b$, and that F maps $(\alpha^+)^\circ$, $(\delta_1)^\circ$, and β° homeomorphically onto the intervals $(-\infty, a^+)$, (a^+, b), and (b, ∞) respectively.

We show that $\mathbb{R} \subseteq \partial F(\mathcal{A}^-)$. Since $F(\beta^\circ) = (b, \infty)$, we obtain $(b, \infty) \subset \partial F(\mathcal{A}^-)$. Moreover, we infer from (5) and the conformality of F on $\mathcal{A}^-$ that there exist "reals" a_- and a^- such that $-\infty \leq a_- < a^- < b$, and that F maps $(\alpha^-)^\circ$ and $(\delta_2^{-1})^\circ$ homeomorphically onto the intervals (a_-, a^-) and (a^-, b) respectively.

We show that $a_- = -\infty$. To do this, fix a point $\xi \in \alpha^\circ$, and denote by $\xi^+ \in (\alpha^+)^\circ$ and $\xi^- \in (\alpha^-)^\circ$ the copies ξ as viewed from $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively. Then, by using (5), we can write

$$f(\xi) = F(\xi^\pm) + 2\,\mathrm{Re}\{g(\xi) + \overline{A}\log_\alpha \xi^\pm\}.$$

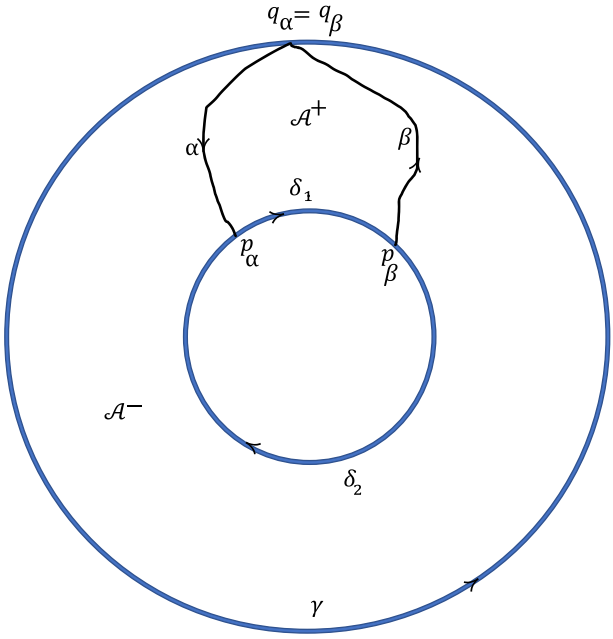

Fig. 1 Case 1(a): $p_\alpha \neq p_\beta$

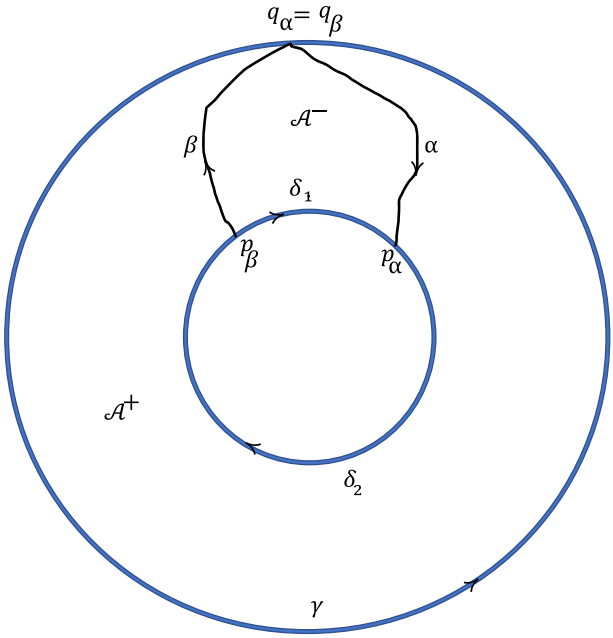

Fig. 2 Case 1(b): $p_\alpha \neq p_\beta$

It follows that

$$F(\xi^+) - F(\xi^-) = -2\,\mathrm{Re}\left\{\overline{A}\left(\log_\alpha \xi^+ - \log_\alpha \xi^-\right)\right\} = -4\pi\ \mathrm{Im}\, A, \tag{7}$$

which is bounded. Hence, our contention that $a_- = -\infty$ is true, and $\mathbb{R} \subseteq \partial F(\mathcal{A}^-)$.

Now we establish the desired contradiction for the proof of the lemma for 1(a). Because F is a univalent function in $\mathcal{A}^-$, it admits finite angular limits almost everywhere on $\partial\mathcal{A}^-$ [16, Cor. 2.17(i)]. The same also holds for f by [1, Thm. 2] and [6, pp. 15–17]. Let $\zeta \in \gamma^\circ$ be a point at which the finite angular limits of F and f exist and are ω and ω' respectively. Because $\mathbb{R} \subseteq \partial F(\mathcal{A}^-)$, it follows that $\omega \in \overline{\mathbb{H}^-}$. But $\omega \notin \mathbb{R}$, otherwise there exists a point in $\partial\mathcal{A}^- \setminus \gamma$ at which the angular limit of F is also ω which implies that F is not univalent in $\mathcal{A}^-$. Thus $\omega \in \mathbb{H}^-$. Let (z_n) be a sequence in $\mathcal{A}^-$ converging non-tangentially to ζ; then $\lim f(z_n) = \omega'$. Since $F \circ f^{-1} : \mathbb{H}^- \to F(\mathcal{A}^-)$ is a homeomorphism, we conclude that

$$\omega = \lim F(z_n) = \lim F \circ f^{-1}\left(f(z_n)\right) = F \circ f^{-1}\left(\lim f(z_n)\right) = F \circ f^{-1}(\omega'),$$

and consequently $\omega \in F(\mathcal{A}^-)$. But $\omega \in \partial F(\mathcal{A}^-)$. Therefore, F has no finite angular derivatives on γ°, which is false.

This completes the proof of the lemma for case 1(a). The proof of the lemma for 1(b) is essentially the same.

2. $p_\alpha = p_\beta$. Let δ be the circular arc obtained by traversing $\mathbb{T}$ clockwise from p_α back to itself. Exactly four possibilities arise:

(a.i) $\mathcal{A}^+$ is the Jordan domain whose boundary is $\alpha\beta$, and $\mathcal{A}^-$ is the quadrilateral whose boundary is $\alpha\delta^{-1}\beta\gamma^{-1}$; see Fig. 3.

(a.ii) $\mathcal{A}^+$ is the quadrilateral whose boundary is $\alpha\delta\beta\gamma$, and $\mathcal{A}^-$ is the Jordan domain whose boundary is $\alpha\beta$; see Fig. 4.

(b.i) $\mathcal{A}^+$ is the triangle whose boundary is $\alpha\delta\beta$, and $\mathcal{A}^-$ is the triangle whose boundary is $\alpha\beta\gamma^{-1}$; see Fig. 5.

(b.ii) $\mathcal{A}^+$ is the triangle whose boundary is $\alpha\beta\gamma$ and $\mathcal{A}^-$ is the triangle whose boundary is $\alpha\delta^{-1}\beta$; see Fig. 6.

The proofs that the four possibilities 2(a.i), 2(a.ii), 2(b.i), and 2(b.ii) fail to hold are essentially analogous to that of 1(a). This completes the proof of the lemma. □

Assume now that $q_\alpha \neq q_\beta$. Let $\mathcal{A}^+$, $\mathcal{A}^-$, δ_1, δ_2, and δ be defined as above. Let γ_1 be the circular arc obtained by traversing $\mathbb{T}_R$ in the counterclockwise direction from q_α to q_β, and let γ_2 be the circular arc obtained by traversing $\mathbb{T}_R$ in the counterclockwise direction from q_β to q_α.

Lemma 3 leads at once to the following.

Lemma 4 *Exactly one of the following two cases holds.*

(A) $p_\alpha \neq p_\beta$ *and* $q_\alpha \neq q_\beta$. *In this case, there is exactly one possibility for which* $\mathcal{A}^+$ *is the quadrilateral whose boundary is* $\alpha\delta_1\beta\gamma_2$, *and* $\mathcal{A}^-$ *is the quadrilateral whose boundary is* $\alpha\delta_2^{-1}\beta\gamma_1^{-1}$; *see Fig. 7.*

(B) $p_\alpha = p_\beta$ *and* $q_\alpha \neq q_\beta$. *In this case, exactly one of the following two cases holds.*

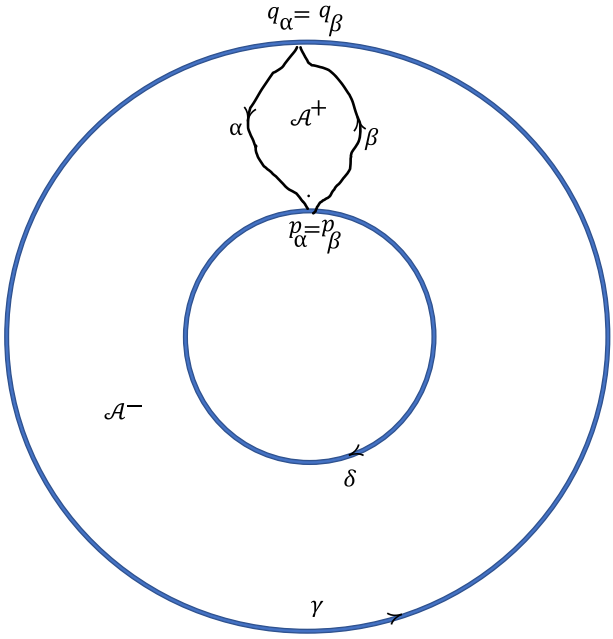

Fig. 3 Case 2(a.i): $p_\alpha = p_\beta$

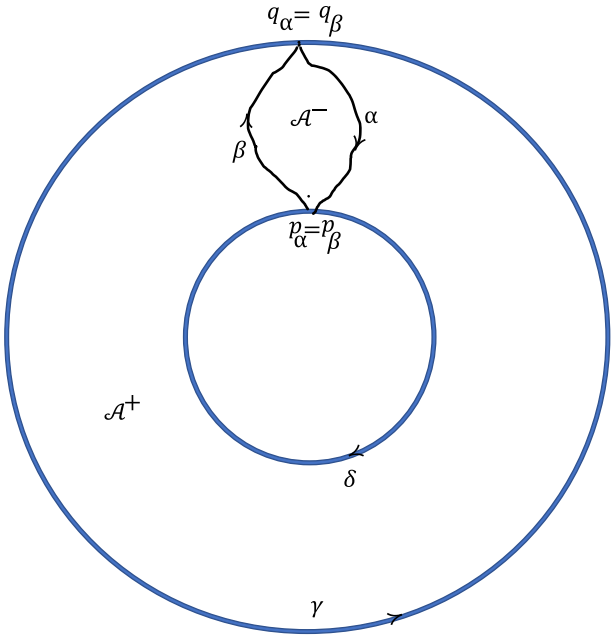

Fig. 4 Case 2(a.ii): $p_\alpha = p_\beta$

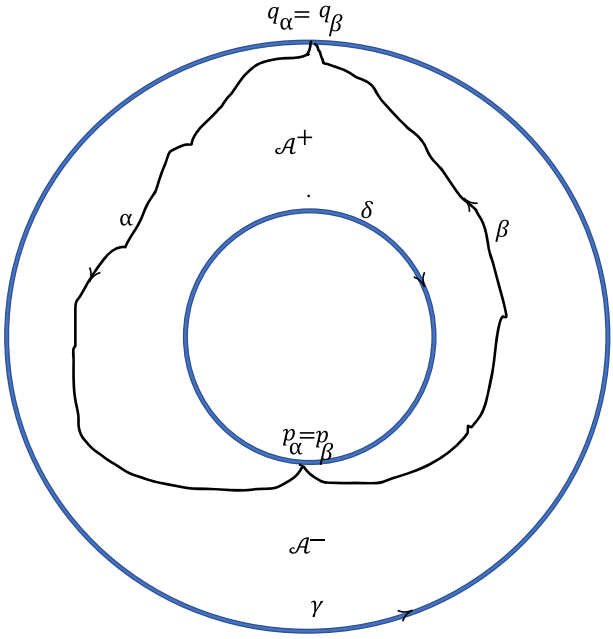

Fig. 5 Case 2(b.i): $p_\alpha = p_\beta$

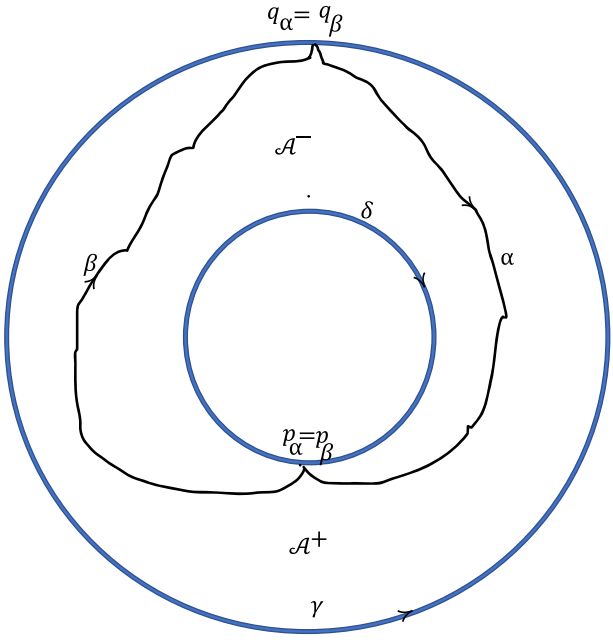

Fig. 6 Case 2(b.ii): $p_\alpha = p_\beta$

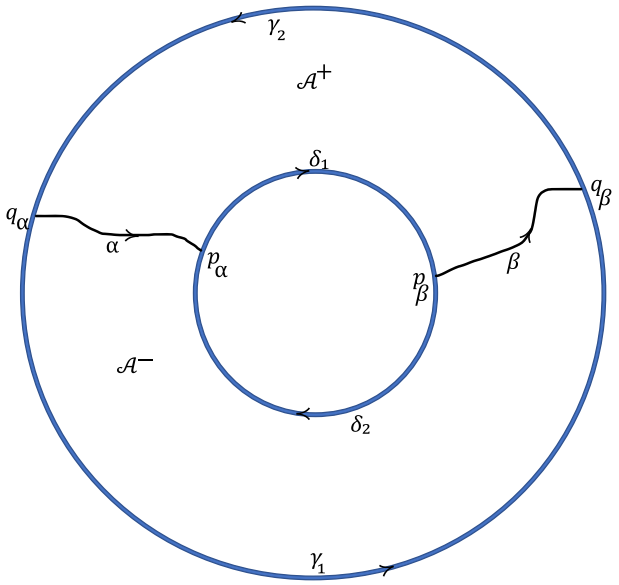

Fig. 7 Case (A): $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$

(i) *$\mathcal{A}^+$ is the quadrilateral whose boundary $\alpha\delta\beta\gamma_2$, and $\mathcal{A}^-$ is the triangle whose boundary is $\alpha\beta\gamma_1^{-1}$; see Fig. 8.*
(ii) *$\mathcal{A}^+$ is the triangle whose boundary is $\alpha\beta\gamma_2$, and $\mathcal{A}^-$ is the quadrilateral whose boundary is $\alpha\delta^{-1}\beta\gamma_1^{-1}$; see Fig. 9.*

Remark 1 In the forthcoming Lemma 7, we shall prove that (B) of Lemma 4 fails to hold. Meanwhile, however, since (B.i) and (B.ii) are similar, we assume that either (A) or (B.i) holds and that in either case $\mathcal{A}^+$ is a quadrilateral.

3 The Teichmüller–Nitsche Problem

The purpose of this section is to tackle the Teichmüller–Nitsche Problem by establishing for a given $s > 1$ a necessary condition for the existence of a harmonic homeomorphism $f : \mathcal{T}(s) \to \mathcal{T}(t)$.

Note that (A) and (B), including (B.i) and (B.ii), refer henceforth to the cases mentioned in Lemma 4.

Our first result here describes the behavior of the horizontal shear F of f.

Lemma 5 *Let F be the horizontal shear of f defined in (6). Then*

(a) $F(\mathcal{A}^\pm) = \mathbb{H}^\pm$.
(b) *$f \circ F^{-1}$ maps every horizontal line other than the real line homeomorphically onto its counterpart in the image plane.*
(c) *There exist reals a^+, a^-, b, and c that satisfy the following properties:*

(i) $-\infty < a^+ < b < c < \infty$ *and* $-\infty < a^- \leq b < c < \infty$;

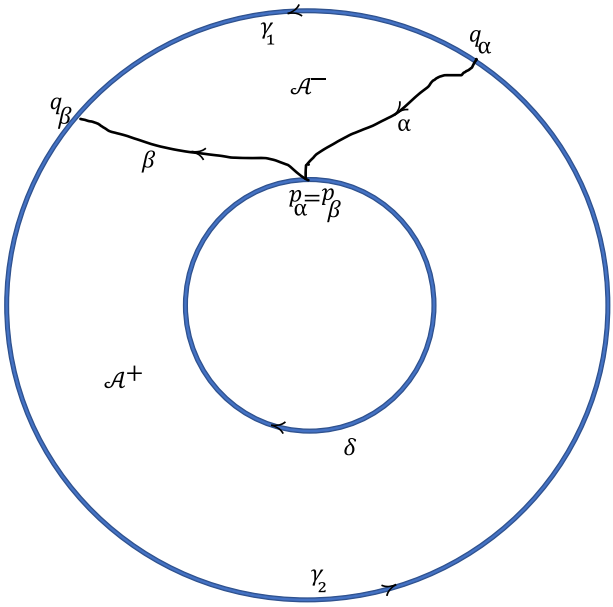

Fig. 8 Case (B.i): $p_\alpha = p_\beta$ and $q_\alpha \neq q_\beta$

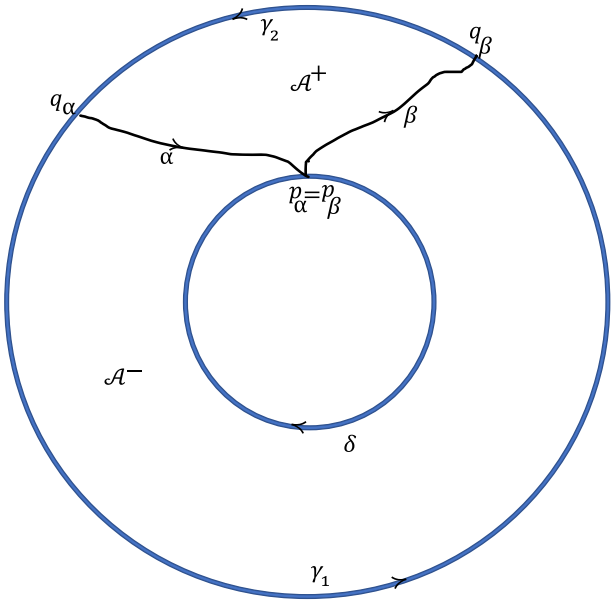

Fig. 9 Case (B.ii): $p_\alpha = p_\beta$ and $q_\alpha \neq q_\beta$

(ii) $a^- < b$ *if (A) holds and* $a^- = b$ *if (B.i) holds;*
(iii) $F : (\alpha^{\pm})^\circ \to (-\infty, a^{\pm})$ *and* $F : \beta^\circ \to (b, c)$ *are analytic homeomorphisms.*

In addition, because of (a) and the conformality of F, the following mappings are also analytic homeomorphisms:

- $F : \delta_j^\circ \to (a^{\pm}, b)$, $j = 1, 2$, if (A) holds;
- $F : \delta^\circ \to (a^+, b)$, if (B.i) holds;
- $F : \gamma_j^\circ \to (c, \infty)$, $j = 1, 2$, if either (A) or (B.i) holds.

Proof (a) Recall that F is a univalent function of $\mathbb{A}_R \setminus \alpha$, and that $F(\mathbb{A}_R \setminus \alpha)$, $F(\mathcal{A}^+)$, and $F(\mathcal{A}^-)$ are CHD. Also, recall that $F(\mathcal{A}^{\pm}) \subseteq \mathbb{H}^{\pm}$.

Assume now that (A) holds in which case $\mathcal{A}^+$ is the quadrilateral whose boundary is $\alpha\delta_1\beta\gamma_2$. Observe that by applying the last argument used in proving Lemma 3(1.a), we conclude that F admits only finite real angular limits almost everywhere on γ_2. We contend that $\partial F(\mathcal{A}^+) \subseteq \mathbb{R}$, which implies $F(\mathcal{A}^+) = \mathbb{H}^+$. Suppose to the contrary that $\omega \in \partial F(\mathcal{A}^+) \bigcap \mathbb{H}^+$. Let L be a horizontal line passing through ω. It is immediate that L meets $F(\mathcal{A}^+)$ in an open semi-line or segment $\widetilde{L}$ terminating at a finite boundary point of $F(\mathcal{A}^+)$, We may assume that this point is ω. Then [16, Prop. 2.14] implies that $F^{-1}(\widetilde{L})$ is a Jordan arc in $\mathcal{A}^+$ terminating at a point $\zeta \in \gamma$ at which F admits ω as a non-real angular limit [16, Cor. 2.17]. This proves our contention and $F(\mathcal{A}^+) = \mathbb{H}^+$.

Analogous proofs apply to show that $F(\mathcal{A}^-) = \mathbb{H}^-$ when (A) holds, and to show as well that $F(\mathcal{A}^{\pm}) = \mathbb{H}^{\pm}$ when (B.i) holds. This completes the proof of (a).

(b) This follows at once from the definition of the horizontal shear F of f.

(c) By using (a) once again and invoking Carathéodary's theorem [16, p. 18], we conclude that F restricted to $\mathcal{A}^{\pm}$ extends to a homeomorphism from $\partial F(\mathcal{A}^{\pm}) \subset \widehat{\mathbb{C}}$ onto $\widehat{\mathbb{R}}$. This implies that there exist "reals" $a_1^+, a_1^-, a_2^+, a_2^-, b$, and c that satisfy the following properties:

(i) $-\infty \le a_1^+ < a_2^+ < b < c \le \infty$ and $-\infty \le a_1^- < a_2^- \le b < c \le \infty$;
(ii) $a_2^- < b$ if (A) holds and $a_2^- = b$ if (B.i) holds;
(iii) $a_1^{\pm}$ and c cannot be simultaneously infinite;
(iv) $F : (\alpha^{\pm})^\circ \to (a_1^{\pm}, a_2^{\pm})$ and $F : \beta^\circ \to (b, c)$ are analytic homeomorphisms.

Assume that either (A) or (B) holds. We show that $a_1^+ = -\infty$. Suppose to the contrary that a_1^+ is real. Since $F : \mathcal{A}^+ \to \mathbb{H}^+$ is a conformal homeomorphism, we conclude that there exists an open subarc γ_2' of γ_2 with one endpoint at q_α such that $F : \gamma_2' \to (-\infty, a_1^+)$ is an analytic homeomorphism. For $\epsilon > 0$, let $L_\epsilon = \{\zeta : \text{Im } \zeta = \epsilon\}$ and $\ell_\epsilon = f \circ F^{-1}(L_\epsilon)$; $f \circ F^{-1} : L_\epsilon \to \ell_\epsilon$ is a sense-preserving homeomorphism. Choose $x_1 \in (-\infty, a_1^+)$ and $x_2 \in (a_1^+, a_2^+)$, and let $y_{1,\epsilon} = x_1 + i\epsilon$ and $y_{2,\epsilon} = x_2 + i\epsilon$. Then $y_{1,\epsilon}, y_{2,\epsilon} \in L_\epsilon$ and $\text{Re } f \circ F^{-1}(y_{1,\epsilon}) < \text{Re } f \circ F^{-1}(y_{2,\epsilon})$. It follows that

$$\limsup_{\epsilon \to 0} \text{Re } f \circ F^{-1}(y_{1,\epsilon}) \le \lim_{\epsilon \to 0} \text{Re } f \circ F^{-1}(y_{2,\epsilon}) = f \circ F^{-1}(x_2) \in (-\infty, -1).$$

But $x_2 \in (a_1^+, a_2^+)$ was chosen arbitrarily; hence $\lim_{\epsilon \to 0} \text{Re } f \circ F^{-1}(y_{1,\epsilon})$ exists and $\lim_{\epsilon \to 0} \text{Re } f \circ F^{-1}(y_{1,\epsilon}) = -\infty$.

It follows that f admits an infinite angular limit at every point of γ_2' which contradicts [1, Thm. 2] and [6, pp. 15–17]. Thus $a_1^+ = -\infty$ as desired. The proof that $a_1^- = -\infty$ follows from the argument of Lemma 3 involving (7). Therefore, (c) follows by taking $a^\pm = a_2^\pm$ and keeping b and c the same.

This completes the proof of the lemma. □

For the rest of the paper we let $\varphi = f \circ F^{-1}$.

By recalling Lemma 5(c), let

$$\widetilde{\mathcal{T}}^\pm(a^\pm, b, c) = \mathbb{H}^\pm \cup (-\infty, a^\pm) \cup (b, c),$$

and let $U^\pm = \text{Re } \varphi$ in $\widetilde{\mathcal{T}}^\pm(a^\pm, b, c)$. We have:

Lemma 6 *Let $a^\pm, b$, and c be the reals appearing in Lemma 5(c). Then there exist real harmonic functions U^+ and U^- of $\widetilde{\mathcal{T}}^+(a^+, b, c)$ and $\widetilde{\mathcal{T}}^-(a^-, b, c)$, respectively, such that*

(i) *U^+ and U^- are harmonic continuations of each other across (b, c);*
(ii) *$U^\pm$ map every horizontal line in $\widetilde{\mathcal{T}}^\pm(a^\pm, b, c)$ homeomorphically onto $\mathbb{R}$;*
(iii) *$U^\pm : (-\infty, a^\pm) \to (-\infty, -1)$ and $U^\pm : (b, c) \to (1, t)$ are harmonic homeomorphisms in the sense that at every point of $(-\infty, a^\pm) \cup (b, c)$ there exists an open disc in which $U^\pm$ is harmonic;*
(iv) *$U_x^\pm > 0$.*

Proof Lemma 5 leads at once to (i), (ii), and (iii). For the proof of (iv), the truth of (ii) implies that the functions $U^+(z) + iy$ and $U^-(z) + iy$, where $z = x + iy$, are sense-preserving harmonic homeomorphisms; hence their dilatations U_x^+ and U_x^- are strictly positive [12]. This completes the proof of the lemma. □

Further properties of $U^\pm$ are included in the following lemma.

Lemma 7 *The following hold true:*

(a) *(B) does not exist;*
(b) *For any reals x and y, $x < y$, and cluster points x', y' of $U^\pm$ at x and y respectively, we have $x' \le y'$;*
(c) *The cluster set of $U^\pm$ at every real x is either a point or a closed interval;*
(d) *The cluster set of $U^\pm$ along $\mathbb{R}$ is $\mathbb{R}$ itself.*

Proof (a) It suffices to show that (B.i) does not hold. Suppose otherwise in which case $a^- = b$. By invoking Lemma 6(iii), we conclude that $F(\mathcal{A}^-)$ contains the vertical strip $\{w : |\text{Re } w| < 1\}$ which contradicts Lemma 5(a). This proves (a).

It suffices to show that (b), (c), and (d) hold for U^+ as the respective proofs for U^- are analogous.

(b) It follows from [1, Thm. 2] and [6, pp. 15–17] that φ admits finite angular limits almost everywhere in $\mathbb{R}$. This implies that there exists r, $x < r < y$, at which the finite angular limit, ω, of φ exists. Consider the semi-line $L : r + i\tau$, $\tau \ge 0$, and let $\ell = \varphi(L)$; then ℓ is an unbounded Jordan curve starting from ω and lying otherwise in $\mathbb{H}^+$. Let D and D' be the wedges bounded by $(-\infty, r] \cup L$ and $(r, \infty] \cup L$ respectively, and let

G and G' be the simply connected domains bounded by $(-\infty, \omega] \cup \ell$ and $(\omega, \infty] \cup \ell$ respectively; then $\varphi(D) = G$ and $\varphi(D') = G'$. It follows that the cluster sets of φ at x and y lie in $(-\infty, \omega]$ and $[\omega, \infty)$ respectively. But the cluster set of φ and U^+ at every real coincide. This concludes the proof of (b).

(c) It follows from (b) that the cluster sets of U^+ at distinct reals concur in at most one single point. Because the cluster set of U^+ at any real is either a point or an interval, we conclude that there are at most countably many reals for which these cluster sets are intervals. It follows that the cluster set of U^+ is a singleton almost everywhere in $\mathbb{R}$, and the proof of (c) is complete.

(d) Fix y_0 and M such that $y_0 > M > t$. Consider the semi-line $\ell : y_0 + i\tau$, $\tau \geq 0$, and let $L = \varphi^{-1}(\ell)$; then L is an unbounded Jordan curve starting from some point $x_0 \in \mathbb{R}$ and lying otherwise in $\mathbb{H}^+$. Denote by D and G the simply connected subdomains of $\mathbb{H}^+$ bounded by $L \cup \{x : x \geq x_0\}$ and $\ell \cup \{y : y \geq y_0\}$ respectively. Then the mapping $\varphi : D \to G$ a harmonic homeomorphism, and consequently it assumes finite angular limits almost everywhere on ∂D; in particular the finite angular limits associated with the semi-line $\{x : x > x_0\}$ lie in the semi-line $\{y : y > y_0\}$. Since the cluster set of φ along $\mathbb{R}$ is connected, we conclude that it contains M. But the cluster set of φ and U^+ at every real coincide. Hence the cluster set of U^+ contains M which proves (d). This completes the proof of the lemma. □

Now we define the function

$$U(z) = \begin{cases} U^+(z), \ z \in \mathbb{H}^+ \cup (b, c), \\ U^-(z), \ z \in \mathbb{H}^- \cup (b, c). \end{cases}$$

In view of Lemmas 6 and 7, U satisfies the following properties:

- U is harmonic and $U_x > 0$ in $\mathbb{C} \setminus [(-\infty, b] \cup [c, \infty)]$;
- U maps every horizontal line different from the real line homeomorphically onto $\mathbb{R}$;
- U continues harmonically in $\mathbb{H}^{\pm}$ to $(-\infty, a^{\pm})$ such that $U : (-\infty, a^{\pm}) \to (-\infty, -1)$ are diffeomorphisms;
- $U : (b, c) \to (1, t)$ is a diffeomorphism.

Let $a = \max\{a^+, a^-\}$; then Lemma 7(a) implies at once that $a < b$. We introduce the function

$$\mathcal{U}(z) = \frac{U(z) + U(\overline{z})}{2}, \qquad z \in \mathbb{C} \setminus [(-\infty, b] \cup [c, \infty)].$$

In view of the above properties of U, we conclude:

Lemma 8 *$\mathcal{U}$ satisfies the following properties:*

(a) *$\mathcal{U}$ is a real harmonic function for which the cluster set at every $x \in (-\infty, a)$ lies in $(-\infty, 0)$ and is either a point or a closed interval;*
(b) $\mathcal{U}((b, c)) = (1, t)$;
(c) *$\mathcal{U}$ is symmetric;*
(d) $\mathcal{U}_x > 0$ *in* $\mathbb{C} \setminus [(-\infty, b] \cup [c, \infty)]$.

Denote by $\mathcal{V}$ the harmonic conjugate of $\mathcal{U}$ which is zero at some point in (b, c), and let $\Phi = \mathcal{U} + i\,\mathcal{V}$ in $\mathbb{C}\backslash[(-\infty, b] \cup [c, \infty)]$. Then we have:

Lemma 9 *Φ satisfies the following properties:*

(a) *Φ is a symmetric analytic function;*
(b) *Φ is convex in the positive direction of the imaginary axis in $\mathbb{H}^+ \cup (b, c)$;*
(c) *The cluster set of Φ restricted to $\mathbb{H}^+$ at every point $x \in (-\infty, a)$ lies in the open half-plane $\{w : \mathrm{Re}\ w < 0\}$, and $\Phi((b, c)) = (1, t)$;*
(d) $\Phi(\mathbb{H}^+) \subset \mathbb{C}\backslash\{w : 1 \leq \mathrm{Re}\ w \leq t, \mathrm{Im}\ w \leq 0\}$.

Proof (a) This follows at once from [3, Lem. 1].
(b) The proof is analogous to that of [3, Lem. 3(iii)].
(c) This is immediate from (a) and Lemma 8(a).
(d) This follows at once from (b) and (c).
This completes the proof. □

Recall the quadrilateral domain Π_t introduced in §1. We shall need the quadrilateral domain

$$\Pi^t = \{w : \mathrm{Re}\ w > 0\}\} \setminus \{w : 1 \leq \mathrm{Re}\ w \leq t, \mathrm{Im}\ w \leq 0\}.$$

Observe that the linear transformation $\kappa(z) = 2z - 1$ satisfies $\kappa(\Pi^t) = \Pi_{2t-1}$.
A requisite result from *loc. cit.* is the following.

Lemma 10 *For $s > 1$, let $S : \overline{\mathbb{H}^+} \to \overline{\Pi_t}$ be the Schwarz–Christoffel transformation satisfying $S(-\infty) = -1 + \infty\, i$, $S(-1) = -1 - \infty\, i$, $S(1) = 1$, and $S(s) = t$. Then*

$$t = \nu^{-1}(s) = \frac{s+3}{\sqrt{2(s+1)}} - 1. \tag{8}$$

Now we prove Theorem 1.

Proof Let $\mathbb{A}_R$ be the annulus that is conformally equivalent to $\mathcal{T}(s)$. Because the pre-composition of a harmonic homeomorphism with a conformal map is also a harmonic homeomorphism, we assume $f : \mathbb{A}_R \to \mathcal{T}(t)$.

Recall the Jordan arcs α and β, the quadrilaterals $\mathcal{A}^+$ and $\mathcal{A}^-$, the horizontal shear F of f; the triplet a, b, c, the univalent function Φ, the quadrilateral domain Π^t, and the linear transformation κ.

We define the following families of rectifiable paths.

(a) The family Γ of paths separating the boundary components of $\mathbb{A}_R$;
(b) The family Γ^+ of paths starting from a point in α°, terminating at a point in β°, and lying otherwise in $\mathcal{A}^+$;
(c) The family Γ^- of paths starting from a point in α°, terminating at a point in β°, and lying otherwise in $\mathcal{A}^-$;
(d) The family Σ of paths starting from a point in $(-\infty, a)$, terminating at a point in (b, c), and lying otherwise in $\mathbb{H}^+$;

(e) The family Δ of paths starting from a point in Re $w = 0$, terminating at a point in $(1, t)$, and lying otherwise in Π^t;
(f) The family $\widetilde{\Delta}$ of paths $\tilde{\delta} \subset \Phi(\mathbb{H}^+)$ starting from a point in Re $w = 0$, terminating at a point in $(1, t)$, and lying otherwise in Π^t;
(g) The family Υ of paths starting from a point in $(-\infty, -1)$, terminating at a point in $(1, \nu(2t-1))$, and lying otherwise in $\mathbb{H}^+$;
(h) The family Λ of paths separating the boundary components of $\mathcal{T}(2\nu(t)-1)$.

For any path γ let $\overline{\gamma}$ be its reflection with respect to the real axis, and for any family $\mathcal{F}$ of paths γ let $\overline{\mathcal{F}} = \{\overline{\gamma} : \gamma \in \mathcal{F}\}$.

Let $\sigma = \sigma(\tau)$, $0 \le \tau \le 1$, be a path in Σ. By virtue of Lemma 9(c), the curve $\Phi(\sigma)$ travels first in the left half-plane Re $w < 0$, then proceeds in $\mathbb{C}\backslash\{w : 1 \le \text{Re } w \le t, \text{Im } w \le 0\}$, and meets the imaginary axis for the last time at a unique point $\Phi(\sigma(\tau_0))$, $\tau_0 \in (0, 1)$, before it terminates in $(1, t)$. Let $\tilde{\sigma} : \sigma = \sigma(\tau), \tau_0 \le \tau \le 1$, and let $\widetilde{\Sigma}$ be the set of all paths $\tilde{\sigma}$. It is immediate from the definition of $\tilde{\sigma}$ that $\Phi(\widetilde{\Sigma}) \subseteq \widetilde{\Delta}$. Also, $\widetilde{\Delta} \subseteq \Phi(\widetilde{\Sigma})$. For if $\alpha' \in \widetilde{\Delta}$ is a path starting from a point $q \in \Phi(\mathbb{H}^+)$ on the imaginary axis, then Lemma 9(b) yields a path β' starting from a boundary point of $\Phi(\mathbb{H}^+)$ in the left half-plane, terminating at q, and lying otherwise in $\Phi(\mathbb{H}^+)$. Then $\alpha' \in \Phi(\widetilde{\Sigma})$ since $\Phi^{-1}(\beta'\alpha') \in \Sigma$ and $\Phi^{-1}(\alpha') \in \widetilde{\Sigma}$.

Hence $\lambda(\Phi(\widetilde{\Sigma})) = \lambda(\widetilde{\Delta})$. By invoking the conformal invariance of extremal length [2, pp. 220–223] and the conformality of the functions F, Φ, and κ, we conclude that $\lambda(\Gamma^+) = \lambda(F(\Gamma^+))$, $\lambda(\widetilde{\Sigma}) = \lambda(\Phi(\widetilde{\Sigma})) = \lambda(\widetilde{\Delta})$, $\lambda(\Delta) = \lambda(\kappa(\Delta)) = \lambda(\Upsilon)$. Also, by invoking the comparison principle [2, p. 222] and the facts $F(\Gamma^+) \subseteq \Sigma$, $\widetilde{\Sigma} < \Sigma$ (every path $\sigma \in \Sigma$ contains a path $\tilde{\sigma} \in \widetilde{\Sigma}$), and $\widetilde{\Delta} \subseteq \Delta$, we obtain $\lambda(F(\Gamma^+)) \ge \lambda(\Sigma) \ge \lambda(\widetilde{\Sigma})$ and $\lambda(\widetilde{\Delta}) \ge \lambda(\Delta)$. It follows that

$$\lambda(\Gamma^+) = \lambda(F(\Gamma^+)) \ge \lambda(\Sigma) \ge \lambda(\widetilde{\Sigma}) = \lambda(\Phi(\widetilde{\Sigma})) \ge \lambda(\Delta) = \lambda(\kappa(\Delta)) = \lambda(\Upsilon).$$

Thus $\lambda(\Gamma^+) \ge \lambda(\Upsilon)$. A similar argument gives $\lambda(\Gamma^-) \ge \lambda(\overline{\Upsilon})$. But $\lambda(\Upsilon) = \lambda(\overline{\Upsilon})$ because of symmetry. Then a composition law [2, p. 222] and an application of [11, Lem. 6.5] yield

$$\lambda(\Gamma) \ge \lambda(\Gamma^+) + \lambda(\Gamma^-) \ge \lambda(\Upsilon) + \lambda(\overline{\Upsilon}) = 2\lambda(\Upsilon) = \lambda(\Lambda).$$

Thus $\lambda(\Gamma) \ge \lambda(\Lambda)$, equivalently $\text{Mod}(\mathbb{A}_R) \le \text{Mod}(\mathcal{T}(\nu(2t-1)))$. But $\text{Mod}(\mathbb{A}_R) = \text{Mod}(\mathcal{T}(s))$; hence $\text{Mod}(\mathcal{T}(s)) \le \text{Mod}(\mathcal{T}(\nu(2t-1))$, which by the comparison principle *loc. cit.* gives $s \le \nu(2t-1)$, or $t \ge (\nu^{-1}(s)+1)/2$ since ν is an increasing function. Therefore, (3) follows from Lemma 10, and the proof is complete. □

Theorem 1 leads at once to the following. If $s = 2$, then $t \ge 5\sqrt{2}/(4\sqrt{3}) > 1.02062$; if $s = 3$, then $t \ge 3\sqrt{2}/4 > 1.06066$, if $s = 4$, then $t \ge 7\sqrt{2}/(4\sqrt{5}) > 1.10679$, if $s = 7$, then $t \ge 1.25$, and if $s = 99$, then $t > 3.60624$. Also, a lower bound for t is asymptotic to $(\sqrt{2}/4)\sqrt{s}$ as $s \to \infty$.

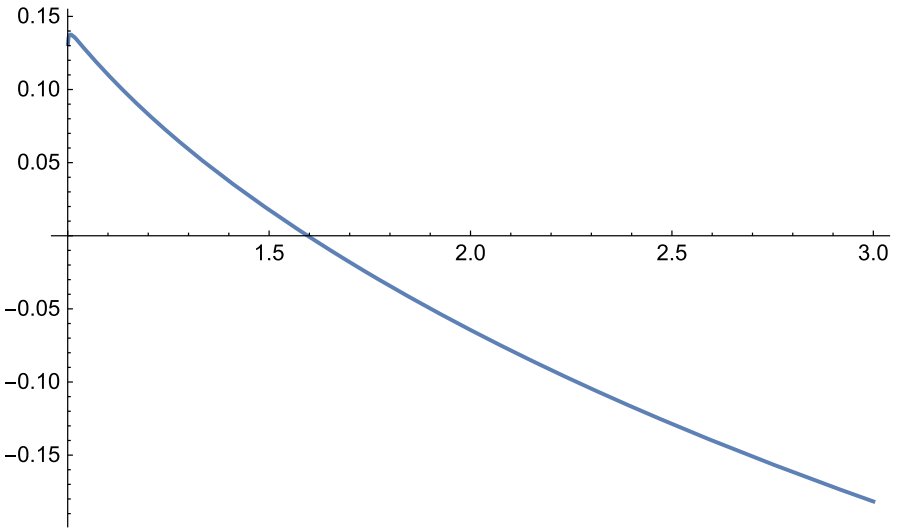

Fig. 10 The graph of $C(s)$

4 Comparing Theorem 1 and Conjecture 1

In this section, we compare for a harmonic homeomorphism $f : \mathcal{T}(s) \to \mathcal{T}(t)$ the lower bound of t proposed in Conjecture 1 [8] with the one obtained in Theorem 1.

We use henceforth the notation $\mu(r)$, $0 < r < 1$, for the modulus of the Grötzsch's ring domain obtained by cutting the open the unit disc along the line segment $[0, r]$; see [11, p. 53].

Let $f : \mathcal{T}(s) \to \mathcal{T}(t)$ be a harmonic homeomorphism. Theorem 1 asserts that

$$\mu\left(\sqrt{\frac{2}{t+1}}\right) \geq \mu\left(\sqrt{\frac{8\sqrt{s+1}}{\sqrt{2}(s+3)+4\sqrt{s+1}}}\right) = A(s). \tag{9}$$

On the other hand, Conjecture 1 suggests that

$$\mu\left(\sqrt{\frac{2}{t+1}}\right) \geq \log\cosh\left[\mu\left(\sqrt{\frac{2}{s+1}}\right)\right] = B(s). \tag{10}$$

It is noteworthy that $\mathrm{Mod}(\mathcal{T}(t)) = \mu(\sqrt{2/(t+1)})$.

We compare $A(s)$ and $B(s)$ for all s, $1 < s \leq 3$, by graphing the function $C(s) = A(s) - B(s)$ via Mathematica; see Fig. 5 and, for a closer look, Fig. 6. Clearly, there exists a value $s_0 \approx 1.6$ such that $C(s)$ is positive whenever $1 < s < s_0$ and negative whenever $s_0 < s \leq 3$. Note that whether $C(s)$ is positive or negative implies respectively whether the $\inf_f t$ on all homeomorphisms f is larger than or smaller than that proposed by the conjecture (Figs. 10, 11).

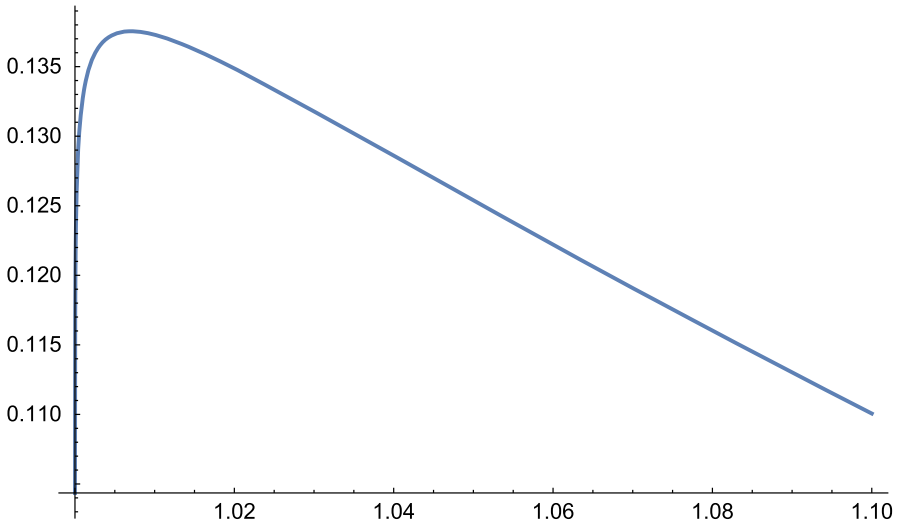

Fig. 11 A closer view of the graph of $C(s)$

As $1 < s < s_0$, then $A(s) > B(s)$. Even though this does not disprove the conjecture, it however indicates a strong doubt about its validity. As $s_0 < s < 3$, then $A(s) < B(s)$ (in fact this inequality holds for all values greater than s_0). This would disprove the conjecture by providing a smaller lower bound for $\mu(\sqrt{2/(t+1)})$ than the proposed one, given that there exists a value

$$t \geq \left(\sqrt{\frac{8\sqrt{s+1}}{\sqrt{2}(s+3)+4\sqrt{s+1}}} \right)$$

for which $A(s) \leq \mu(\sqrt{2/(t+1)}) < B(s)$ for some s. Whether such values s and t exist remains open.

We conclude the paper with the following question.

Question 2 *Let $f : \mathcal{T}(s) \to \mathcal{T}(t)$ be a harmonic homeomorphism. Then (i) is (3) sharp for every $s > 1$? If otherwise, (ii) is (3) sharp for some s?*

Data Availability Not applicable.

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